Linear Algebra and its Applications 471 (2015) 203–223

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Partial orders in rings based on generalized inverses – unified theory ☆

Dragan S. Rakić [a,*], Dragan S. Djordjević [b]

[a] *Faculty of Mechanical Engineering, University of Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia*
[b] *Faculty of Sciences and Mathematics, University of Niš, P.O. Box 95, 18000 Niš, Serbia*

A R T I C L E   I N F O

Article history:
Received 20 August 2014
Accepted 3 January 2015
Available online xxxx
Submitted by Y. Wei

MSC:
06A06
15A09
16U99

Keywords:
Matrix partial ordering
$\mathcal{G}$-based order relation
Minus partial order
Star partial order
Sharp partial order
Core partial order
Generalized inverse
Moore–Penrose inverse
Group inverse
Core inverse
Ring
Dedekind finite ring

A B S T R A C T

The unified theory for matrix partial orders based on generalized inverses has already been done by Mitra. We consider a special kind of ring R, which generalizes the ring of linear operators on finite dimensional vector space, and extend Mitra's approach to it. Thus, we find necessary and sufficient condition for a G-based relation to be a partial order on R. Also, some known results are generalized and some new results are proved.

☆ The authors are supported by the Ministry of Science, Republic of Serbia, grant No. 174007.

* Corresponding author.
E-mail addresses: rakic.dragan@gmail.com (D.S. Rakić), dragandjordjevic70@gmail.com (D.S. Djordjević).

http://dx.doi.org/10.1016/j.laa.2015.01.004
0024-3795/

1. Introduction and preliminaries

The original idea of introducing the matrix partial orders comes from the papers arising in the middle of last century in which several partial orders were defined in the context of semigroups. In 1952, Wagner introduced the notion of inverse semigroup and natural partial ordering on it [30]:

$$a < b \iff a^{-1}a = a^{-1}b,$$

where a^{-1} is generalized inverse of a in the sense that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Later on, Clifford and Preston [3] and Lyapin [16] introduced partial ordering on the set of idempotents in arbitrary semigroups:

$$e < f \iff e = ef = fe.$$

It was natural to ask how to extend this relation on a larger class of elements. In 1977, Drazin considered the problem on semigroup S with proper involution and he defined the binary relation on S [5]:

$$a < b \iff a^*a = a^*b \quad \text{and} \quad aa^* = ba^*,$$

which is known as star partial order. On the set of Moore–Penrose invertible elements this relation coincides with the relation

$$a < b \iff aa^\dagger = ba^\dagger \quad \text{and} \quad a^\dagger a = a^\dagger b.$$

In 1980, Hartwig [10] and Nambooripad [25] independently introduced the minus partial order on semigroup:

$$a < b \iff aa^{(1)} = ba^{(1)} \quad \text{and} \quad a^{(1)}a = a^{(1)}b,$$

for some $a^{(1)}$ such that $aa^{(1)}a = a$. This order is a partial order relation on regular semigroup. Mitsch [23] further generalized the definition of minus order to arbitrary semigroups S:

$$a < b \iff xb = xa = a = ay = by \quad \text{for some } x, y \in S^1.$$

Mitsch's order is a partial order relation on arbitrary semigroup and coincides with minus partial order when S is regular.

After that the other partial orders, such as sharp, core and one-sided orders, are introduced on the set of complex matrices. For thorough treatment of the subject of matrix partial orders we refer the reader to monograph [22], articles [1,17,18,8,9] and the references given there. The unification of these orders on the set of complex matrices has already been done by Mitra in [20]. The aim of this article is to extend Mitra's approach to the ring case.

Although the minus and star order was originally defined on semigroups, the most of the theory consider the matrix partial orders. But, in recent years, a number of papers was published considering the generalized inverses and associated partial orders in rings, see for example [14]. Furthermore, some new generalized inverses, such as core and dual core inverse (see [1]), (b, c)-inverse (see [4]) and an inverse along an element (see [19]), are introduced. For that reason there is a need of unified theory of partial orders based on generalized inverses in rings.

In Section 2 we will introduce a special kind of Dedekind finite ring by requiring additional condition on its idempotent elements. This ring, which may be called the finite dimensional ring (FD ring for short), is a generalization of the ring of linear operators on finite dimensional vector space. The necessary and sufficient condition that makes G-based matrix relation a partial order was found in [20] and [22, Chapter 7]. We will show in Section 3 that the same result holds for an arbitrary FD ring. A number of results will be generalized and we will also prove some new results. The connection with some known partial orders will be established.

