Mixed-type reverse order laws for the group inverses in rings with involution

Dijana Mosić and Dragan S. Djordjević*

Abstract

We investigate some equivalent conditions for the reverse order laws $(ab)^{\#} = b^{\dagger}a^{\#}$ and $(ab)^{\#} = b^{\#}a^{\dagger}$ in rings with involution. Similar results for $(ab)^{\#} = b^{\#}a^{*}$ and $(ab)^{\#} = b^{*}a^{\#}$ are presented too.

Key words and phrases: Group inverse; Moore–Penrose inverse; Reverse order law.

2010 *Mathematics subject classification*: 16B99, 15A09, 46L05.

1 Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1, and let $a \in \mathcal{R}$. Then a is *group invertible* if there is $a^{\#} \in \mathcal{R}$ such that

$$(1)\ aa^{\#}a = a, \quad (2)\ a^{\#}aa^{\#} = a^{\#}, \quad (5)\ aa^{\#} = a^{\#}a;$$

$a^{\#}$ is a group inverse of a and it is uniquely determined by these equations. The group inverse $a^{\#}$ double commutes with a, that is, $ax = xa$ implies $a^{\#}x = xa^{\#}$ [1]. Denote by $\mathcal{R}^{\#}$ the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^{*}$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^{*})^{*} = a, \quad (a+b)^{*} = a^{*} + b^{*}, \quad (ab)^{*} = b^{*}a^{*}.$$

An element $a \in \mathcal{R}$ is self-adjoint (or Hermitian) if $a^{*} = a$.

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $a^{\dagger} \in \mathcal{R}$, if the following equations hold [9]:

$$(1)\ aa^{\dagger}a = a, \quad (2)\ a^{\dagger}aa^{\dagger} = a^{\dagger}, \quad (3)\ (aa^{\dagger})^{*} = aa^{\dagger}, \quad (4)\ (a^{\dagger}a)^{*} = a^{\dagger}a.$$

*The authors are supported by the Ministry of the Ministry of Education and Science, Republic of Serbia, grant no. 174007.

There is at most one $a^\dagger$ such that above conditions hold. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

If $\delta \subset \{1,2,3,4,5\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denoted by $a\{\delta\}$. Notice that $a\{1,2,5\} = \{a^\#\}$ and $a\{1,2,3,4\} = \{a^\dagger\}$. If a is invertible, then $a^\#$ and $a^\dagger$ coincide with the ordinary inverse a^{-1} of a. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{-1}$.

For $a \in \mathcal{R}$ consider two annihilators

$$a^\circ = \{x \in \mathcal{R} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{R} : xa = 0\}.$$

For invertible elements $a, b \in \mathcal{R}$, the inverse of the product ab satisfied the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. A natural consideration is to see what will be obtained if we replace the inverse by other type of generalized inverses. The reverse order laws for various generalized inverses yield a class of interesting problems which are fundamental in the theory of generalized inverses. They have attracted considerable attention since the middle 1960s, and many interesting results have been obtained [1, 2, 3, 4, 5, 6].

C.Y. Deng [3] presented some necessary and sufficient conditions concerning the reverse order law $(ab)^\# = b^\# a^\#$ for the group invertible linear bounded operators a and b on a Hilbert space. He used the matrix form of operators induced by some natural decomposition of Hilbert spaces.

Inspired by [3], in this paper we present equivalent conditions which are related to the reverse order laws for the group inverses in rings with involution. In particular, we obtain equivalent conditions for $(ab)^\# = b^\# a^\dagger$ and $(ab)^\# = b^\dagger a^\#$ to hold. We also characterize the rules $(ab)^\# = b^\# a^*$ and $(ab)^\# = b^* a^\#$. Assuming that a is Moore-Penrose invertible, and that b is group invertible, we study the reverse order laws $(ab)^\# = (a^\dagger ab)^\# a^\dagger$, $(ab)^\# = (a^* ab)^\# a^*$, $(a^\dagger ab)^\# = b^\# a^\dagger a$, $(a^* ab)^\# = b^\# a^\dagger a$, $(a^\dagger ab)^\# a^\dagger = b^\# a^\dagger$ and $(a^* ab)^\# a^* = b^\# a^*$. When we suppose that a is group invertible and b is Moore-Penrose invertible, we get similar results for the reverse order laws $(ab)^\# = b^\dagger (abb^\dagger)^\#$, $(ab)^\# = b^*(abb^*)^\#$, $(abb^\dagger)^\# = bb^\dagger a^\#$, $(abb^*)^\# = bb^\dagger a^\#$, $b^\dagger(abb^\dagger)^\# = b^\dagger a^\#$ and $b^*(abb^*)^\# = b^* a^\#$. Also, we show that $(ab)\{5\} \subseteq (a^\dagger ab)\{1,5\} \cdot a^\dagger$ is equivalent to $(ab)\{5\} = (a^\dagger ab)\{1,5\} \cdot a^\dagger$ and similar statements for $(ab)\{5\} \subseteq (a^* ab)\{1,5\} \cdot a^*$, $(ab)\{5\} \subseteq b^\dagger \cdot (abb^\dagger)\{1,5\}$ and $(ab)\{5\} \subseteq b^* \cdot (abb^*)\{1,5\}$.

2 Reverse order laws involving triple products

Several equivalent conditions for $(ab)^{\#} = (a^{\dagger}ab)^{\#}a^{\dagger}$ and $(ab)^{\#} = (a^{*}ab)^{\#}a^{*}$ to hold are presented in the following theorems.

Theorem 2.1. *Let $b \in \mathcal{R}$ and $a \in \mathcal{R}^{\dagger}$. If $a^{\dagger}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) *$ab \in \mathcal{R}^{\#}$ and $(ab)^{\#} = (a^{\dagger}ab)^{\#}a^{\dagger}$,*

(ii) *$(a^{\dagger}ab)^{\#}a^{\dagger} \in (ab)\{5\}$,*

(iii) *$abaa^{\dagger} = ab$ and $(a^{\dagger}ab)^{\#}a^{\dagger}aba = ab(a^{\dagger}ab)^{\#}a^{\dagger}a$,*

(iv) *$(a^{\dagger}ab)\{1,5\} \cdot a^{\dagger} \subseteq (ab)\{5\}$.*

Proof. (i) $\Rightarrow$ (ii): Obviously.

