Mixed-type reverse order laws for the group inverses in rings with involution

Dijana Mosić and Dragan S. Djordjević*

Abstract
We investigate some equivalent conditions for the reverse order laws \((ab)^\# = b^a^\#\) and \((ab)^\# = b^\#_a^\#\) in rings with involution. Similar results for \((ab)^\# = b^\#a^\#\) and \((ab)^\# = b^*a^\#\) are presented too.

Key words and phrases: Group inverse; Moore–Penrose inverse; Reverse order law.

1 Introduction
Let \(R\) be an associative ring with the unit 1, and let \(a \in R\). Then \(a\) is group invertible if there is \(a^\# \in R\) such that

1. \(aa^\#a = a\), 
2. \(a^\#aa^\# = a^\#\), 
5. \(aa^\# = a^\#a\);

\(a^\#\) is a group inverse of \(a\) and it is uniquely determined by these equations. The group inverse \(a^\#\) double commutes with \(a\), that is, \(ax = xa\) implies \(a^\#x = xa^\#\ [1]\). Denote by \(R^\#\) the set of all group invertible elements of \(R\).

An involution \(a \mapsto a^*\) in a ring \(R\) is an anti-isomorphism of degree 2, that is,

\((a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*\).

An element \(a \in R\) is self-adjoint (or Hermitian) if \(a^* = a\).

The Moore–Penrose inverse (or MP-inverse) of \(a \in R\) is the element \(a^\dagger \in R\), if the following equations hold [9]:

1. \(aa^\dagger a = a\), 
2. \(a^\dagger aa^\dagger = a^\dagger\), 
3. \((aa^\dagger)^* = aa^\dagger\), 
4. \((a^\dagger a)^* = a^\dagger a\).

*The authors are supported by the Ministry of the Ministry of Education and Science, Republic of Serbia, grant no. 174007.
There is at most one $a^\dagger$ such that above conditions hold. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^1$.

If $\delta \subset \{1, 2, 3, 4, 5\}$ and $b$ satisfies the equations $(i)$ for all $i \in \delta$, then $b$ is an $\delta$–inverse of $a$. The set of all $\delta$–inverse of $a$ is denoted by $a\{\delta\}$. Notice that $a\{1, 2, 5\} = \{a^\#\}$ and $a\{1, 2, 3, 4\} = \{a^\dagger\}$. If $a$ is invertible, then $a^\#$ and $a^\dagger$ coincide with the ordinary inverse $a^{-1}$ of $a$. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{-1}$.

For $a \in \mathcal{R}$ consider two annihilators

$$a^\circ = \{x \in \mathcal{R} : ax = 0\}, \quad a^\circ = \{x \in \mathcal{R} : xa = 0\}.$$

For invertible elements $a, b \in \mathcal{R}$, the inverse of the product $ab$ satisfies the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. A natural consideration is to see what will be obtained if we replace the inverse by other type of generalized inverses. The reverse order laws for various generalized inverses yield a class of interesting problems which are fundamental in the theory of generalized inverses. They have attracted considerable attention since the middle 1960s, and many interesting results have been obtained [1, 2, 3, 4, 5, 6].

C.Y. Deng [3] presented some necessary and sufficient conditions concerning the reverse order law $(ab)^\# = b^\#a^\#$ for the group invertible linear bounded operators $a$ and $b$ on a Hilbert space. He used the matrix form of operators induced by some natural decomposition of Hilbert spaces.

Inspired by [3], in this paper we present equivalent conditions which are related to the reverse order laws for the group inverses in rings with involution. In particular, we obtain equivalent conditions for $(ab)^\# = b^\#a^\dagger$ and $(ab)^\# = b^\dagger a^\#$ to hold. We also characterize the rules $(ab)^\# = b^\#a^\ast$ and $(ab)^\# = b^\ast a^\#$. Assuming that $a$ is Moore-Penrose invertible, and that $b$ is group invertible, we study the reverse order laws $(ab)^\# = (a^\dagger ab)^\#a^\dagger$, $(ab)^\# = (a^*ab)^\#a^\ast$, $(a^\dagger ab)^\# = b^\#a^\dagger a$, $(a^*ab)^\# = b^\#a^\dagger a$, $(a^\dagger ab)^\#a^\dagger = b^\#a^\dagger$ and $(a^*ab)^\#a^\ast = b^\#a^\ast$. When we suppose that $a$ is group invertible and $b$ is Moore-Penrose invertible, we get similar results for the reverse order laws $(ab)^\# = b^\dagger(ab^\ast)^\#$, $(ab)^\# = b^\ast(ab^\dagger)^\#$, $(ab^\dagger)^\# = bb^\dagger a^\#$, $(ab^\ast)^\# = bb^\ast a^\#$, $b^\dagger(ab^\dagger)^\# = b^\dagger a^\#$ and $b^\ast(ab^\ast)^\# = b^\ast a^\#$. Also, we show that $(ab)^\{5\} \subseteq (a^\dagger ab)^\{1, 5\} \cdot a^\dagger$ is equivalent to $(ab)^\{5\} = (a^\dagger ab)^\{1, 5\} \cdot a^\dagger$ and similar statements for $(ab)^\{5\} \subseteq (a^*ab)^\{1, 5\} \cdot a^\ast$, $(ab)^\{5\} \subseteq b^\dagger \cdot (bb^\dagger)^\{1, 5\}$ and $(ab)^\{5\} \subseteq b^\ast \cdot (bb^\ast)^\{1, 5\}$. 

2
2 Reverse order laws involving triple products

Several equivalent conditions for \((ab)^\# = (a^\dagger ab)^\# a^\dagger\) and \((ab)^\# = (a^* ab)^\# a^*\) to hold are presented in the following theorems.

**Theorem 2.1.** Let \(b \in \mathcal{R}\) and \(a \in \mathcal{R}^\dagger\). If \(a^\dagger ab \in \mathcal{R}^\#,\) then the following statements are equivalent:

(i) \(ab \in \mathcal{R}^\#\) and \((ab)^\# = (a^\dagger ab)^\# a^\dagger\),

(ii) \((a^\dagger ab)^\# a^\dagger \in (ab)\{5\},\)

(iii) \(abaa^\dagger = ab\) and \((a^\dagger ab)^\# aba = ab(a^\dagger ab)^\# a^\dagger a,\)

(iv) \((a^\dagger ab)\{1, 5\} \cdot a^\dagger \subseteq (ab)\{5\}.

**Proof.**

(i) \(\Rightarrow\) (ii): Obviously.