Unless otherwise stated, R is an arbitrary ring with identity 1 (with or without involution – depending on the context). For the reader's convenience we recall definitions of some known types of generalized inverses and associated partial orders. An element $a \in R$ is von Neumann regular (regular for short) if there exists an $x \in R$ such that $axa = a$ in which case x is called an inner generalized inverse of a. If $axa = a$ and $xax = x$ then x is reflexive generalized inverse of a. The set of all inner generalized inverses of a is denoted by $a\{1\}$ and the set of all reflexive generalized inverses of a is denoted by $a\{1, 2\}$. An element $a \in R$ has Moore–Penrose inverse if there exists an $x \in R$ such that the following equations hold [26,24]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa.$$

In this case x is unique and denoted by $a^\dagger$. An element $a \in R$ has group inverse if there exists an $x \in R$ such that following equations hold [6,2]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (5)\ ax = xa.$$

In this case x is unique and denoted by $a^\#$. Recently, Baksalary and Trenkler [1] introduced a new kind of matrix generalized inverse, called core inverse. Let M_n denote the algebra of all $n \times n$ complex matrices. The matrix $A^{\oplus} \in M_n$ is the core inverse of matrix $A \in M_n$ if it satisfies

$$AA^{\oplus} = P_A \quad \text{and} \quad \mathcal{R}(A^{\oplus}) \subseteq \mathcal{R}(A),$$

where $\mathcal{R}(A)$ is the range (column space) of A. It is shown in [28] that $X \in M_n$ is the core inverse of A if and only if

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (6)\ XA^2 = A, \quad (7)\ AX^2 = X.$$

Now we can give the definition in ring case. An element $a \in R$ has core inverse if there exists an $x \in R$ such that the following equations hold [28]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (6)\ xa^2 = a, \quad (7)\ ax^2 = x.$$

In this case x is unique and denoted by $a^{\oplus}$. Similarly, the matrix $A_{\oplus} \in M_n$ is dual core inverse of matrix $A \in M_n$ if it satisfies

$$A_{\oplus}A = P_{A^*} \quad \text{and} \quad \mathcal{R}(A_{\oplus}) \subseteq \mathcal{R}\big(A^*\big).$$

As well as the core inverse, the dual core inverse can be characterized by a set of equations. An element $a \in R$ has dual core inverse if there exists an $x \in R$ such that the following equations hold [28]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (4)\ (xa)^* = xa, \quad (8)\ a^2x = a, \quad (9)\ x^2a = x.$$

Let $a\{i, j, \dots, k\}$ denote the set of all elements $x \in R$ which satisfy equations $(i), (j), \dots, (k)$ among equations (1)–(9). Using these inverses, several partial orders can be defined:

- the minus partial order [10]: $a <^- b$ if there exists an $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)}a = a^{(1)}b$;
- the star partial order [5]: $a <^* b$ if $aa^\dagger = ba^\dagger$ and $a^\dagger a = a^\dagger b$;
- the sharp partial order [21]: $a <^\# b$ if $aa^\# = ba^\#$ and $a^\# a = a^\# b$;
- the core partial order [1]: $a <^{\oplus} b$ if $aa^{\oplus} = ba^{\oplus}$ and $a^{\oplus}a = a^{\oplus}b$;
- the dual core partial order [1]: $a <_{\oplus} b$ if $aa_{\oplus} = ba_{\oplus}$ and $a_{\oplus}a = a_{\oplus}b$.

From these orders one can define appropriate one-sided orders [22]. The minus and star partial orders are originally defined on a semigroup and others on the set of complex matrices. In [27] and [29] these orders are considered in the context of arbitrary ring with involution. It is shown that these orders are partial orders on appropriate subsets of R. One can notice an obvious similarity in definitions of partial orders based on generalized inverses.

In the theory of matrix partial orders, the most basic binary relation is the space pre-order, [20]. For $A, B \in M_n$, $A <^s B$ if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$. The appropriate definition of space pre-order in the ring case is as follows (see Remark 1.6):

$$a <^s b \quad \text{if } aR \subseteq bR \text{ and } Ra \subseteq Rb.$$

The relation $<^s$ is pre-order, i.e. it is reflexive and transitive but it is not antisymmetric.