(ii) $\Rightarrow$ (iii): From the condition $(a^{\dagger}ab)^{\#}a^{\dagger} \in (ab)\{5\}$, we have $ab(a^{\dagger}ab)^{\#}a^{\dagger} = (a^{\dagger}ab)^{\#}a^{\dagger}ab$. So, $ab(a^{\dagger}ab)^{\#}a^{\dagger}a = (a^{\dagger}ab)^{\#}a^{\dagger}aba$. Observe that $(a^{\dagger}ab)^{\#}a^{\dagger} \in (ab)\{1\}$, by

$$ab(a^{\dagger}ab)^{\#}a^{\dagger}ab = a(a^{\dagger}ab(a^{\dagger}ab)^{\#}a^{\dagger}ab) = aa^{\dagger}ab = ab. \tag{1}$$

Now, we get

$$abaa^{\dagger} = abab(a^{\dagger}ab)^{\#}a^{\dagger}aa^{\dagger} = abab(a^{\dagger}ab)^{\#}a^{\dagger} = ab.$$

(iii) $\Rightarrow$ (iv): Assume that $abaa^{\dagger} = ab$ and $(a^{\dagger}ab)^{\#}a^{\dagger}aba = ab(a^{\dagger}ab)^{\#}a^{\dagger}a$. If $(a^{\dagger}ab)^{(1,5)} \in (a^{\dagger}ab)\{1,5\}$, then

$$\begin{aligned} a^{\dagger}ab(a^{\dagger}ab)^{(1,5)} &= (a^{\dagger}ab)^{\#}a^{\dagger}ab(a^{\dagger}ab(a^{\dagger}ab)^{(1,5)}) = (a^{\dagger}ab)^{\#}(a^{\dagger}ab(a^{\dagger}ab)^{(1,5)}a^{\dagger}ab) \\ &= (a^{\dagger}ab)^{\#}a^{\dagger}ab. \end{aligned} \tag{2}$$

Using the equalities (2) and (iii), we obtain that $(a^{\dagger}ab)^{(1,5)}a^{\dagger} \in (ab)\{5\}$:

$$\begin{aligned} ab(a^{\dagger}ab)^{(1,5)}a^{\dagger} &= a(a^{\dagger}ab(a^{\dagger}ab)^{(1,5)})a^{\dagger} = aa^{\dagger}ab(a^{\dagger}ab)^{\#}a^{\dagger} \\ &= (ab(a^{\dagger}ab)^{\#}a^{\dagger}a)a^{\dagger} = (a^{\dagger}ab)^{\#}a^{\dagger}(abaa^{\dagger}) \\ &= (a^{\dagger}ab)^{\#}a^{\dagger}ab = (a^{\dagger}ab)^{(1,5)}a^{\dagger}ab. \end{aligned}$$

Hence, for any $(a^{\dagger}ab)^{(1,5)} \in (a^{\dagger}ab)\{1,5\}$, $(a^{\dagger}ab)^{(1,5)}a^{\dagger} \in (ab)\{5\}$ and the statement (iv) holds.

(iv) $\Rightarrow$ (i): Since $(a^{\dagger}ab)^{\#} \in (a^{\dagger}ab)\{1,5\}$, by (iv), $(a^{\dagger}ab)^{\#}a^{\dagger} \in (ab)\{5\}$. The equalities (1) and

$$((a^{\dagger}ab)^{\#}a^{\dagger}ab(a^{\dagger}ab)^{\#})a^{\dagger} = (a^{\dagger}ab)^{\#}a^{\dagger}$$

imply $(a^{\dagger}ab)^{\#}a^{\dagger} \in (ab)\{1,2\}$ and the condition (i) is satisfied. $\square$

Theorem 2.2. *Let $b \in \mathcal{R}$ and $a \in \mathcal{R}^{\dagger}$. If $a^*ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = (a^*ab)^{\#}a^*$,

(ii) $(a^*ab)^{\#}a^* \in (ab)\{5\}$,

(iii) $abaa^{\dagger} = ab$ *and* $(a^*ab)^{\#}a^*aba = ab(a^*ab)^{\#}a^*a$,

(iv) $(a^*ab)\{1,5\} \cdot a^* \subseteq (ab)\{5\}$.

Proof. Using $a = (a^{\dagger})^*a^*a$ and $a^* = a^*aa^{\dagger}$, we verify this result similarly as in Theorem 2.1. □

The following results concerning $(ab)^{\#} = b^{\dagger}(abb^{\dagger})^{\#}$ and $(ab)^{\#} = b^*(abb^*)^{\#}$ are actually dual to Theorems 2.1 and 2.2, where dual means "working in the opposite ring $(\mathcal{R}, \circ)$ with reverse multiplication $a \circ b = ba$".

Corollary 2.1. *Let $a \in \mathcal{R}$ and $b \in \mathcal{R}^{\dagger}$. If $abb^{\dagger} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = b^{\dagger}(abb^{\dagger})^{\#}$,

(ii) $b^{\dagger}(abb^{\dagger})^{\#} \in (ab)\{5\}$,

(iii) $b^{\dagger}bab = ab$ *and* $babb^{\dagger}(abb^{\dagger})^{\#} = bb^{\dagger}(abb^{\dagger})^{\#}ab$,

(iv) $b^{\dagger} \cdot (abb^{\dagger})\{1,5\} \subseteq (ab)\{5\}$.

Corollary 2.2. *Let $a \in \mathcal{R}$ and $b \in \mathcal{R}^{\dagger}$. If $abb^* \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = b^*(abb^*)^{\#}$,

(ii) $b^*(abb^*)^{\#} \in (ab)\{5\}$,

(iii) $b^{\dagger}bab = ab$ *and* $babb^*(abb^*)^{\#} = bb^*(abb^*)^{\#}ab$,

(iv) $b^* \cdot (abb^*)\{1,5\} \subseteq (ab)\{5\}$.

In the following theorem, we prove that $(ab)\{5\} \subseteq (a^{\dagger}ab)\{1,5\} \cdot a^{\dagger}$ is equivalent to $(ab)\{5\} = (a^{\dagger}ab)\{1,5\} \cdot a^{\dagger}$.

Theorem 2.3. *Let $b \in \mathcal{R}$ and $a \in \mathcal{R}^{\dagger}$. If $ab, a^{\dagger}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)\{5\} \subseteq (a^{\dagger}ab)\{1,5\} \cdot a^{\dagger}$,

(ii) $(ab)\{5\} = (a^\dagger ab)\{1,5\} \cdot a^\dagger$.

Proof. (i) $\Rightarrow$ (ii): Assume that $(ab)\{5\} \subseteq (a^\dagger ab)\{1,5\} \cdot a^\dagger$. Because $(ab)^\# \in (ab)\{5\}$, then there exists $(a^\dagger ab)^{(1,5)} \in (a^\dagger ab)\{1,5\}$ such that $(ab)^\# = (a^\dagger ab)^{(1,5)} a^\dagger$. Since the equalities (2) hold again, we obtain

$$(a^\dagger ab)^\# = (a^\dagger ab)^\# a^\dagger ab (a^\dagger ab)^\# = (a^\dagger ab)^{(1,5)} a^\dagger ab (a^\dagger ab)^{(1,5)}$$

which implies

$$(a^\dagger ab)^\dagger a^\dagger = ((a^\dagger ab)^{(1,5)} a^\dagger) ab ((a^\dagger ab)^{(1,5)} a^\dagger) = (ab)^\# ab (ab)^\# = (ab)^\#.$$

By Theorem 2.1, we deduce that $(a^\dagger ab)\{1,5\} \cdot a^\dagger \subseteq (ab)\{5\}$. Hence, the condition (ii) holds.