(ii) \(\Rightarrow\) (iii): From the condition \((a^\dagger ab)^\# a^\dagger \in (ab)\{5\},\) we have \(ab(a^\dagger ab)^\# a^\dagger = (a^\dagger ab)^\# a^\dagger ab.\) So, \(ab(a^\dagger ab)^\# a^\dagger a = (a^\dagger ab)^\# a^\dagger aba.\) Observe that \((a^\dagger ab)^\# a^\dagger \in (ab)\{1\},\) by

\[
ab(a^\dagger ab)^\# a^\dagger = a(a^\dagger ab(a^\dagger ab)^\# a^\dagger ab) = aa^\dagger ab = ab. \tag{1}
\]

Now, we get

\[
abaa^\dagger = abab(a^\dagger ab)^\# a^\dagger aa^\dagger = abab(a^\dagger ab)^\# a^\dagger = ab.
\]

(iii) \(\Rightarrow\) (iv): Assume that \(abaa^\dagger = ab\) and \((a^\dagger ab)^\# aba = ab(a^\dagger ab)^\# a^\dagger a.\) If \((a^\dagger ab)^{(1, 5)} \in (a^\dagger ab)\{1, 5\},\) then

\[
a^\dagger ab(a^\dagger ab)^{(1, 5)} = (a^\dagger ab)^\# a^\dagger ab(a^\dagger ab(a^\dagger ab)^{(1, 5)} a^\dagger ab) = (a^\dagger ab)^\# a^\dagger ab. \tag{2}
\]

Using the equalities (2) and (iii), we obtain that \((a^\dagger ab)^{(1, 5)} a^\dagger \in (ab)\{5\}:

\[
ab(a^\dagger ab)^{(1, 5)} a^\dagger = a(a^\dagger ab(a^\dagger ab)^{(1, 5)}) a^\dagger = aa^\dagger ab(a^\dagger ab)^\# a^\dagger
\]

\[
= (ab(a^\dagger ab)^\# a^\dagger a)a^\dagger = (a^\dagger ab)^\# a^\dagger (abaa^\dagger)
\]

\[
= (a^\dagger ab)^\# a^\dagger ab = (a^\dagger ab)^{(1, 5)} a^\dagger ab.
\]

Hence, for any \((a^\dagger ab)^{(1, 5)} \in (a^\dagger ab)\{1, 5\},\) \((a^\dagger ab)^{(1, 5)} a^\dagger \in (ab)\{5\}\) and the statement (iv) holds.

(iv) \(\Rightarrow\) (i): Since \((a^\dagger ab)^\# \in (a^\dagger ab)\{1, 5\},\) by (iv), \((a^\dagger ab)^\# a^\dagger \in (ab)\{5\}.\)

The equalities (1) and

\[
((a^\dagger ab)^\# a^\dagger ab(a^\dagger ab)^\#) a^\dagger = (a^\dagger ab)^\# a^\dagger
\]

imply \((a^\dagger ab)^\# a^\dagger \in (ab)\{1, 2\}\) and the condition (i) is satisfied. \(\square\)
Theorem 2.2. Let $b \in \mathcal{R}$ and $a \in \mathcal{R}^\dagger$. If $a^*ab \in \mathcal{R}^#$, then the following statements are equivalent:

(i) $ab \in \mathcal{R}^#$ and $(ab)^# = (a^*ab)^#a^*$,
(ii) $(a^*ab)^#a^* \in (ab)\{5\}$,
(iii) $abaa^\dagger = ab$ and $(a^*ab)^#a^*aba = ab(a^*ab)^#a^*a,$
(iv) $(a^*ab)\{1,5\} \cdot a^* \subseteq (ab)\{5\}.$

Proof. Using $a = (a^\dagger)^*a$ and $a^* = a^*aa^\dagger$, we verify this result similarly as in Theorem 2.1. □

The following results concerning $(ab)^# = b^\dagger(ab^\dagger)^#$ and $(ab)^* = b^*(abb^*)^#$ are actually dual to Theorems 2.1 and 2.2, where dual means working in the opposite ring $(\mathcal{R}, \circ)$ with reverse multiplication $a \circ b = ba$.

Corollary 2.1. Let $a \in \mathcal{R}$ and $b \in \mathcal{R}^\dagger$. If $abb^\dagger \in \mathcal{R}^#$, then the following statements are equivalent:

(i) $ab \in \mathcal{R}^#$ and $(ab)^# = b^\dagger(ab^\dagger)^#$,
(ii) $b^\dagger(ab^\dagger)^# \in (ab)\{5\}$,
(iii) $b^\dagger bab = ab$ and $bab^\dagger(ab^\dagger)^# = bb^\dagger(ab^\dagger)^#ab,$
(iv) $b^\dagger \cdot (ab^\dagger)\{1,5\} \subseteq (ab)\{5\}.$

Corollary 2.2. Let $a \in \mathcal{R}$ and $b \in \mathcal{R}^\dagger$. If $abb^* \in \mathcal{R}^#$, then the following statements are equivalent:

(i) $ab \in \mathcal{R}^#$ and $(ab)^* = b^*(abb^*)^#,$
(ii) $b^*(abb^*)^# \in (ab)\{5\}$,
(iii) $b^*bab = ab$ and $bab^*(abb^*)^# = bb^*(abb^*)^#ab,$
(iv) $b^* \cdot (abb^*)\{1,5\} \subseteq (ab)\{5\}.$

In the following theorem, we prove that $(ab)\{5\} \subseteq (a^\dagger ab)\{1,5\} \cdot a^\dagger$ is equivalent to $(ab)\{5\} = (a^\dagger ab)\{1,5\} \cdot a^\dagger.$

Theorem 2.3. Let $b \in \mathcal{R}$ and $a \in \mathcal{R}^\dagger$. If $ab, a^\dagger ab \in \mathcal{R}^#$, then the following statements are equivalent:

(i) $(ab)\{5\} \subseteq (a^\dagger ab)\{1,5\} \cdot a^\dagger,$
(ii) \((ab)\{5\} = (a^\dagger ab)\{1,5\} \cdot a^\dagger\).

**Proof.** (i) \(\Rightarrow\) (ii): Assume that \((ab)\{5\} \subseteq (a^\dagger ab)\{1,5\} \cdot a^\dagger\). Because \((ab)^\# \in (ab)\{5\}\), then there exists \((a^\dagger ab)^{(1,5)} \in (a^\dagger ab)\{1,5\}\) such that \((ab)^\# = (a^\dagger ab)^{(1,5)}a^\dagger\). Since the equalities (2) hold again, we obtain

\[(a^\dagger ab)^\# = (a^\dagger ab)^\# a^\dagger ab(a^\dagger ab)^\# = (a^\dagger ab)^{(1,5)}a^\dagger ab(a^\dagger ab)^{(1,5)}\]

which implies

\[(a^\dagger ab)^\dagger a^\dagger = ((a^\dagger ab)^{(1,5)}a^\dagger)ab((a^\dagger ab)^{(1,5)}a^\dagger) = (ab)^\# ab(ab)^\# = (ab)^\#\].