As we pointed out, the unified theory of matrix partial orders was done by Mitra, see [20] and [22]. Our aim is to present the unified theory in the ring context. For further exposition we need some definitions. The notation used in Chapter 7 of [22] can also be applied in the ring setting. Thus, we follow this notation.

Definition 1.1. Let $\mathcal{P}(R)$ denote the power set (class of all subsets) of R. A g-map is a map

$$\mathcal{G} : R \longrightarrow \mathcal{P}(R)$$

such that for each $a \in R$, $\mathcal{G}(a)$ is a certain subset of $a\{1\}$. The set

$$\Omega_{\mathcal{G}} = \big\{a \in R : \mathcal{G}(a) \neq \emptyset\big\}$$

is called the support of the g-map $\mathcal{G}$. We write $\mathcal{G}_r(a)$ for the set $\mathcal{G}(a) \cap a\{1,2\}$.

Definition 1.2. Let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map. For $a, b \in R$, we say

$$a <^{\mathcal{G}} b \quad \text{if } a \in \Omega_{\mathcal{G}},\ ga = gb \text{ and } ag = bg \text{ for some } g \in \mathcal{G}(a).$$

The order relation $<^{\mathcal{G}}$ is called $\mathcal{G}$-based order relation.

An element $a \in R$ is said to be $\mathcal{G}$-maximal if for any $b \in R$, $a <^{\mathcal{G}} b$ implies $a = b$. Observe that the above $\mathcal{G}$-based order relation concept covers as special cases the minus, star, sharp, core and dual core partial orders:

- Let $\mathcal{G}(a) = a\{1\}$. Then the order relation $<^{\mathcal{G}}$ is the minus order.
- If $\mathcal{G}(a) = \{a^\dagger\}$, then the order relation $<^{\mathcal{G}}$ is the star order.
- If $\mathcal{G}(a) = \{a^\#\}$, then the order relation $<^{\mathcal{G}}$ is the sharp order.
- If $\mathcal{G}(a) = \{a^{\textcircled{\#}}\}$, then the order relation $<^{\mathcal{G}}$ is the core order.
- If $\mathcal{G}(a) = \{a_{\textcircled{\#}}\}$, then the order relation $<^{\mathcal{G}}$ is the dual core order.

We see at once that a $\mathcal{G}$-based order relation is reflexive on the support $\Omega_{\mathcal{G}}$. Also, for $a, b \in R$, $a <^{\mathcal{G}} b$ implies $a <^- b$. It is well known that the relation $<^-$ is a partial order, see [10]. Therefore, $<^{\mathcal{G}}$ is always antisymmetric. Our main objective is to examine when a $\mathcal{G}$-based order relation is transitive and thus a partial order. We need the following definitions.

Definition 1.3. Let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map and $a \in R$. The class

$$\tilde{\mathcal{G}}(a) = \big\{g : ga = a^{(1)}a,\ ag = aa^{(1)} \text{ for some } a^{(1)} \in \mathcal{G}(a)\big\}$$

is called the completion of $\mathcal{G}(a)$. We say that $\mathcal{G}(a)$ is complete if $\mathcal{G}(a) = \tilde{\mathcal{G}}(a)$. If $\mathcal{G}(a)$ is complete for each $a \in R$, we say that $\mathcal{G}$ is complete.

We see at once that $\tilde{\mathcal{G}}(a) \subseteq a\{1\}$ and $\mathcal{G}(a) \subseteq \tilde{\mathcal{G}}(a)$ for each $a \in R$.

Definition 1.4. Let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map. For $a, b \in R$ let

$$\mathcal{G}(a, b) = \{hah : h \in \mathcal{G}(b)\}.$$

A pair (a, b) is said to satisfy the (T)-condition if $\mathcal{G}(a, b) \subseteq \mathcal{G}(a)$.

Definition 1.5. For a g-map $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ and $a \in R$, the set $\mathcal{G}(a)$ is said to be semi-complete if $\mathcal{G}(a, a) \subseteq \mathcal{G}(a)$. If for each $a \in R$, $\mathcal{G}(a)$ is semi-complete, we say the g-map $\mathcal{G}$ is semi-complete.