(ii) $\Rightarrow$ (i): This is obvious. $\square$

Analogously to Theorem 2.3, we obtain the following theorem.

Theorem 2.4. *Let $b \in \mathcal{R}$ and $a \in \mathcal{R}^\dagger$. If $ab, a^*ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)\{5\} \subseteq (a^*ab)\{1,5\} \cdot a^*$,

(ii) $(ab)\{5\} = (a^*ab)\{1,5\} \cdot a^*$.

Applying Theorems 2.3 and 2.4 to the opposite ring $(\mathcal{R}, \circ)$, we get the dual statements.

Corollary 2.3. *Let $a \in \mathcal{R}$ and $b \in \mathcal{R}^\dagger$. If $ab, abb^\dagger \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)\{5\} \subseteq b^\dagger \cdot (abb^\dagger)\{1,5\}$,

(ii) $(ab)\{5\} = b^\dagger \cdot (abb^\dagger)\{1,5\}$.

Corollary 2.4. *Let $a \in \mathcal{R}$ and $b \in \mathcal{R}^\dagger$. If $ab, abb^* \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)\{5\} \subseteq b^* \cdot (abb^*)\{1,5\}$,

(ii) $(ab)\{5\} = b^* \cdot (abb^*)\{1,5\}$.

Now, we consider the conditions which ensure that the reverse order laws $(a^\dagger ab)^\# = b^\# a^\dagger a$ and $(abb^\dagger)^\# = bb^\dagger a^\#$ hold.

Theorem 2.5. *If $a \in \mathcal{R}^\dagger$ and $b \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $a^\dagger ab \in \mathcal{R}^\#$ *and* $(a^\dagger ab)^\# = b^\# a^\dagger a$,

(ii) $a^\dagger ab = ba^\dagger a$.

Proof. (i) $\Rightarrow$ (ii): From the assumption $(a^\dagger ab)^\# = b^\# a^\dagger a$, we obtain

$$a^\dagger abb^\# a^\dagger a = b^\# a^\dagger aa^\dagger ab = b^\# a^\dagger ab \tag{3}$$

and

$$b^\# a^\dagger a = b^\# a^\dagger a(a^\dagger ab)b^\# a^\dagger a = b^\#(a^\dagger abb^\# a^\dagger a) = b^\# b^\# a^\dagger ab. \tag{4}$$

The equalities (3) and (4) imply

$$ba^\dagger a = b^2(b^\# a^\dagger a) = b^2 b^\# b^\# a^\dagger ab = bb^\# a^\dagger ab. \tag{5}$$

and

$$(a^\dagger abb^\# a^\dagger a)b = b^\# a^\dagger abb = b(b^\# b^\# a^\dagger ab)b = bb^\# a^\dagger ab. \tag{6}$$

Since

$$a^\dagger ab = a^\dagger ab(a^\dagger ab)^\# a^\dagger ab = a^\dagger abb^\# a^\dagger aa^\dagger ab = a^\dagger abb^\# a^\dagger ab,$$

by (6) and (5), we get

$$a^\dagger ab = bb^\# a^\dagger ab = ba^\dagger a.$$

Hence, the condition (ii) holds.

(ii) $\Rightarrow$ (i): Assume that $a^\dagger ab = ba^\dagger a$. Because the group inverse $b^\#$ double commutes with b, we deduce that $a^\dagger ab^\# = b^\# a^\dagger a$ and $a^\dagger abb^\# = bb^\# a^\dagger a$. We can easily verify that $b^\# a^\dagger a \in (a^\dagger ab)\{1, 2, 5\}$. $\square$

Remark 2.1 Applying Theorem 2.5 with a projection $p = a^\dagger a$ (hence $p = p^\#$), for $b \in \mathcal{R}^\#$, we recover the equivalence $pb \in \mathcal{R}^\#$ and $(pb)^\# = b^\# p$ $\Leftrightarrow pb = bp$.

Dually to Theorem 2.5, we can check the following result.

Corollary 2.5. *If $a \in \mathcal{R}^\#$ and $b \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(i) $abb^\dagger \in \mathcal{R}^\#$ *and* $(abb^\dagger)^\# = bb^\dagger a^\#$,

(ii) $abb^\dagger = bb^\dagger a$.

Notice that the condition (ii) of Theorem 2.5 can be written as $a_l^\pi b = ba_l^\pi$, where $a_l^\pi = 1 - a^\dagger a$. The condition $abb^\dagger = bb^\dagger a$ of Corollary 2.5 is equivalent to $ab_r^\pi = b_r^\pi a$, where $b_r^\pi = 1 - bb^\dagger$. If a is EP element ($a \in \mathcal{R}^\dagger$ and $a^\dagger a = aa^\dagger$ or equivalently $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $a^\dagger = a^\#$), then $a^\pi = a_l^\pi = a_r^\pi$ is the spectral idempotent of the element a.

The following results give the equivalent conditions to $(a^*ab)^\# = b^\# a^\dagger a$ and $(abb^*)^\# = bb^\dagger a^\#$.

Theorem 2.6. *If $a \in \mathcal{R}^\dagger$ and $b, a^*ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^*ab)^\# = b^\# a^\dagger a$,

(ii) $a^*ab = ba^\dagger a$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $(a^*ab)^\# = b^\# a^\dagger a$. Then

$$b^\# a^\dagger a = b^\# a^\dagger a(a^*ab)b^\# a^\dagger a = b^\#(a^*abb^\# a^\dagger a) = b^\# b^\# a^\dagger aa^*ab = b^\# b^\# a^*ab$$

gives

$$\begin{aligned} a^*ab &= (a^*ab)^\# a^*aba^*ab = b^\# a^\dagger aa^*aba^*ab = b(b^\# b^\# a^*ab)a^*ab \\ &= bb^\# a^\dagger aa^*ab = bb^\# a^*ab = bb(b^\# b^\# a^*ab) = bbb^\# a^\dagger a = ba^\dagger a. \end{aligned}$$

(ii) $\Rightarrow$ (i): If $a^*ab = ba^\dagger a$, we get

$$a^*ab = ba^\dagger a = bb^\#(ba^\dagger a) = bb^\# a^*ab.$$

Now, from

$$\begin{aligned} (a^*ab)^\# &= (a^*ab)[(a^*ab)^\#]^2 = bb^\# a^*ab[(a^*ab)^\#]^2 \\ &= bb^\# a^\dagger a(a^*ab[(a^*ab)^\#]^2) = bb^\# a^\dagger a(a^*ab)^\# \end{aligned}$$

and

$$\begin{aligned} b^\# a^\dagger a &= b^\# b^\#(ba^\dagger a) = b^\# b^\#(a^*ab) = b^\# b^\#(a^*ab)a^*ab(a^*ab)^\# \\ &= b^\# b^\# ba^\dagger aa^*ab(a^*ab)^\# = b^\#(a^*ab)(a^*ab)^\# = b^\# ba^\dagger a(a^*ab)^\#, \end{aligned}$$

we obtain that $(a^*ab)^\# = b^\# a^\dagger a$. $\square$

The dual statement to Theorem 2.6 also holds.