By Theorem 2.1, we deduce that \((a^\dagger ab)\{1,5\} \cdot a^\dagger \subseteq (ab)\{5\}\). Hence, the condition (ii) holds.

(ii) \(\Rightarrow\) (i): This is obvious. \(\square\)

Analogously to Theorem 2.3, we obtain the following theorem.

**Theorem 2.4.** Let \(b \in \mathcal{R}\) and \(a \in \mathcal{R}^\dagger\). If \(ab, a^* ab \in \mathcal{R}^\#\), then the following statements are equivalent:

(i) \((ab)\{5\} \subseteq (a^* ab)\{1,5\} \cdot a^*,\)

(ii) \((ab)\{5\} = (a^* ab)\{1,5\} \cdot a^*\).

Applying Theorems 2.3 and 2.4 to the opposite ring \((\mathcal{R}, \circ)\), we get the dual statements.

**Corollary 2.3.** Let \(a \in \mathcal{R}\) and \(b \in \mathcal{R}^\dagger\). If \(ab, abb^\dagger \in \mathcal{R}^\#\), then the following statements are equivalent:

(i) \((ab)\{5\} \subseteq b^\dagger \cdot (abb^\dagger)\{1,5\},\)

(ii) \((ab)\{5\} = b^\dagger \cdot (abb^\dagger)\{1,5\}.\)

**Corollary 2.4.** Let \(a \in \mathcal{R}\) and \(b \in \mathcal{R}^\dagger\). If \(ab, abb^* \in \mathcal{R}^\#\), then the following statements are equivalent:

(i) \((ab)\{5\} \subseteq b^* \cdot (abb^*)\{1,5\},\)

(ii) \((ab)\{5\} = b^* \cdot (abb^*)\{1,5\}.\)

Now, we consider the conditions which ensure that the reverse order laws \((a^\dagger ab)^\# = b^\# a^\dagger a\) and \((abb^\dagger)^\# = bb^\dagger a^\#\) hold.
Theorem 2.5. If $a \in \mathcal{R}^\dagger$ and $b \in \mathcal{R}^\#$, then the following statements are equivalent:

(i) $a^\dagger ab \in \mathcal{R}^\#$ and $(a^\dagger ab)^\# = b^\# a^\dagger a$,

(ii) $a^\dagger ab = ba^\dagger a$.

Proof. (i) $\Rightarrow$ (ii): From the assumption $(a^\dagger ab)^\# = b^\# a^\dagger a$, we obtain

$$a^\dagger abb^\# a^\dagger a = b^\# a^\dagger a a^\dagger ab = b^\# a^\dagger ab$$

and

$$b^\# a^\dagger a = b^\# a^\dagger (a^\dagger ab)b^\# a^\dagger a = b^\# (a^\dagger abb^\# a^\dagger a) = b^\# b^\# a^\dagger ab. \tag{4}$$

The equalities (3) and (4) imply

$$ba^\dagger a = b^2 (b^\# a^\dagger a) = b^2 b^\# a^\dagger ab = bb^\# a^\dagger ab. \tag{5}$$

and

$$(a^\dagger abb^\# a^\dagger a)b = b^\# a^\dagger abb = b(b^\# b^\# a^\dagger ab)b = bb^\# a^\dagger ab. \tag{6}$$

Since

$$a^\dagger ab = a^\dagger ab(a^\dagger ab)^\dagger a^\dagger ab = a^\dagger abb^\# a^\dagger a a^\dagger ab = a^\dagger abb^\# a^\dagger ab,$$

by (6) and (5), we get

$$a^\dagger ab = bb^\# a^\dagger ab = ba^\dagger a.$$

Hence, the condition (ii) holds.

(ii) $\Rightarrow$ (i): Assume that $a^\dagger ab = ba^\dagger a$. Because the group inverse $b^\#$ double commutes with $b$, we deduce that $a^\dagger ab^\# = b^\# a^\dagger a$ and $a^\dagger abb^\# = bb^\# a^\dagger a$. We can easily verify that $b^\# a^\dagger a \in (a^\dagger ab)\{1, 2, 5\}$. \hfill $\square$

Remark 2.1 Applying Theorem 2.5 with a projection $p = a^\dagger a$ (hence $p = p^\#$), for $b \in \mathcal{R}^\#$, we recover the equivalence $pb \in \mathcal{R}^\#$ and $(pb)^\# = b^\# p \iff pb = bp$.

Dually to Theorem 2.5, we can check the following result.

Corollary 2.5. If $a \in \mathcal{R}^\#$ and $b \in \mathcal{R}^\dagger$, then the following statements are equivalent:

(i) $abb^\dagger \in \mathcal{R}^\#$ and $(abb^\dagger)^\# = bb^\dagger a^\#$,

(ii) $abb^\dagger = bb^\dagger a$. 

6
Notice that the condition (ii) of Theorem 2.5 can be written as \( a_\pi^\pi b = ba_\pi^\pi \), where \( a_\pi^\pi = 1 - a^\dagger a \). The condition \( ab_\pi^\pi = bb_\pi^\pi a \) of Corollary 2.5 is equivalent to \( ab_\pi^\pi = b_\pi^\pi a \), where \( b_\pi^\pi = 1 - bb_\pi^\star \). If \( a \) is EP element \((a \in \mathcal{R}_1 \text{ and } a^\dagger a = aa^\dagger \text{ or equivalently } a \in \mathcal{R}_1 \cap \mathcal{R}_\# \text{ and } a^\dagger = a^\#)\), then \( a^\pi = a_\pi^\pi = a_\pi^\star \) is the spectral idempotent of the element \( a \).

The following results give the equivalent conditions to \((a^\ast ab)^\# = b^\# a^\dagger a \) and \((ab^\ast)^\# = bb_\pi^\dagger a^\# \).

**Theorem 2.6.** If \( a \in \mathcal{R}_1 \) and \( b, a^\ast ab \in \mathcal{R}_\# \), then the following statements are equivalent:

(i) \((a^\ast ab)^\# = b^\# a^\dagger a \),

(ii) \( a^\ast ab = ba^\dagger a \).