Thus, $\mathcal{G}(a)$ is semi-complete if and only if (a, a) satisfy the (T)-condition.

Remark 1.6. The proofs of many results stated in [20] are purely algebraic and can be applied in a ring case. Some comments are still necessary. Note that condition $\mathcal{C}(A) \subseteq \mathcal{C}(B)$, where $\mathcal{C}(A)$ is column space of matrix A, is equivalent to $AM_n \subseteq BM_n$. Similarly, $\mathcal{C}(A^*) \subseteq \mathcal{C}(B^*)$ is equivalent to $M_n A \subseteq M_n B$. Thus, when we consider a ring case, the conditions $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(A^*) \subseteq \mathcal{C}(B^*)$ must be replaced by $aR \subseteq bR$ and $Ra \subseteq Rb$, respectively.

The main objective in unified theory is to find the necessary and sufficient condition that makes G-based order relation a partial order. The proof of this result uses linear algebra techniques that cannot be used in ring case. Instead of that we use two-sided Peirce decompositions of R relative to the appropriate sets of idempotents (see [11]). The notion will be explained in the following remark.

Remark 1.7. An element $e \in R$ is idempotent if $e^2 = e$. The set of all idempotents of R is denoted by $R^\bullet = \{e \in R : e^2 = e\}$. An equality $1 = e_1 + e_2 + \cdots + e_n$ where $e_i \in R^\bullet$ and $e_i e_j = 0$ for $i \neq j$ is called the decomposition of the identity of the ring R. If $1 = e_1 + e_2 + \cdots + e_n$ and $1 = f_1 + f_2 + \cdots + f_n$ are two decompositions of the identity of the ring R then any $x \in R$ can be represented in the form

$$x = \left(\sum_{i=1}^{n} e_i\right) x \left(\sum_{j=1}^{n} f_j\right) = \sum_{i,j=1}^{n} e_i x f_j = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f},$$

where $x_{ij} = e_i x f_j$. Let $y = [y_{ij}]_{e \times f}$ and $z = [z_{ij}]_{f \times e}$. As $f_i f_j = 0$ for $i \neq j$, one can check that the usual algebraic operations $x + y$ and xz can be interpreted as operations between appropriate $n \times n$ matrices over R.

2. FD ring

Let R be an arbitrary ring with identity 1. For $e, f \in R^\bullet$ we write

$$e < f \quad \text{if } e = ef = fe.$$

This is the well-known partial order on the set of idempotents. For $a \in R$ we will denote by aR the set $\{ax : x \in R\}$. Similarly, $Ra = \{xa : x \in R\}$ and $aRb = \{axb : x \in R\}$. We need the notion of equivalent idempotents, see [12].

Definition 2.1. (See [12].) Idempotents $e, f \in R^\bullet$ are equivalent, written $e \sim f$, if there exist elements $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$.

For $e \in R^\bullet$, from Definition 2.1, we obtain

$$e \sim 0 \quad \Longrightarrow \quad e = 0. \tag{2.1}$$

Remark 2.2. It is not difficult to show that idempotents e and f are equivalent if and only if eR and fR are isomorphic as right R-modules if and only if Re and Rf are isomorphic as left R-modules. For the proof see Theorem 14 in [12]. From this, it follows that $\sim$ is an equivalence relation on $R^\bullet$, see [12].

Remark 2.3. It is easily seen that $e \sim f$ if and only if there exist $x, y \in R$ such that $xy = e$ and $yx = f$. Indeed, suppose that $xy = e$ and $yx = f$ and set $x_1 = exf$ and $y_1 = fye$. It is easy to show that $x_1 y_1 = e$ and $y_1 x_1 = f$.

The ring R is called Dedekind finite (directly finite) if for every $x, y \in R$, $xy = 1$ implies $yx = 1$, see [13,7]. We see at once that R is Dedekind finite if and only if for every idempotent $e \in R^\bullet$,

$$e \sim 1 \quad \Longrightarrow \quad e = 1. \tag{2.2}$$

Let V, W be arbitrary complex vector spaces. We use $\mathcal{L}(V, W)$ to denote the set of all linear transformations from V to W. Also, $\mathcal{L}(V) = \mathcal{L}(V, V)$. Identifying a complex $n \times n$ matrix A with linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$, we conclude that the set of all complex $n \times n$ matrices is equal to $\mathcal{L}(\mathbb{C}^n)$. It is customary to write M_n rather than $\mathcal{L}(\mathbb{C}^n)$.