Corollary 2.6. *If $b \in \mathcal{R}^\dagger$ and $a, abb^* \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(abb^*)^\# = bb^\dagger a^\#$,

(ii) $abb^* = bb^\dagger a$.

In the following theorem, we give necessary and sufficient conditions for $(a^\dagger ab)^\# a^\dagger = b^\# a^\dagger$ to be satisfied.

Theorem 2.7. *If $a \in \mathcal{R}^\dagger$ and $b, a^\dagger ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^\dagger ab)^\# a^\dagger = b^\# a^\dagger$,

(ii) $ba^\dagger a = a^\dagger aba^\dagger a$,

(iii) $ba^\dagger a\mathcal{R} \subseteq a^*\mathcal{R}$ *(or* $^\circ(a^*) \subseteq {}^\circ(ba^\dagger a)$ *).*

Proof. (i) $\Rightarrow$ (ii): Let $(a^\dagger ab)^\# a^\dagger = b^\# a^\dagger$. Now the equality

$$a^\dagger ab = a^\dagger ab(a^\dagger ab)^\# a^\dagger ab = ((a^\dagger ab)^\# a^\dagger)aba^\dagger ab = b^\# a^\dagger aba^\dagger ab$$

implies

$$\begin{aligned} (a^\dagger ab)a^\dagger a &= b^\# a^\dagger aba^\dagger aba^\dagger a = b^\# b(b^\# a^\dagger aba^\dagger ab)a^\dagger a \\ &= b^\# ba^\dagger aba^\dagger a = b(b^\# a^\dagger)aba^\dagger a = b((a^\dagger ab)^\# a^\dagger ab)a^\dagger a \\ &= ba^\dagger ab((a^\dagger ab)^\# a^\dagger)a = ba^\dagger abb^\# a^\dagger a. \end{aligned}$$

Using this equality and

$$b^\# a^\dagger = (a^\dagger ab)^\# a^\dagger = (a^\dagger ab)^\# a^\dagger ab(a^\dagger ab)^\# a^\dagger = b^\# a^\dagger abb^\# a^\dagger,$$

we obtain

$$a^\dagger aba^\dagger a = ba^\dagger abb^\# a^\dagger a = b^2(b^\# a^\dagger abb^\# a^\dagger)a = b^2 b^\# a^\dagger a = ba^\dagger a.$$

So, the statement (ii) is satisfied.

(ii) $\Rightarrow$ (i): Applying the hypothesis $ba^\dagger a = a^\dagger aba^\dagger a$, we get

$$\begin{aligned} b^\# a^\dagger &= b^\# b^\# (ba^\dagger a)a^\dagger = b^\# b^\# (a^\dagger ab)a^\dagger aa^\dagger = b^\# b^\# a^\dagger ab((a^\dagger ab)^\# a^\dagger ab)a^\dagger \\ &= b^\# b^\# (a^\dagger aba^\dagger a)b(a^\dagger ab)^\# a^\dagger = b^\# b^\# ba^\dagger ab(a^\dagger ab)^\# a^\dagger \\ &= b^\# a^\dagger ab(a^\dagger ab)^\# a^\dagger. \end{aligned} \tag{7}$$

Since

$$(a^\dagger ab)^\# = a^\dagger ab[(a^\dagger ab)^\#]^2 = a^\dagger a(a^\dagger ab[(a^\dagger ab)^\#]^2) = a^\dagger a(a^\dagger ab)^\#,$$

then

$$\begin{aligned}(a^\dagger ab)^\# a^\dagger &= (a^\dagger aba^\dagger a)b[(a^\dagger ab)^\#]^3 a^\dagger = ba^\dagger ab[(a^\dagger ab)^\#]^3 a^\dagger \\ &= bb^\#(ba^\dagger a)b[(a^\dagger ab)^\#]^3 a^\dagger = bb^\# a^\dagger aba^\dagger ab[(a^\dagger ab)^\#]^3 a^\dagger \\ &= bb^\# a^\dagger a(a^\dagger aba^\dagger ab[(a^\dagger ab)^\#]^3)a^\dagger = b^\#(ba^\dagger a)(a^\dagger ab)^\# a^\dagger \\ &= b^\# a^\dagger ab(a^\dagger a(a^\dagger ab)^\#)a^\dagger = b^\# a^\dagger ab(a^\dagger ab)^\# a^\dagger,\end{aligned}$$

which yields, by (7), $(a^\dagger ab)^\# a^\dagger = b^\# a^\dagger$.

(ii) $\Leftrightarrow$ (iii): The condition $ba^\dagger a = a^\dagger aba^\dagger a$ gives $ba^\dagger a\mathcal{R} \subseteq a^\dagger\mathcal{R} = a^*\mathcal{R}$. Conversely, from $ba^\dagger a\mathcal{R} \subseteq a^*\mathcal{R}$, we conclude that $ba^\dagger a = a^*x$ for some $x \in \mathcal{R}$. Now, $ba^\dagger a = a^*x = a^\dagger a(a^*x) = a^\dagger aba^\dagger a$. □

Obviously, for condition (ii) of Theorem 2.7, we have $ba^\dagger a = a^\dagger aba^\dagger a \Leftrightarrow a_l^\pi b(1-a_l^\pi) = 0 \Leftrightarrow a_l^\pi b(1-a_l^\pi) = (1-a_l^\pi)a_l^\pi b$.

The following theorem can be proved in the similar manner as Theorem 2.7.

Theorem 2.8. *If $a \in \mathcal{R}^\dagger$ and $b, a^*ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^*ab)^\# a^* = b^\# a^*$,

(ii) $ba^*a = a^*aba^*a$ *(or* $ba^\dagger a = a^*aba^\dagger a$ *).*

Using Theorems 2.7 and 2.8 to the opposite ring, we obtain the dual results.

Corollary 2.7. *If $b \in \mathcal{R}^\dagger$ and $a, abb^\dagger \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $b^\dagger(abb^\dagger)^\# = b^\dagger a^\#$,

(ii) $bb^\dagger a = bb^\dagger abb^\dagger$,

(iii) $\mathcal{R}bb^\dagger a \subseteq \mathcal{R}b^*$ *(or* $(b^*)^\circ \subseteq (bb^\dagger a)^\circ$*).*

Note that $bb^\dagger a = bb^\dagger abb^\dagger \Leftrightarrow (1-b_r^\pi)ab_r^\pi = 0 \Leftrightarrow (1-b_r^\pi)ab_r^\pi = ab_r^\pi(1-b_r^\pi)$.