**Proof.** (i) \(\Rightarrow\) (ii): Suppose that \((a^\ast ab)^\# = b^\# a^\dagger a \). Then

\[
\begin{align*}
    b^\# a^\dagger a &= b^\# a^\dagger a(a^\ast ab)b^\# a^\dagger a = b^\# (a^\ast ab b^\# a^\dagger a) = b^\# b^\# a^\dagger a a^\ast ab = b^\# b^\# a^\ast ab \\
    &= b^\# a^\dagger a a^\ast ab = bb^\# a^\ast ab = bb(b^\# b^\# a^\ast ab) = bbb^\# a^\dagger a = ba^\dagger a.
\end{align*}
\]

(ii) \(\Rightarrow\) (i): If \( a^\ast ab = ba^\dagger a \), we get

\[ a^\ast ab = ba^\dagger a = bb^\# (ba^\dagger a) = bb^\# a^\ast ab. \]

Now, from

\[
(a^\ast ab)^\# = (a^\ast ab)(a^\# (a^\ast ab)^\#)^2 = bb^\# a^\ast ab[(a^\ast ab)^\#]^2 = bb^\# a^\dagger a (a^\ast ab)^\#
\]

and

\[
b^\# a^\dagger a = b^\# b^\# (ba^\dagger a) = b^\# b^\# (a^\ast ab) = b^\# b^\# (a^\ast ab) a^\ast ab(a^\ast ab)^\# = b^\# b^\# ba^\dagger a a^\ast ab(a^\ast ab)^\# = b^\# (a^\ast ab)(a^\ast ab)^\# = b^\# ba^\dagger a (a^\ast ab)^\#, \]

we obtain that \((a^\ast ab)^\# = b^\# a^\dagger a \).

The dual statement to Theorem 2.6 also holds.

**Corollary 2.6.** If \( b \in \mathcal{R}_1 \) and \( a, abb^\ast \in \mathcal{R}_\# \), then the following statements are equivalent:
In the following theorem, we give necessary and sufficient conditions for \((a^\dagger ab)^\#a^\dagger = b^\#a^\dagger\) to be satisfied.

**Theorem 2.7.** If \(a \in \mathcal{R}^\dagger\) and \(b, a^\dagger ab \in \mathcal{R}^\#\), then the following statements are equivalent:

(i) \((a^\dagger ab)^\#a^\dagger = b^\#a^\dagger\),

(ii) \(ba^\dagger a = a^\dagger aba^\dagger a\),

(iii) \(ba^\dagger a \mathcal{R} \subseteq a^\dagger \mathcal{R}\) (or \(\circ(a^*) \subseteq \circ(ba^\dagger a)\)).

**Proof.** (i) \(\Rightarrow\) (ii): Let \((a^\dagger ab)^\#a^\dagger = b^\#a^\dagger\). Now the equality

\[a^\dagger ab = a^\dagger ab(a^\dagger ab)^\#a^\dagger ab = ((a^\dagger ab)^\#a^\dagger)aba^\dagger ab = b^\#a^\dagger aba^\dagger ab\]

implies

\[(a^\dagger ab)^\dagger a = b^\#a^\dagger aba^\dagger a = b^\#b(b^\#a^\dagger aba^\dagger ab)a^\dagger a = b^\#ba^\dagger aba^\dagger a = b((a^\dagger ab)^\#a^\dagger ab)a^\dagger a = ba^\dagger ab((a^\dagger ab)^\#a^\dagger) a = ba^\dagger ab(b^\#a^\dagger a).\]

Using this equality and

\[b^\#a^\dagger = (a^\dagger ab)^\#a^\dagger = (a^\dagger ab)^\#a^\dagger ab(a^\dagger ab)^\#a^\dagger = b^\#a^\dagger abb^\#a^\dagger,\]

we obtain

\[a^\dagger aba^\dagger a = ba^\dagger abb^\#a^\dagger a = b^2(b^\#a^\dagger abb^\#a^\dagger) a = b^2b^\#a^\dagger a = ba^\dagger a.\]

So, the statement (ii) is satisfied.

(ii) \(\Rightarrow\) (i): Applying the hypothesis \(ba^\dagger a = a^\dagger aba^\dagger a\), we get

\[b^\#a^\dagger = b^\#b^\#(ba^\dagger a)a^\dagger = b^\#b^\#(a^\dagger ab)a^\dagger aa^\dagger = b^\#b^\#a^\dagger ab((a^\dagger ab)^\#a^\dagger ab)a^\dagger = b^\#b^\#a^\dagger ab(a^\dagger ab)^\#a^\dagger = b^\#a^\dagger ab(a^\dagger ab)^\#a^\dagger.\]  

(7)

Since

\[(a^\dagger ab)^\# = a^\dagger ab[(a^\dagger ab)^\#]^2 = a^\dagger a(a^\dagger ab[(a^\dagger ab)^\#]^2) = a^\dagger a(a^\dagger ab)^\#,\]

8
then

\[(a^1ab)^\#a^1 = (a^1aba^1a)b[(a^1ab)^\#]^3a^1 = ba^1ab[(a^1ab)^\#]^3a^1 \]
\[= bb^\#(ba^1a)b[(a^1ab)^\#]^3a^1 = bb^\#a^1ab[(a^1ab)^\#]^3a^1 \]
\[= bb^\#a^1a(1)a^1ab[(a^1ab)^\#]^3a^1 = b^\#(ba^1a)(a^1ab)^\#a^1 \]
\[= b^\#a^1ab(a^1a(1)a^1)^\#a^1 = b^\#a^1ab(a^1ab)^\#a^1, \]

which yields, by (7), \((a^1ab)^\#a^1 = b^\#a^1.\)

(ii) \(\Leftrightarrow\) (iii): The condition \(ba^1a = a^1aba^1a\) gives \(ba^1aR \subseteq a^1R = a^*R.\) Conversely, from \(ba^1aR \subseteq a^*R,\) we conclude that \(ba^1a = a^*x\) for some \(x \in R.\) Now, \(ba^1a = a^*x = a^1a(a^*x) = a^1aba^1a.\)

Obviously, for condition (ii) of Theorem 2.7, we have \(ba^1a = a^1aba^1a \Leftrightarrow a^*b(1 - a^*f) = 0 \Leftrightarrow a^*b(1 - a^*f) = (1 - a^*f)a^*b.\)

The following theorem can be proved in the similar manner as Theorem 2.7.

**Theorem 2.8.** If \(a \in R^1\) and \(b, a^*ab \in R^\#,\) then the following statements are equivalent:

(i) \((a^*ab)^\#a^* = b^\#a^*,\)

(ii) \(ba^*a = a^*aba^*a\) (or \(ba^1a = a^*aba^1a\)).

Using Theorems 2.7 and 2.8 to the opposite ring, we obtain the dual results.