For $A \in \mathcal{L}(V, W)$, we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote the range and the null-space of A, respectively. A linear transformation $P \in \mathcal{L}(V)$ which is idempotent, that is $P^2 = P$, is called a projection. A class of operators related to idempotent elements on Hilbert spaces was studied in [15].

Remark 2.4. It is well known that the complex vector space V is finite dimensional if and only if $\mathcal{L}(V)$ is a Dedekind finite ring, see for example [7, pp. 165–166].

For our purpose we need slightly different characterization of finite dimensional vector spaces.

Theorem 2.5. *Let V be an arbitrary vector space. The following conditions are equivalent:*

(i) $\dim V < \infty$;
(ii) *For arbitrary projections $P, Q \in \mathcal{L}(V)$ the following holds:*

$$P \sim Q \quad \Longrightarrow \quad I - P \sim I - Q.$$

Proof. (i) ⇒ (ii): Suppose that $\dim V = n < \infty$ and let $P, Q \in \mathcal{L}(V)$ be two equivalent idempotents. By Remark 2.3, there exist $X, Y \in \mathcal{L}(V)$ such that $P = XY$ and $Q = YX$. It is easy to see that $P = XQY$ and $Q = YPX$ so $\operatorname{rank}(P) = \operatorname{rank}(Q)$. By the rank-nullity theorem it follows that $\operatorname{rank}(I-P) = \operatorname{rank}(I-Q)$, so there exists an isomorphism $A_1 \in \mathcal{L}(\mathcal{R}(I-P), \mathcal{R}(I-Q))$. Let $A, B \in \mathcal{L}(V)$ be defined by $Av = A_1(I-P)v$ and $Bv = A_1^{-1}(I-Q)v$, $v \in V$. It is easy to see that $AB = I - Q$ and $BA = I - P$, so, by Remark 2.3, $I - P \sim I - Q$.

(ii) ⇒ (i): By (2.1), it is clear that the condition (ii) implies the condition $P \sim I \Rightarrow P = I$. Thus $\mathcal{L}(V)$ is a Dedekind finite ring, so, by Remark 2.4, it follows that $\dim V < \infty$. □

Remark 2.6. Let V be an arbitrary vector space and let $P, Q \in \mathcal{L}(V)$ be two projections. It is not difficult to prove that $P \sim Q$ if and only if the subspaces $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ are isomorphic.

The previous characterization of finite dimensionality is purely algebraic, as one may see. Inspired by Theorem 2.5, we introduce a special kind of ring, which may be called finite dimensional ring or FD ring for short.

Definition 2.7. A ring R is FD ring if for arbitrary idempotent elements $e, f \in R^\bullet$ the following holds:

$$e \sim f \quad \Longrightarrow \quad 1 - e \sim 1 - f. \tag{2.3}$$

Remark 2.8. The notion of FD ring is new, as far as we know, and, as we can see, every FD ring is Dedekind finite ring. So, by (2.2), if R is an FD ring then for $e \in R^\bullet$ the following holds:

$$e \sim 1 \quad \Longrightarrow \quad e = 1. \tag{2.4}$$

It follows by Theorem 2.5 that M_n is an FD ring, but $\mathcal{L}(V)$, where $\dim V = \infty$, is not an FD ring.

Remark 2.9. From Remark 2.2, it follows that the following conditions are equivalent:

(i) R is FD ring.
(ii) For each idempotents $e, f \in R^\bullet$ the following holds:
If eR and fR are isomorphic as right R-modules then $(1-e)R$ and $(1-f)R$ are isomorphic as right R-modules.
(iii) For each idempotents $e, f \in R^\bullet$ the following holds:
If Re and Rf are isomorphic as left R-modules then $R(1-e)$ and $R(1-f)$ are isomorphic as left R-modules.

When R is FD ring then the following theorem shows that a condition stronger than (2.3) is satisfied.