Corollary 2.8. *If $b \in \mathcal{R}^\dagger$ and $a, abb^* \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $b^*(abb^*)^\# = b^*a^\#$,

(ii) $bb^*a = bb^*abb^*$ *(or* $bb^\dagger a = bb^\dagger abb^*$ *).*

Notice that the conditions of Theorem 2.5 (Theorem 2.6, Corollary 2.5, Corollary 2.6, respectively) imply the conditions of Theorem 2.7 (Theorem 2.8, Corollary 2.7, Corollary 2.8, respectively)

3 Reverse order laws $(ab)^\# = b^\# a^\dagger$ and $(ab)^\# = b^\# a^*$

Assuming that a is Moore-Penrose invertible, and that b is group invertible in a ring with involution, equivalent conditions to the reverse order law $(ab)^\# = b^\# a^\dagger$ are presented in the following theorem.

Theorem 3.1. *If $a \in \mathcal{R}^\dagger$ and $b, ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = b^\# a^\dagger$,

(ii) $(ab)^\# a = b^\# a^\dagger a$ *and* $a^* ab = a^* abaa^\dagger$,

(iii) $(ab)^\# a = b^\# a^\dagger a$ *and* $a^\dagger ab = a^\dagger abaa^\dagger$,

(iv) $b(ab)^\# = bb^\# a^\dagger$ *and* $abb^\# = bb^\# abb^\#$.

Proof. (i) $\Rightarrow$ (ii): The hypothesis $(ab)^\# = b^\# a^\dagger$ gives $(ab)^\# a = b^\# a^\dagger a$ and

$$\begin{aligned} a^* ab &= a^* ab((ab)^\# ab) = a^* abab(ab)^\# = a^* ababb^\# a^\dagger \\ &= a^*(ababb^\# a^\dagger) aa^\dagger = a^* abaa^\dagger. \end{aligned}$$

Hence, the condition (ii) holds.

(ii) $\Rightarrow$ (iii): Because $(ab)^\# a = b^\# a^\dagger a$ and $a^* ab = a^* abaa^\dagger$, then

$$a^\dagger ab = a^\dagger (a^\dagger)^* (a^* ab) = a^\dagger (a^\dagger)^* a^* abaa^\dagger = a^\dagger abaa^\dagger.$$

So, (iii) is satisfied.

(iii) $\Rightarrow$ (i): Suppose that $(ab)^\# a = b^\# a^\dagger a$ and $a^\dagger ab = a^\dagger abaa^\dagger$. First, we show that $b^\# a^\dagger \in (ab)\{5\}$:

$$\begin{aligned} (b^\# a^\dagger a)b &= (ab)^\# ab = (ab)^\# a(a^\dagger ab) = (ab)^\# aa^\dagger abaa^\dagger \\ &= ((ab)^\# ab)aa^\dagger = ab((ab)^\# a)a^\dagger = abb^\# a^\dagger aa^\dagger \\ &= abb^\# a^\dagger. \end{aligned}$$

Further, from

$$ab = ab((ab)^\# a)b = abb^\# a^\dagger ab$$

and

$$\begin{aligned} b^\# a^\dagger &= (b^\# a^\dagger a)a^\dagger = (ab)^\# aa^\dagger = ((ab)^\# a)b((ab)^\# a)a^\dagger \\ &= b^\# a^\dagger abb^\# a^\dagger aa^\dagger = b^\# a^\dagger abb^\# a^\dagger, \end{aligned}$$

we deduce that $b^\# a^\dagger \in (ab)\{1, 2\}$, i.e. $(ab)^\# = b^\# a^\dagger$.

(i) $\Leftrightarrow$ (iv): This equivalence can be proved similarly as previous parts. $\square$

The condition $a^\dagger ab = a^\dagger abaa^\dagger$ in Theorem 3.1 can be replaced with equivalent conditions $\mathcal{R}a^\dagger ab \subseteq \mathcal{R}a^*$ or $(a^*)^\circ \subseteq (a^\dagger ab)^\circ$. Also, the condition $abb^\# = bb^\# abb^\#$ in Theorem 3.1 can be replaced with equivalent conditions $abb^\#\mathcal{R} \subseteq b\mathcal{R}$ or ${}^\circ b \subseteq {}^\circ(abb^\#)$.

Similarly as in the proof of Theorem 3.1, we get necessary and sufficient conditions which ensure that $(ab)^\# = b^\# a^*$ is satisfied.

Theorem 3.2. *If $a \in \mathcal{R}^\dagger$ and $b, ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = b^\# a^*$,

(ii) $(ab)^\# a = b^\# a^* a$ *and* $a^* ab = a^* abaa^\dagger$,

(iii) $(ab)^\# a = b^\# a^* a$ *and* $a^\dagger ab = a^\dagger abaa^\dagger$,

(iv) $b(ab)^\# = bb^\# a^*$ *and* $abb^\# = bb^\# abb^\#$.

If we suppose that a is EP element in Theorem 3.1 or that $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ and $a^* = a^\#$ in Theorem 3.2, we obtain new characterizations of the classical reverse order law $(ab)^\# = b^\# a^\#$.

Dually to Theorems 3.1 and 3.2, equivalent conditions for $(ab)^\# = b^\dagger a^\#$ and $(ab)^\# = b^* a^\#$ are presented.

Corollary 3.1. *If $b \in \mathcal{R}^\dagger$ and $a, ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = b^\dagger a^\#$,

(ii) $(ab)^\# a = b^\dagger a^\# a$ *and* $a^\# ab = a^\# abaa^\#$,

(iii) $b(ab)^\# = bb^\dagger a^\#$ *and* $abb^\dagger = b^\dagger babb^\dagger$,

(iv) $b(ab)^\# = bb^\dagger a^\#$ *and* $abb^* = b^\dagger babb^*$.

In Corollary 3.1, the condition $a^\# ab = a^\# abaa^\#$ can be replaced with $\mathcal{R}a^\# ab \subseteq \mathcal{R}a$ or $a^\circ \subseteq (a^\# ab)^\circ$, and the condition $abb^\dagger = b^\dagger babb^\dagger$ can be replaced with $abb^\dagger\mathcal{R} \subseteq b^*\mathcal{R}$ or ${}^\circ(b^*) \subseteq {}^\circ(abb^\dagger)$.

Corollary 3.2. *If $b \in \mathcal{R}^\dagger$ and $a, ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = b^* a^\#$,

(ii) $(ab)^\# a = b^* a^\# a$ *and* $a^\# ab = a^\# abaa^\#$,

(iii) $b(ab)^{\#} = bb^*a^{\#}$ *and* $abb^\dagger = b^\dagger babb^\dagger$,

(iv) $b(ab)^{\#} = bb^*a^{\#}$ *and* $abb^* = b^\dagger babb^*$.

Several sufficient conditions for the reverse order law $(ab)^{\#} = b^{\#}a^\dagger$ are presented in the next results.