**Corollary 2.7.** If \(b \in R^1\) and \(a, abb^1 \in R^\#,\) then the following statements are equivalent:

(i) \(b^1(abb^1)^\# = b^1a^\#,\)

(ii) \(bb^1a = bb^1abb^1,\)

(iii) \(Rbb^1a \subseteq Rb^*\) (or \((b^*)^\circ \subseteq (bb^1a)^\circ).\)

Note that \(bb^1a = bb^1abb^1 \Leftrightarrow (1 - b^*f)ab^*f = 0 \Leftrightarrow (1 - b^*f)ab^*f = ab^*f(1 - b^*f).\)

**Corollary 2.8.** If \(b \in R^1\) and \(a, abb^* \in R^\#,\) then the following statements are equivalent:

(i) \(b^*(abb^*)^\# = b^*a^\#,\)

(ii) \(bb^*a = bb^*abb^*\) (or \(bb^1a = bb^1abb^*\)).

Notice that the conditions of Theorem 2.5 (Theorem 2.6, Corollary 2.5, Corollary 2.6, respectively) imply the conditions of Theorem 2.7 (Theorem 2.8, Corollary 2.7, Corollary 2.8, respectively)
3 Reverse order laws \((ab)^\# = b^# a^\dagger\) and \((ab)^\# = b^# a^*\)

Assuming that \(a\) is Moore-Penrose invertible, and that \(b\) is group invertible in a ring with involution, equivalent conditions to the reverse order law \((ab)^\# = b^# a^\dagger\) are presented in the following theorem.

**Theorem 3.1.** If \(a \in \mathcal{R}^\dagger\) and \(b, ab \in \mathcal{R}^\#\), then the following statements are equivalent:

(i) \((ab)^\# = b^# a^\dagger\),

(ii) \((ab)^\# a = b^# a^\dagger a\) and \(a^* ab = a^* abaa^\dagger\),

(iii) \((ab)^\# a = b^# a^\dagger a\) and \(a^\dagger ab = a^\dagger abaa^\dagger\),

(iv) \(b(ab)^\# = bb^# a^\dagger\) and \(abb^# = bb^# abb^#\).

**Proof.** (i) \(\Rightarrow\) (ii): The hypothesis \((ab)^\# = b^# a^\dagger\) gives \((ab)^\# a = b^# a^\dagger a\) and

\[
\begin{align*}
a^* ab &= a^* ab((ab)^\# ab) = a^* abab(ab)^\# = a^* ababb^# a^\dagger \\
&= a^* (ababb^# a^\dagger) aa^\dagger = a^* abaa^\dagger.
\end{align*}
\]

Hence, the condition (ii) holds.

(ii) \(\Rightarrow\) (iii): Because \((ab)^\# a = b^# a^\dagger a\) and \(a^* ab = a^* abaa^\dagger\), then

\[
a^\dagger ab = a^\dagger (a^\dagger)^* (a^* ab) = a^\dagger (a^\dagger)^* a^* abaa^\dagger = a^\dagger abaa^\dagger.
\]

So, (iii) is satisfied.

(iii) \(\Rightarrow\) (i): Suppose that \((ab)^\# a = b^# a^\dagger a\) and \(a^\dagger ab = a^\dagger abaa^\dagger\). First, we show that \(b^# a^\dagger \in (ab)^\{1,2\}\):

\[
\begin{align*}
(b^# a^\dagger a)b &= (ab)^# ab = (ab)^# a(a^\dagger ab) = (ab)^# a a^\dagger abaa^\dagger \\
&= ((ab)^# ab) aa^\dagger = ab((ab)^# a) a^\dagger = abb^# a^\dagger aa^\dagger \\
&= abb^# a^\dagger.
\end{align*}
\]

Further, from

\[
ab = ab((ab)^# a)b = abb^# a^\dagger ab
\]

and

\[
b^# a^\dagger = (b^# a^\dagger a)a^\dagger = (ab)^# aa^\dagger = ((ab)^# a)b((ab)^# a)a^\dagger \\
= b^# a^\dagger abbb^# a^\dagger aa^\dagger = b^# a^\dagger abbb^# a^\dagger.
\]

we deduce that \(b^# a^\dagger \in (ab)^\{1,2\}\), i.e. \((ab)^# = b^# a^\dagger\).

(i) \(\Leftrightarrow\) (iv): This equivalence can be proved similarly as previous parts. \(\square\)
The condition $a^\dagger ab = a^\dagger abaa^\dagger$ in Theorem 3.1 can be replaced with equivalent conditions $Ra^\dagger ab \subseteq Ra^*$ or $(a^*)^\circ \subseteq (a^\dagger ab)^\circ$. Also, the condition $abb^\# = bb^\# abb^\#$ in Theorem 3.1 can be replaced with equivalent conditions $abb^\# R \subseteq bR$ or $b^* \subseteq (abb^\#)$.

Similarly as in the proof of Theorem 3.1, we get necessary and sufficient conditions which ensure that $(ab)^\# = b^# a^*$ is satisfied.

**Theorem 3.2.** If $a \in R^\dagger$ and $b, ab \in R^\#$, then the following statements are equivalent:

(i) $(ab)^\# = b^# a^*$,
(ii) $(ab)^\# a = b^# a^* a$ and $a^* ab = a^# abaa^\dagger$,
(iii) $(ab)^\# a = b^# a^* a$ and $a^\dagger ab = a^\dagger abaa^\dagger$,
(iv) $b(ab)^\# = bb^\# a^*$ and $abb^\# = bb^\# abb^\#$.

If we suppose that $a$ is EP element in Theorem 3.1 or that $a \in R^\dagger \cap R^\#$ and $a^* = a^#$ in Theorem 3.2, we obtain new characterizations of the classical reverse order law $(ab)^\# = b^# a^*$.

Dually to Theorems 3.1 and 3.2, equivalent conditions for $(ab)^\# = b^\dagger a^#$ and $(ab)^\# = b^* a^#$ are presented.

**Corollary 3.1.** If $b \in R^\dagger$ and $a, ab \in R^\#$, then the following statements are equivalent:

(i) $(ab)^\# = b^\dagger a^\#$,
(ii) $(ab)^\# a = b^\dagger a^\# a$ and $a^\# ab = a^\# abaa^\#$,
(iii) $b(ab)^\# = bb^\dagger a^\#$ and $abb^\dagger = b^\dagger babb^\dagger$,
(iv) $b(ab)^\# = bb^\dagger a^\#$ and $abb^* = b^\dagger babb^*$.