Theorem 2.10. *Let R be an FD ring and let $e_1, e_2, f_1, f_2 \in R^\bullet$ be idempotents in R. Then the following implication holds:*

$$(e_1 \sim e_2,\ f_1 \sim f_2,\ e_1 < f_1,\ e_2 < f_2) \quad \Longrightarrow \quad f_1 - e_1 \sim f_2 - e_2. \tag{2.5}$$

Proof. Suppose that $e_1 \sim e_2$, $f_1 \sim f_2$ and

$$e_1 f_1 = f_1 e_1 = e_1, \qquad e_2 f_2 = f_2 e_2 = e_2. \tag{2.6}$$

It is easy to see that $f_1 - e_1$ and $f_2 - e_2$ are idempotents. Set

$$\begin{aligned} p_1 &= e_1, & p_2 &= f_1 - e_1, & p_3 &= 1 - f_1 \\ q_1 &= e_2, & q_2 &= f_2 - e_2, & q_3 &= 1 - f_2. \end{aligned}$$

Using (2.6), an easy verification shows that $1 = p_1 + p_2 + p_3$ and $1 = q_1 + q_2 + q_3$ are two decompositions of the identity of the ring R. From $p_1 = e_1 \sim e_2 = q_1$, we have that there exist $x \in p_1 R q_1$ and $y \in q_1 R p_1$ such that $xy = p_1$ and $yx = q_1$. Since $x = p_1 x q_1$ and $y = q_1 y p_1$ we have the following representations:

$$x = \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{p \times q} \quad \text{and} \quad y = \begin{bmatrix} y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{q \times p}.$$

Since R is FD ring, $f_1 \sim f_2$ implies $1 - f_1 \sim 1 - f_2$, i.e. $p_3 \sim q_3$. So there exist $u \in p_3 R q_3$ and $v \in q_3 R p_3$ such that $uv = p_3$ and $vu = q_3$. Since $u = p_3 u q_3$ and $v = q_3 v p_3$ we have:

$$u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u \end{bmatrix}_{p \times q} \quad \text{and} \quad v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{bmatrix}_{q \times p}.$$

It follows that

$$(x+u)(y+v) = \begin{bmatrix} xy & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & uv \end{bmatrix}_{p\times p} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_3 \end{bmatrix}_{p\times p}.$$

Thus, $(x+u)(y+v) = p_1 + p_3$. Similarly we can show that $(y+v)(x+u) = q_1 + q_3$. We conclude that $p_1+p_3 \sim q_1+q_3$. As R is FD ring it follows that $1-(p_1+p_3) \sim 1-(q_1+q_3)$, i.e. $f_1 - e_1 \sim f_2 - e_2$. $\square$

Note that when we set $f_1 = f_2 = 1$ in Theorem 2.10 then the condition (2.5) actually reduces to condition (2.3).

3. Main result

The main objective of this section is to find a necessary and sufficient condition for a $\mathcal{G}$-based relation to be a partial order on R. First we give same introductory results. Recall that $a <^{\mathcal{G}} b$, where $<^{\mathcal{G}}$ is an arbitrary $\mathcal{G}$-based order relation, implies $a <^{-} b$. We now give some elementary properties of minus partial order which will be used in the sequel. If $a <^{-} b$ then, see [27]:

$$a = bb^{(1)}a = ab^{(1)}b = ab^{(1)}a, \tag{3.1}$$

for each $b^{(1)} \in b\{1\}$. It follows that $a <^{-} b$ implies $b\{1\} \subseteq a\{1\}$. Also, from (3.1) we obtain

$$(b-a)b^{(1)}(b-a) = b - a. \tag{3.2}$$

Theorem 3.1. *Let R be a ring and $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map. Let $a, b \in \Omega_{\mathcal{G}}$. Then the following hold:*

(i) *Suppose that $a <^{\mathcal{G}} b$. Fix $h \in \mathcal{G}(b)$ and set*

$$e_1 = ah, \qquad e_2 = (b-a)h, \qquad e_3 = 1 - bh$$
$$f_1 = ha, \qquad f_2 = h(b-a), \qquad f_3 = 1 - hb.$$

Then

$$1 = e_1 + e_2 + e_3, \quad \text{and} \quad 1 = f_1 + f_2 + f_3$$

are two decompositions of the identity of the ring R with respect to which a and b have the following matrix forms:

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}, \qquad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}. \tag{3.3}$$