Theorem 3.3. *Suppose that* $a \in \mathcal{R}^\dagger$ *and* $b, ab, a^\dagger ab, abb^{\#}, a^*ab \in \mathcal{R}^{\#}$. *Then each of the following conditions is sufficient for* $(ab)^{\#} = b^{\#}a^\dagger$ *to hold:*

(i) $(ab)^{\#}a = b^{\#}a^\dagger a$ *and* $a^\dagger ab = baa^\dagger$,

(ii) $(ab)^{\#} = (a^\dagger ab)^{\#}a^\dagger$ *and* $(a^\dagger ab)^{\#} = b^{\#}a^\dagger a$,

(iii) $(ab)^{\#} = b^{\#}(abb^{\#})^{\#}$ *and* $(abb^{\#})^{\#} = bb^{\#}a^\dagger$,

(iv) $b(ab)^{\#} = bb^{\#}a^\dagger = (abb^{\#})^{\#}$,

(v) $(ab)^{\#} = (a^*ab)^{\#}a^*$ *and* $(a^*ab)^{\#} = b^{\#}(a^*a)^{\#}$.

Proof. (i) Assume that $(ab)^{\#}a = b^{\#}a^\dagger a$ and $a^\dagger ab = baa^\dagger$. As $a^\dagger a$ is idempotent, then $a^\dagger ab = a^\dagger abaa^\dagger$ and $(ab)^{\#} = b^{\#}a^\dagger$ by (iii) of Theorem 3.1.

(ii) From the hypothesis $(ab)^{\#} = (a^\dagger ab)^{\#}a^\dagger$ and $(a^\dagger ab)^{\#} = b^{\#}a^\dagger a$, we get

$$(ab)^{\#} = (a^\dagger ab)^{\#}a^\dagger = b^{\#}a^\dagger aa^\dagger = b^{\#}a^\dagger.$$

(iii) It follows as part (ii).

(iv) Suppose that $b(ab)^{\#} = bb^{\#}a^\dagger = (abb^{\#})^{\#}$. Then

$$abb^{\#} = (abb^{\#})^{\#}(abb^{\#})^2 = bb^{\#}a^\dagger(abb^{\#})^2 = (bb^{\#})^2a^\dagger(abb^{\#})^2 = bb^{\#}abb^{\#}.$$

By part (iv) of Theorem 3.1, $(ab)^{\#} = b^{\#}a^\dagger$.

(v) The condition $a \in \mathcal{R}^\dagger$ implies $a^*a \in \mathcal{R}^{\#}$ and $a^\dagger = (a^*a)^{\#}a^*$ (see [8]). The rest of this part follows as (ii). $\square$

The following theorem can be proved in the similar way as Theorem 3.3.

Theorem 3.4. *Suppose that* $a \in \mathcal{R}^\dagger$ *and* $b, ab, a^\dagger ab, abb^{\#}, a^*ab \in \mathcal{R}^{\#}$. *Then each of the following conditions is sufficient for* $(ab)^{\#} = b^{\#}a^*$ *to hold:*

(i) $(ab)^{\#}a = b^{\#}a^*a$ *and* $a^\dagger ab = baa^\dagger$,

(ii) $(ab)^{\#} = (a^*ab)^{\#}a^*$ *and* $(a^*ab)^{\#} = b^{\#}a^\dagger a$,

(iii) $(ab)^{\#} = b^{\#}(abb^{\#})^{\#}$ *and* $(abb^{\#})^{\#} = bb^{\#}a^*$,

(iv) $b(ab)^\# = bb^\# a^* = (abb^\#)^\#$,

(v) $(ab)^\# = (a^\dagger ab)^\# a^\dagger$ *and* $(a^\dagger ab)^\# = b^\# a^* a$.

Notice that the dual results to Theorems 3.3 and 3.4 are satisfied too.

Corollary 3.3. *Suppose that* $b \in \mathcal{R}^\dagger$ *and* $a, ab, a^\# ab, abb^\dagger, abb^* \in \mathcal{R}^\#$. *Then each of the following conditions is sufficient for* $(ab)^\# = b^\dagger a^\#$ *to hold:*

(i) $b(ab)^\# = bb^\dagger a^\#$ *and* $b^\dagger ba = abb^\dagger$,

(ii) $(ab)^\# = (a^\# ab)^\# a^\#$ *and* $(a^\# ab)^\# = b^\dagger a^\# a$,

(iii) $(ab)^\# = b^\dagger (abb^\dagger)^\#$ *and* $(abb^\dagger)^\# = bb^\dagger a^\#$,

(iv) $(ab)^\# a = b^\dagger a^\# a = (a^\# ab)^\#$,

(v) $(ab)^\# = b^* (abb^*)^\#$ *and* $(abb^*)^\# = (bb^*)^\# a^\#$.

Corollary 3.4. *Suppose that* $b \in \mathcal{R}^\dagger$ *and* $a, ab, a^\# ab, abb^\dagger, abb^* \in \mathcal{R}^\#$. *Then each of the following conditions is sufficient for* $(ab)^\# = b^* a^\#$ *to hold:*

(i) $b(ab)^\# = bb^* a^\#$ *and* $b^\dagger ba = abb^\dagger$,

(ii) $(ab)^\# = (a^\# ab)^\# a^\#$ *and* $(a^\# ab)^\# = b^* a^\# a$,

(iii) $(ab)^\# = b^* (abb^*)^\#$ *and* $(abb^*)^\# = bb^\dagger a^\#$,

(iv) $(ab)^\# a = b^* a^\# a = (a^\# ab)^\#$,

(v) $(ab)^\# = b^\dagger (abb^\dagger)^\#$ *and* $(abb^*)^\# = bb^* a^\#$.

Remark 3.1. Combining the conditions of Theorem 2.1 and Theorem 2.7, we get the sufficient conditions for the reverse order law $(ab)^\# = b^\# a^\dagger$ to hold. If we combine the conditions of Corollary 2.1 and Corollary 2.7, we obtain the sufficient conditions for $(ab)^\# = b^\dagger a^\#$ to be satisfied.

Sufficient conditions for the reverse order law $(ab)^\# = b^\# a^*$ ($(ab)^\# = b^* a^\#$) to hold can be obtained combining the conditions of Theorem 2.2 and Theorem 2.8 (Corollary 2.2 and Corollary 2.8).

4 Other results

More specific results are proved in this section.

Theorem 4.1. *If $a \in \mathcal{R}^\dagger$ and $b, ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $b^\# = (ab)^\# a$,

(ii) $b = aa^\dagger b = ba^\dagger a$ *and* $abb^\# = bb^\# a$,

(iii) $b\mathcal{R} \subseteq a\mathcal{R}$, $aa^\dagger b = ba^\dagger a$ *and* $abb^\# = bb^\# a$,

(iv) $a^\circ \subseteq b^\circ$, $aa^\dagger b = ba^\dagger a$ *and* $abb^\# = bb^\# a$.

Proof. (i) $\Rightarrow$ (ii): Using the equality $b^\# = (ab)^\# a$, we observe that

$$ba^\dagger a = b^2 b^\# a^\dagger a = b^2 (ab)^\# a a^\dagger a = b^2 (ab)^\# a = b^2 b^\# = b$$

and $b = b^\# b^2 = ((ab)^\# ab)b = ab(ab)^\# b$ which yields

$$b = ab(ab)^\# b = aa^\dagger (ab(ab)^\# b) = aa^\dagger b.$$

Also, by (i), we get

$$abb^\# = (ab(ab)^\#)a = ((ab)^\# a)ba = b^\# ba.$$

So, the condition (ii) holds.