In Corollary 3.1, the condition $a^\# ab = a^\# abaa^\#$ can be replaced with $Ra^\# ab \subseteq Ra$ or $a^\circ \subseteq (a^\# ab)^\circ$, and the condition $abb^\dagger = b^\dagger babb^\dagger$ can be replaced with $abb^\dagger R \subseteq b^* R$ or $(b^*)^\circ \subseteq (abb^\dagger)^\circ$.

**Corollary 3.2.** If $b \in R^\dagger$ and $a, ab \in R^\#$, then the following statements are equivalent:

(i) $(ab)^\# = b^* a^#$,
(ii) $(ab)^\# a = b^* a^\# a$ and $a^\# ab = a^\# abaa^\#$,
Suppose that

(i) Assume that

Proof. (i) Assume that \((ab)^\# = b^*a^#\) and \(abb^\dagger = b^\dagger b a^\dagger b^\dagger\),

(ii) From the hypothesis \((ab)^\# = b^#a^\dagger\) and \((a^\dagger a)^\# = b^*a^#,\) we get

(iii) It follows as part (ii).

(iv) Suppose that \(b(ab)^\# = bb^#a^\dagger = (abb^\#)^\#.\) Then

\[ abb^\# = (ab^\#)^\#(ab^\#)^2 = bb^#a^\dagger(ab^\#)^2 = (bb^#)^2a^\dagger(ab^\#)^2 = bb^#abb^#.\]

By part (iv) of Theorem 3.1, \((ab)^\# = b^\#a^\dagger\).

(v) The condition \(a \in \mathcal{R}^1\) implies \(a^\dagger a \in \mathcal{R}^\#\) and \(a^\dagger = (a^*a)^\#a^*\) (see [8]). The rest of this part follows as (ii).

The following theorem can be proved in the similar way as Theorem 3.3.

Theorem 3.4. Suppose that \(a \in \mathcal{R}^1\) and \(b, ab, a^\dagger ab, abb^\#, a^*ab \in \mathcal{R}^\#.\) Then each of the following conditions is sufficient for \((ab)^\# = b^#a^*\) to hold:

(i) \((ab)^\# = b^*a^\#\) and \(abb^\dagger = b^\dagger b a^\dagger b^\dagger\),

(ii) \((ab)^\# = b^#a^\dagger\) and \((a^\dagger a)^\# = b^#a^\dagger\),

(iii) \((ab)^\# = b^#(abb^\#)^\#\) and \((abb^\#)^\# = bb^#a^*\),
(iv) \(b(ab)^\# = bb^\# a^* = (abb^\#)^\#\),

(v) \((ab)^\# = (a^\dagger ab)^\# a^\dagger\) and \((a^\dagger ab)^\# = b^\# a^* a\).

Notice that the dual results to Theorems 3.3 and 3.4 are satisfied too.

**Corollary 3.3.** Suppose that \(b \in R^\dagger\) and \(a, ab, a^\# ab, abb^\dagger, ab^{*} \in R^\#\). Then each of the following conditions is sufficient for \((ab)^\# = b^\dagger a^\#\) to hold:

(i) \(b(ab)^\# = bb^\dagger a^\#\) and \(b^\dagger ba = abb^\dagger\),

(ii) \((ab)^\# = (a^\# ab)^\# a^\#\) and \((a^\# ab)^\# = b^\dagger a^\# a\),

(iii) \((ab)^\# = b^\dagger (abb^\dagger)^\#\) and \((abb^\dagger)^\# = bb^\dagger a^\#\),

(iv) \((ab)^\# a = b^\dagger a^\# a = (a^\# ab)^\#\),

(v) \((ab)^\# = b^{*} (abb^* )^\#\) and \((abb^* )^\# = (bb^*)^\# a^\#\).

**Corollary 3.4.** Suppose that \(b \in R^\dagger\) and \(a, ab, a^\# ab, abb^\dagger, ab^{*} \in R^\#\). Then each of the following conditions is sufficient for \((ab)^\# = b^* a^\#\) to hold:

(i) \(b(ab)^\# = bb^* a^\#\) and \(b^\dagger ba = abb^\dagger\),

(ii) \((ab)^\# = (a^\# ab)^\# a^\#\) and \((a^\# ab)^\# = b^* a^\# a\),

(iii) \((ab)^\# = b^* (abb^* )^\#\) and \((abb^* )^\# = bb^\dagger a^\#\),

(iv) \((ab)^\# a = b^* a^\# a = (a^\# ab)^\#\),

(v) \((ab)^\# = b^\dagger (abb^\dagger)^\#\) and \((abb^\dagger)^\# = bb^* a^\#\).

**Remark 3.1.** Combining the conditions of Theorem 2.1 and Theorem 2.7, we get the sufficient conditions for the reverse order law \((ab)^\# = b^\# a^\dagger\) to hold. If we combine the conditions of Corollary 2.1 and Corollary 2.7, we obtain the sufficient conditions for \((ab)^\# = b^\dagger a^\#\) to be satisfied.

Sufficient conditions for the reverse order law \((ab)^\# = b^\# a^*\) ((\(ab)^\# = b^{*} a^\#\)) to hold can be obtained combining the conditions of Theorem 2.2 and Theorem 2.8 (Corollary 2.2 and Corollary 2.8).
4 Other results

More specific results are proved in this section.

**Theorem 4.1.** If \( a \in \mathcal{R}^\dagger \) and \( b, ab \in \mathcal{R}^\# \), then the following statements are equivalent:

(i) \( b^\# = (ab)^\# a \),

(ii) \( b = aa^\dagger b = ba^\dagger a \) and \( abb^\# = bb^\# a \),

(iii) \( b \mathcal{R} \subseteq a \mathcal{R} \), \( aa^\dagger b = ba^\dagger a \) and \( abb^\# = bb^\# a \),

(iv) \( a^\circ \subseteq b^\circ \), \( aa^\dagger b = ba^\dagger a \) and \( abb^\# = bb^\# a \).

**Proof.** (i) \( \Rightarrow \) (ii): Using the equality \( b^\# = (ab)^\# a \), we observe that

\[
ba^\dagger a = b^2 b^\# a^\dagger a = b^2(ab)^\# a = b^2 b^\# = b
\]

and

\[
b = b^\# b^2 = ((ab)^\# ab)b = ab(ab)^\# b
\]

which yields

\[
b = ab(ab)^\# b = aa^\dagger(ab(ab)^\# b) = aa^\dagger b.
\]

Also, by (i), we get

\[
abb^\# = (ab(ab)^\#)a = ((ab)^\# a)ba = b^\# ba.
\]

So, the condition (ii) holds.