(ii) *If $\mathcal{G}$ is semi-complete and a and b have representations (3.3) where $1 = e_1 + e_2 + e_3$ and $1 = f_1 + f_2 + f_3$ are two decompositions of the identity of the ring R such that $e_1 = ag$ and $f_1 = ga$ for some $g \in \mathcal{G}(a)$, then $a <^{\mathcal{G}} b$.*

Proof. The proof of (i) proceeds along the same lines as the proof of Theorem 3.5 in [27]. For the proof of (ii) suppose that a and b have representations (3.3) where $e_1 = ag$ and $f_1 = ga$ for some $g \in \mathcal{G}(a)$. Let $g' = gag$. Since $\mathcal{G}$ is semi-complete, we obtain $g' \in \mathcal{G}(a)$. We have

$$f_1 g' e_1 = gagagag = gag = g',$$

so

$$g' = \begin{bmatrix} g' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f\times e}.$$

Now it is easy to see that

$$ag' = bg' = \begin{bmatrix} ag' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \quad \text{and} \quad g'a = g'b = \begin{bmatrix} g'a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f\times f}.$$

By Definition 1.2, $a <^{\mathcal{G}} b$. $\square$

In the next theorem we will characterize all elements which are above some element a under the G-based order relation. See Theorem 3.4.3 in [22] for the characterization concerning the minus partial order on the set of matrices.

Theorem 3.2. *Let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a semi-complete g-map and $a \in \Omega_{\mathcal{G}}$. Then*

$$\{b \in R : a <^{\mathcal{G}} b\} = \{a + (1 - ag)d(1 - ga) : g \in \mathcal{G}(a),\ d \in R\}. \tag{3.4}$$

Proof. Let S denote the set on the right hand side of (3.4). Suppose that $a <^{\mathcal{G}} b$. Then $ag = bg$ and $ga = gb$ for some $g \in \mathcal{G}(a)$. Therefore, $(b - a)g = 0$ and $g(b - a) = 0$. It is easy to check that

$$b = a + b - a = a + (1 - ag)(b - a)(1 - ga),$$

so $b \in S$. Suppose now that $b \in S$, i.e. $b = a + (1 - ag)d(1 - ga)$ for some $g \in \mathcal{G}(a)$ and $d \in R$. Let $g' = gag$. Since $\mathcal{G}$ is semi-complete, we conclude that $g' \in \mathcal{G}(a)$. As $aga = a$, it is easily seen that $bg' = ag'$ and $g'b = g'a$. Thus $a <^{\mathcal{G}} b$. $\square$

From Theorem 3.2 we obtain the following corollary.

Corollary 3.3. *Let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a semi-complete g-map and $a \in \Omega_{\mathcal{G}}$, $b \in R$. Then $a <^{\mathcal{G}} b$ if and only if*

$$b = \begin{bmatrix} a & 0 \\ 0 & v \end{bmatrix}_{p \times q},$$

where $p = ag$, $q = ga$ for some $g \in \mathcal{G}(a)$ and $v \in (1-p)R(1-q)$ is arbitrary.

In the next theorem we will prove that under certain conditions the only maximal elements are invertible elements.

Theorem 3.4. *Let R be an FD ring and let $\mathcal{G} : R \to \mathcal{P}(R)$ be a semi-complete g-map. The element $a \in \Omega_{\mathcal{G}}$ is $\mathcal{G}$-maximal under the $<^{\mathcal{G}}$ if and only if a is invertible.*

Proof. Suppose that $a \in \Omega_G$ is not invertible. We wish to prove that a is not $\mathcal{G}$-maximal. Let $a^{(1)} \in \mathcal{G}(a)$ and $g = a^{(1)}aa^{(1)}$. Since G is semi-complete, we have $g \in \mathcal{G}(a)$. Also, $gag = g$. Set $e = ag$ and $f = ga$. Then $e \sim f$. If $e = 1$ or $f = 1$ then by (2.4), $e = f = 1$ and a is invertible. Therefore, $e \neq 1$ and $f \neq 1$. As R is FD ring, we have $1 - e \sim 1 - f$. Thus, there exist $x \in (1-e)R(1-f)$ and $y \in (1-f)R(1-e)$ such that $xy = 1 - e$ and $yx = 1 - f$. Let $b = a + x$. Since $1 - e \neq 0$, we have $x \neq 0$ so $b \neq a$. Note that

$$(1-f)g = (1-ga)g = 0 \quad \text{and} \quad g(1-e) = g(1-ag) = 0.$$

Therefore, $xg = 0$ and $gx = 0$. It follows that $bg = ag$ and $gb = ga$ so $a <^{\mathcal{G}} b$, which means that a is not $\mathcal{G}$-maximal.