(ii) $\Rightarrow$ (i): Suppose that $b = aa^\dagger b = ba^\dagger a$ and $abb^\# = bb^\# a$. Then, from

$$b(ab)^\# ab = b^\# bb(ab)^\# ab = b^\# ba^\dagger (ab(ab)^\# ab) = b^\# (ba^\dagger a)b = b^\# bb = b$$

and

$$(ab)^\# ab(ab)^\# a = (ab)^\# a,$$

we conclude that $(ab)^\# a \in \ b\{1, 2\}$. Since

$$b(ab)^\# a = b[(ab)^\#]^2 aba = b([(ab)^\#]^2 ab)(bb^\# a) = (b(ab)^\# ab)b^\# = bb^\#$$

and

$$ab(ab)^\# = (abb^\#)b(ab)^\# = bb^\# ab(ab)^\# = b^\# (b(ab)^\# ab) = b^\# b,$$

we have $b(ab)^\# a = ab(ab)^\# = (ab)^\# ab$, that is, $(ab)^\# a \in \ b\{5\}$. Hence, the condition (i) is satisfied.

(ii) $\Leftrightarrow$ (iii): We will show that $b = aa^\dagger b$ is equivalent to $b\mathcal{R} \subseteq a\mathcal{R}$. First, $b = aa^\dagger b$ implies $b\mathcal{R} \subseteq a\mathcal{R}$. Conversely, if $b\mathcal{R} \subseteq a\mathcal{R}$, then, for some $x \in R$, $b = ax$ which gives $b = aa^\dagger (ax) = aa^\dagger b$.

(ii) $\Leftrightarrow$ (iv): It follows from $b = ba^\dagger a$ iff $b(1 - a^\dagger a) = 0$ iff $(1 - a^\dagger a)\mathcal{R} \subseteq b^\circ$ iff $a^\circ \subseteq b^\circ$. $\square$

The next result follows dually to Theorem 4.1.

Theorem 4.2. *If $b \in \mathcal{R}^\dagger$ and $a, ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $a^\# = b(ab)^\#$,

(ii) $aa^\# b = baa^\#$ *and* $a = ab^\dagger b = bb^\dagger a$,

(iii) $a\mathcal{R} \subseteq b\mathcal{R}$, $aa^\# b = baa^\#$ *and* $ab^\dagger b = bb^\dagger a$,

(iv) $b^\circ \subseteq a^\circ$, $aa^\# b = baa^\#$ *and* $ab^\dagger b = bb^\dagger a$.

Some conditions of Theorem 4.1 and Theorem 4.2 can be written as
$aa^\dagger b = ba^\dagger a$ and $abb^\# = bb^\# a \Leftrightarrow a^\pi_r b = ba^\pi_l$ and $ab^\pi = b^\pi a$;
$aa^\# b = baa^\#$ and $ab^\dagger b = bb^\dagger a \Leftrightarrow a^\pi b = ba^\pi$ and $ab^\pi_l = b^\pi_r a$.

Theorem 4.3. *If $a \in \mathcal{R}^\dagger$, $b \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $a^\dagger ab = b(ab)^\# ab$,

(ii) $baba = a^\dagger ababa$,

(iii) $baba\mathcal{R} \subseteq a^*\mathcal{R}$ (*or* ${}^\circ(a^*) \subseteq {}^\circ(baba)$).

Proof. (i) $\Rightarrow$ (ii): Using $a^\dagger ab = b(ab)^\# ab$, we have

$$(a^\dagger ab)aba = b((ab)^\# abab)a = baba.$$

Thus, the equality (ii) is satisfied.

(ii) $\Rightarrow$ (i): Since $baba = a^\dagger ababa$, then

$$a^\dagger ab = a^\dagger abab(ab)^\# = (a^\dagger ababa)b[(ab)^\#]^2 = babab[(ab)^\#]^2 = b(ab)^\# ab.$$

(ii) $\Rightarrow$ (iii): It follows by $a^*\mathcal{R} = a^\dagger\mathcal{R}$.

(iii) $\Rightarrow$ (ii): By the condition $baba\mathcal{R} \subseteq a^*\mathcal{R}$, we see that $baba = a^*x$, for $x \in \mathcal{R}$. Hence, $baba = a^*x = a^\dagger a(a^*x) = a^\dagger ababa$.

□

In the same way as in Theorem 4.3, we prove the following theorem.

Theorem 4.4. *If $a, b \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $a^*ab = b(ab)^\# ab$,

(ii) $baba = a^*ababa$.

Applying Theorems 4.3 and 4.4, we have that the next dual statements hold.

Theorem 4.5. *If $a \in \mathcal{R}$, $b \in \mathcal{R}^\dagger$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $abb^\dagger = ab(ab)^\# a$,

(ii) $baba = bababb^\dagger$,

(iii) $\mathcal{R}baba \subseteq \mathcal{R}b^*$ (*or* $(b^*)^\circ \subseteq (baba)^\circ$).

Theorem 4.6. *If $a, b \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $abb^* = ab(ab)^\# a$,

(ii) $baba = bababb^*$.

Some equivalent conditions for $aa^\# = bb^\dagger$ to hold are given in the following theorem in a ring with involution.

Theorem 4.7. *If $a \in \mathcal{R}^\#$ and $b \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(i) $aa^\# = bb^\dagger$,

(ii) $a\mathcal{R} = b\mathcal{R}$ *and* $a^\circ = (b^*)^\circ$,

(iii) $a + 1 - bb^\dagger \in \mathcal{R}^{-1}$ *and* $aa^\# = aa^\# bb^\dagger = bb^\dagger aa^\#$,

(iv) $a + 1 - bb^\dagger, 1 - aa^\# + bb^\dagger \in \mathcal{R}^{-1}$ *and* $abb^\dagger = bb^\dagger a$,

(v) $a + 1 - bb^\dagger, 1 - aa^\# + bb^\dagger \in \mathcal{R}^{-1}$ *and* $aa^\# bb^\dagger = bb^\dagger aa^\#$.

Proof. (i) $\Rightarrow$ (ii)-(v): This is trivial, when we notice that $(a+1-aa^\#)(a^\# + 1 - aa^\#) = 1$ gives $a + 1 - aa^\# \in \mathcal{R}^{-1}$.

(ii) $\Rightarrow$ (i): Assume that $a\mathcal{R} = b\mathcal{R}$ and $a^\circ = (b^*)^\circ$. Now, we have $b = ax$ for $x \in \mathcal{R}$ and, by $(b^*)^\circ = (1 - bb^\dagger)\mathcal{R}$, $a^\circ = (1 - bb^\dagger)\mathcal{R}$. Further, $b = aa^\#(ax) = aa^\# b$ and $a(1 - bb^\dagger) = 0$. Thus, $bb^\dagger = aa^\# bb^\dagger = a^\#(abb^\dagger) = a^\# a$.