(ii) \( \Rightarrow \) (i): Suppose that \( b = aa^\dagger b = ba^\dagger a \) and \( abb^\# = bb^\# a \). Then, from

\[
b(ab)^\# ab = b^{\#} bb(ab)^\# ab = b^{\#} ba^\dagger (ab(ab)^\# ab) = b^{\#} (ba^\dagger a)b = b^{\#} bb = b
\]

and

\[
(ab)^\# ab(ab)^\# a = (ab)^\# a,
\]

we conclude that \( (ab)^\# a \in b\{1, 2\} \). Since

\[
b(ab)^\# a = b[(ab)^\#]^2 aba = b[(ab)^\#]^2(ab)(bb^\# a) = (b(ab)^\# ab)b^\# = bb^\#
\]

and

\[
ab(ab)^\# = (abb^\#)b(ab)^\# = bb^\# ab(ab)^\# = b^\# (b(ab)^\# ab) = b^\# b,
\]

we have \( b(ab)^\# a = ab(ab)^\# = (ab)^\# ab \), that is, \( (ab)^\# a \in b\{5\} \). Hence, the condition (i) is satisfied.

(ii) \( \Leftrightarrow \) (iii): We will show that \( b = aa^\dagger b \) is equivalent to \( b \mathcal{R} \subseteq a \mathcal{R} \). First, \( b = aa^\dagger b \) implies \( b \mathcal{R} \subseteq a \mathcal{R} \). Conversely, if \( b \mathcal{R} \subseteq a \mathcal{R} \), then, for some \( x \in R \), \( b = ax \) which gives \( b = aa^\dagger(ax) = aa^\dagger b \).

(ii) \( \Leftrightarrow \) (iv): It follows from \( b = ba^\dagger a \) iff \( b(1 - a^\dagger a) = 0 \) iff \( (1 - a^\dagger a) \mathcal{R} \subseteq b^\circ \) iff \( a^\circ \subseteq b^\circ \). \( \square \)
The next result follows dually to Theorem 4.1.

**Theorem 4.2.** If $b \in \mathcal{R}^\dagger$ and $a, ab \in \mathcal{R}^\#$, then the following statements are equivalent:

(i) $a^\# = b(ab)^\#$,

(ii) $aa^\#b = baa^\#$ and $a = ab^\dagger b = bb^\dagger a$,

(iii) $a \mathcal{R} \subseteq b \mathcal{R}$, $aa^\#b = baa^\#$ and $ab^\dagger b = bb^\dagger a$,

(iv) $b^\circ \subseteq a^\circ$, $aa^\#b = baa^\#$ and $ab^\dagger b = bb^\dagger a$.

Some conditions of Theorem 4.1 and Theorem 4.2 can be written as $aa^\dagger b = ba^\dagger a$ and $abb^\# = bb^\# a$. Hence, $a R \subseteq b R$ and $ab^\dagger b = bb^\dagger a$.

**Theorem 4.3.** If $a \in \mathcal{R}^\dagger$, $b \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:

(i) $a^\dagger ab = b(ab)^\# ab$,

(ii) $baba = a^\dagger ababa$,

(iii) $baba \mathcal{R} \subseteq a^\ast \mathcal{R}$ (or $^\circ (a^\ast) \subseteq ^\circ (baba)$).

**Proof.** (i) $\Rightarrow$ (ii): Using $a^\dagger ab = b(ab)^\# ab$, we have $(a^\dagger ab)aba = b((ab)^\# abab)a = baba$.

Thus, the equality (ii) is satisfied.

(ii) $\Rightarrow$ (i): Since $baba = a^\dagger ababa$, then $a^\dagger ab = a^\dagger abab(ab)^\# = (a^\dagger ababa)b[(ab)^\#]^2 = babab[(ab)^\#]^2 = b(ab)^\# ab$.

(ii) $\Rightarrow$ (iii): It follows by $a^\ast \mathcal{R} = a^\dagger \mathcal{R}$.

(iii) $\Rightarrow$ (ii): By the condition $baba \mathcal{R} \subseteq a^\ast \mathcal{R}$, we see that $baba = a^\ast x$, for $x \in \mathcal{R}$. Hence, $baba = a^\ast x = a^\dagger a(a^\ast x) = a^\dagger ababa$.

□

In the same way as in Theorem 4.3, we prove the following theorem.

**Theorem 4.4.** If $a, b \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:

(i) $a^\ast ab = b(ab)^\# ab$,  

15
Applying Theorems 4.3 and 4.4, we have that the next dual statements hold.

**Theorem 4.5.** If \( a \in \mathcal{R} \), \( b \in \mathcal{R}^\dagger \) and \( ab \in \mathcal{R}^\# \), then the following statements are equivalent:

(i) \( abb^\dagger = ab(ab)^\# a \),

(ii) \( baba = bababb^\dagger \),

(iii) \( \mathcal{R}baba \subseteq \mathcal{R}b^\ast \) (or \( (b^\ast)^\circ \subseteq (baba)^\circ \)).

**Theorem 4.6.** If \( a, b \in \mathcal{R} \) and \( ab \in \mathcal{R}^\# \), then the following statements are equivalent:

(i) \( abb^\ast = ab(ab)^\# a \),

(ii) \( baba = bababb^\ast \).

Some equivalent conditions for \( aa^\# = bb^\dagger \) to hold are given in the following theorem in a ring with involution.

**Theorem 4.7.** If \( a \in \mathcal{R}^\# \) and \( b \in \mathcal{R}^\dagger \), then the following statements are equivalent:

(i) \( aa^\# = bb^\dagger \),

(ii) \( a\mathcal{R} = b\mathcal{R} \) and \( a^\circ = (b^\ast)^\circ \),

(iii) \( a + 1 - bb^\dagger \in \mathcal{R}^{-1} \) and \( aa^\# = aa^\# bb^\dagger = bb^\dagger aa^\# \),

(iv) \( a + 1 - bb^\dagger, 1 - aa^\# + bb^\dagger \in \mathcal{R}^{-1} \) and \( abb^\dagger = bb^\dagger a \),

(v) \( a + 1 - bb^\dagger, 1 - aa^\# + bb^\dagger \in \mathcal{R}^{-1} \) and \( aa^\# bb^\dagger = bb^\dagger aa^\# \).

**Proof.** (i) \( \Rightarrow \) (ii)-(v): This is trivial, when we notice that \((a + 1 - aa^\#)(a^\# + 1 - aa^\#) = 1\) gives \( a + 1 - aa^\# \in \mathcal{R}^{-1} \).