(iii) $\Rightarrow$ (i): Let $a + 1 - bb^\dagger \in \mathcal{R}^{-1}$ and $aa^\# = aa^\# bb^\dagger = bb^\dagger aa^\#$. The equalities

$$(a + 1 - bb^\dagger)bb^\dagger = abb^\dagger + bb^\dagger - bb^\dagger = abb^\dagger$$

and

$$(a+1-bb^\dagger)bb^\dagger aa^\# = a(bb^\dagger aa^\#) = aaa^\# bb^\dagger = abb^\dagger,$$

imply $bb^\dagger = bb^\dagger aa^\#$. Hence, we get $bb^\dagger = aa^\#$.

(iv) $\Rightarrow$ (iii): Since $abb^\dagger = bb^\dagger a$, and the group inverse $a^\#$ double commutes with a, then $a^\# bb^\dagger = bb^\dagger a^\#$ and $aa^\# bb^\dagger = bb^\dagger aa^\#$. From

$$(1-aa^\# + bb^\dagger)aa^\# = aa^\# - aa^\# + bb^\dagger aa^\# = bb^\dagger aa^\#,$$

$$(1-aa^\# + bb^\dagger)aa^\# bb^\dagger = bb^\dagger(aa^\# bb^\dagger) = bb^\dagger aa^\#,$$

and the condition $1-aa^\# + bb^\dagger \in \mathcal{R}^{-1}$, we obtain $aa^\# = aa^\# bb^\dagger$. So, the statements (iii) holds.

(v) $\Rightarrow$ (i): This part can be check in the same way as (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i). $\square$

Changing b in previous theorem by $b^\dagger$, by $(b^\dagger)^\dagger = b$, we obtain equivalent conditions for $aa^\# = b^\dagger b$.

Theorem 4.8. *If $a \in \mathcal{R}^\#$ and $b \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(i) $aa^\# = b^\dagger b$,

(ii) $a\mathcal{R} = b^*\mathcal{R}$ *and* $a^\circ = b^\circ$,

(iii) $a+1-b^\dagger b \in \mathcal{R}^{-1}$ *and* $aa^\# = aa^\# b^\dagger b = b^\dagger b aa^\#$,

(iv) $a+1-b^\dagger b, 1-aa^\# + b^\dagger b \in \mathcal{R}^{-1}$ *and* $abb^\dagger = b^\dagger ba$,

(v) $a+1-b^\dagger b, 1-aa^\# + b^\dagger b \in \mathcal{R}^{-1}$ *and* $aa^\# b^\dagger b = b^\dagger b aa^\#$.

5 Characterization of operators on Hilbert space

Let H be a Hilbert space and $\mathcal{L}(H)$ the set of all linear bounded operators on H. In addition, if $T \in \mathcal{L}(H)$, then T^*, $N(T)$ and $R(T)$ stand for the adjoint, the null space and the range of T, respectively.

In the spirit of previous results, we prove the following one.

Theorem 5.1. *Let $A \in \mathcal{L}(H)$ have a closed range and let $B \in \mathcal{L}(H)$.*

(i) *If AB is group invertible, then*

$$I + A^\dagger(B-A) \text{ is invertible } \Leftrightarrow AB(AB)^\# A = A.$$

(ii) *If BA is group invertible, then*

$$I + A^\dagger(B-A) \text{ is invertible } \Leftrightarrow A(BA)^\# BA = A$$

Proof. (i) Since $A \in \mathcal{L}(H)$ have a closed range, there exists the unique Moore–Penrose inverse $A^\dagger \in \mathcal{L}(H)$ of A. The operators A, B and AB have the matrix representations on $H = R(A^*) \oplus N(A)$ of the forms

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} \quad \textit{and} \quad AB = \begin{bmatrix} A_1B_1 & A_1B_3 \\ 0 & 0 \end{bmatrix},$$

where A_1 is invertible. The Moore–Penrose inverse of A is given by

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Analogously as in Theorem [7, Theorem 1] for the matrix case, we can verify that AB is group invertible if and only if A_1B_1 is group invertible and $A_1B_1(A_1B_1)^\# A_1B_3 = A_1B_3$. In this case,

$$(AB)^\# = \begin{bmatrix} (A_1B_1)^\# & [(A_1B_1)^\#]^2 A_1B_3 \\ 0 & 0 \end{bmatrix}.$$

Observe that, $AB(AB)^\# A = A$ iff $A_1B_1(A_1B_1)^\# A_1 = A_1$ iff $A_1B_1(A_1B_1)^\# = I$ iff A_1B_1 is invertible iff B_1 is invertible. Then, by

$$I + A^\dagger(B - A) = \begin{bmatrix} A_1^{-1}B_1 & A_1^{-1}B_3 \\ 0 & I \end{bmatrix},$$

we deduce that $I + A^\dagger(B - A)$ is invertible iff B_1 is invertible.

(ii) Applying (i) to the opposite ring, we get $I + (B - A)A^\dagger$ is invertible $\Leftrightarrow A(BA)^\# BA = A$. But by Jacobson lemma, $I + (B - A)A^\dagger$ is invertible $\Leftrightarrow I + A^\dagger(B - A)$ is invertible. $\square$

6 Conclusions

In this paper we consider necessary and sufficient conditions related to the reverse order laws $(ab)^\# = b^\# a^\dagger$ and $(ab)^\# = b^\dagger a^\#$ in rings with involution, applying a purely algebraic technique. In the case of linear bounded operators on Hilbert spaces, where the method of operator matrices is very useful, similar results for the reverse order law $(ab)^\# = b^\# a^\#$ are given. In a $*$-regular ring $\mathcal{R}$, observe that the assumption $a \in \mathcal{R}^\dagger$ is automatically satisfied. It could be interesting to extend this work to the reverse order laws of a triple product.

References

[1] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.

[2] C. Cao, X. Zhang, X. Tang, *Reverse order law of group inverses of products of two matrices*, Appl. Math. Comput. 158 (2004) 489-495.

[3] C.Y. Deng, *Reverse order law for the group inverses*, J. Math. Anal. Appl. 382 (2) (2011), 663-671.

[4] D. S. Djordjević, *Unified approach to the reverse order rule for generalized inverses*, Acta Sci. Math. (Szeged), 67 (2001), 761-776.

[5] T.N.E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. 8 (1966), 518–521.

[6] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. 34 (1982), 43–52.

[7] R.E. Hartwig, J. Shoaf *Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices*, J. Austral. Math. Soc. 24(A) (1977), 10–34.

[8] J.J. Koliha, P. Patrício *Elements of rings with equal spectral idempotents*, J. Australian Math. Soc. 72 (2002), 137–152.

[9] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

Address:

Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

E-mail

D. Mosić: dijana@pmf.ni.ac.rs
D. S. Djordjević: dragan@pmf.ni.ac.rs