(ii) \( \Rightarrow \) (i): Assume that \( a\mathcal{R} = b\mathcal{R} \) and \( a^\circ = (b^\ast)^\circ \). Now, we have \( b = ax \) for \( x \in \mathcal{R} \) and, by \((b^\ast)^\circ = (1 - bb^\dagger)\mathcal{R}\), \( a^\circ = (1 - bb^\dagger)\mathcal{R} \). Further, \( b = aa^\#(ax) = aa^\# b \) and \( a(1 - bb^\dagger) = 0 \). Thus, \( bb^\dagger = aa^\# bb^\dagger = a^\#(abb^\dagger) = a^\# a \).

(iii) \( \Rightarrow \) (i): Let \( a + 1 - bb^\dagger \in \mathcal{R}^{-1} \) and \( aa^\# = aa^\# bb^\dagger = bb^\dagger aa^\# \). The equalities

\[
(a + 1 - bb^\dagger)bb^\dagger = abb^\dagger + bb^\dagger - bb^\dagger = abb^\dagger
\]
and
\[(a + 1 - bb^\dagger)bb^\dagger aa^\# = a(bb^\dagger aa^\#) = aaa^\# bb^\dagger = abb^\dagger,\]
imply \(bb^\dagger = bb^\dagger aa^\#\). Hence, we get \(bb^\dagger = aa^\#\).

(iv) \(\Rightarrow\) (iii): Since \(abb^\dagger = bb^\dagger a\), and the group inverse \(a^\#\) double commutes with \(a\), then \(a^\# bb^\dagger = bb^\dagger a^\#\) and \(aa^\# bb^\dagger = bb^\dagger aa^\#\). From
\[(1 - aa^\# + bb^\dagger)aa^\# = aa^\# - aa^\# + bb^\dagger aa^\# = bb^\dagger aa^\#,
\[(1 - aa^\# + bb^\dagger)aa^\# bb^\dagger = bb^\dagger(aa^\# bb^\dagger) = bb^\dagger aa^\#,
\]and the condition \(1 - aa^\# + bb^\dagger \in \mathcal{R}^{-1}\), we obtain \(aa^\# = aa^\# bb^\dagger\). So, the statements (iii) holds.

(v) \(\Rightarrow\) (i): This part can be check in the same way as (iv) \(\Rightarrow\) (iii) \(\Rightarrow\) (i).

Changing \(b\) in previous theorem by \(b^\dagger\), by \((b^\dagger)^\dagger = b\), we obtain equivalent conditions for \(aa^\# = b^\dagger b^\dagger\).

**Theorem 4.8.** If \(a \in \mathcal{R}^\#\) and \(b \in \mathcal{R}^\dagger\), then the following statements are equivalent:

(i) \(aa^\# = b^\dagger b\),

(ii) \(a\mathcal{R} = b^*\mathcal{R} \text{ and } a^\circ = b^\circ\),

(iii) \(a + 1 - b^\dagger b \in \mathcal{R}^{-1}\) and \(aa^\# = aa^\# b^\dagger b = b^\dagger baa^\#\),

(iv) \(a + 1 - b^\dagger b, 1 - aa^\# + b^\dagger b \in \mathcal{R}^{-1}\) and \(abb^\dagger = b^\dagger ba\),

(v) \(a + 1 - b^\dagger b, 1 - aa^\# + b^\dagger b \in \mathcal{R}^{-1}\) and \(aa^\# b^\dagger b = b^\dagger baa^\#\).

5 Characterization of operators on Hilbert space

Let \(H\) be a Hilbert space and \(\mathcal{L}(H)\) the set of all linear bounded operators on \(H\). In addition, if \(T \in \mathcal{L}(H)\), then \(T^*\), \(N(T)\) and \(R(T)\) stand for the adjoint, the null space and the range of \(T\), respectively.

In the spirit of previous results, we prove the following one.

**Theorem 5.1.** Let \(A \in \mathcal{L}(H)\) have a closed range and let \(B \in \mathcal{L}(H)\).

(i) If \(AB\) is group invertible, then
\[I + A^\dagger (B - A) \text{ is invertible } \iff AB(AB)^\dagger A = A.\]

(ii) If \(BA\) is group invertible, then
\[I + A^\dagger (B - A) \text{ is invertible } \iff A(BA)^\dagger BA = A.\]
Proof. (i) Since $A \in \mathcal{L}(H)$ have a closed range, there exists the unique Moore–Penrose inverse $A^\dagger \in \mathcal{L}(H)$ of $A$. The operators $A$, $B$ and $AB$ have the matrix representations on $H = R(A^*) \oplus N(A)$ of the forms

$$
A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} A_1 B_1 & A_1 B_3 \\ 0 & 0 \end{bmatrix},
$$

where $A_1$ is invertible. The Moore–Penrose inverse of $A$ is given by

$$
A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.
$$

Analogously as in Theorem [7, Theorem 1] for the matrix case, we can verify that $AB$ is group invertible if and only if $A_1 B_1$ is group invertible and $A_1 B_1 (A_1 B_1)^\# A_1 B_3 = A_1 B_3$. In this case,

$$
(AB)^\# = \begin{bmatrix} (A_1 B_1)^\# & [(A_1 B_1)^\#]^2 A_1 B_3 \\ 0 & 0 \end{bmatrix}.
$$

Observe that, $AB(AB)^\# A = A$ iff $A_1 B_1 (A_1 B_1)^\# A_1 = A_1$ iff $A_1 B_1 (A_1 B_1)^\# = I$ iff $A_1 B_1$ is invertible iff $B_1$ is invertible. Then, by

$$
I + A^\dagger (B - A) = \begin{bmatrix} A_1^{-1} B_1 & A_1^{-1} B_3 \\ 0 & I \end{bmatrix},
$$

we deduce that $I + A^\dagger (B - A)$ is invertible iff $B_1$ is invertible.

(ii) Applying (i) to the opposite ring, we get $I + (B - A) A^\dagger$ is invertible $\Leftrightarrow A(BA)^\# BA = A$. But by Jacobson lemma, $I + (B - A) A^\dagger$ is invertible $\Leftrightarrow I + A^\dagger (B - A)$ is invertible. \hfill \Box

6 Conclusions

In this paper we consider necessary and sufficient conditions related to the reverse order laws $(ab)^\# = b^\# a^\dagger$ and $(ab)^\# = b_1^\dagger a^\#$ in rings with involution, applying a purely algebraic technique. In the case of linear bounded operators on Hilbert spaces, where the method of operator matrices is very useful, similar results for the reverse order law $(ab)^\# = b^\# a^\#$ are given. In a \textit{*}-regular ring $R$, observe that the assumption $a \in R^\dagger$ is automatically satisfied. It could be interesting to extend this work to the reverse order laws of a triple product.

18
References


Address:

Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

E-mail

D. Mosić: dijana@pmf.ni.ac.rs
D. S. Djordjević: dragan@pmf.ni.ac.rs