On the other hand, suppose that $a \in \Omega_{\mathcal{G}}$ is invertible and suppose that $a <^{\mathcal{G}} b$ for some $b \in R$. Since a is invertible, the only inner inverse of a is a^{-1}, so $\mathcal{G}(a) = \{a^{-1}\}$. Now, $a <^{\mathcal{G}} b$ implies $aa^{-1} = ba^{-1}$. Thus, $a = b$. We prove that a is maximal. □

We are now in a position to give a necessary and sufficient condition for a $\mathcal{G}$-based relation to be a partial order on R. It turns out that it is much easier to determine a sufficient condition.

Theorem 3.5. *Let R be a ring and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map. Given any $a, b \in R$ suppose that if $a <^{\mathcal{G}} b$ and b is not maximal then the pair (a, b) satisfies the (T)-condition. Then the binary relation $<^{\mathcal{G}}$ is a partial order on Ω_G.*

Proof. The proof does not differ from the proof in the complex matrix case, see Theorem 7.2.13 in [22]. We give it here for completeness. Since the relation $<^{\mathcal{G}}$ is always reflexive and antisymmetric on Ω_G, it is sufficient to show that $<^{\mathcal{G}}$ is transitive. Suppose that $a <^{\mathcal{G}} b$ and $b <^{\mathcal{G}} c$. If b is maximal then $b = c$ so $a <^{\mathcal{G}} c$. So, let b be not maximal.

Since $b <^{\mathcal{G}} c$, there exists $h \in \mathcal{G}(b)$ such that $bh = ch$ and $hb = hc$. As $a <^{\mathcal{G}} b$ we have that the pair (a, b) satisfies the (T)-condition, so $hah \in \mathcal{G}(a)$. Let $g = hah$. As $a <^{\mathcal{G}} b$ implies $a <^{-} b$, from (3.1) we obtain $a = bha = ahb = aha$. We have

$$ag = ahah = ah = bhah = chah = cg.$$

Similarly, $ga = gc$. Thus, $a <^{\mathcal{G}} c$. □

Corollary 3.6. *Let R be a ring and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a semi-complete g-map. For $a, b \in R$, the following conditions are equivalent:*

(i) *If $a <^{\mathcal{G}} b$ and b is not maximal then the pair (a, b) satisfies the (T)-condition.*
(ii) *If $a <^{\mathcal{G}} b$ and b is not maximal then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*
(iii) *If $a <^{\mathcal{G}} b$ and b is not maximal then $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$ (i.e. for any $h \in \mathcal{G}(b)$ there exists $g \in \mathcal{G}(a)$ such that $ag = ah$ and $ga = ha$).*

If one of the conditions (i)–(iii) is satisfied for each $a, b \in R$ then the relation $<^{\mathcal{G}}$ is a partial order on Ω_G.

Proof. In view of Theorem 3.5 it is sufficient to show the equivalence of (i), (ii) and (iii). If $\mathcal{G}(b) = \emptyset$ then all the conditions are satisfied. Suppose that $\mathcal{G}(b) \neq \emptyset$.

(i) ⇒ (ii): Suppose that (i) is satisfied and let $a <^{\mathcal{G}} b$ and b is not maximal. Let h be an arbitrary element of the set $\tilde{\mathcal{G}}(b)$. This means that there exists $b^{(1)} \in \mathcal{G}(b)$ such that $hb = b^{(1)}b$ and $bh = bb^{(1)}$. Let $g = b^{(1)}ab^{(1)}$. Since (a, b) satisfies the (T)-condition, we have $g \in \mathcal{G}(a)$. Since $a <^{\mathcal{G}} b$ implies $a <^{-} b$, by (3.1), we obtain $a = aua = aub = bua$ for each $u \in b\{1\}$. Therefore, $a = bha = ahb = ab^{(1)}a$. It follows that

$$ha = h(bha) = b^{(1)}bha = b^{(1)}a = b^{(1)}\big(ab^{(1)}a\big) = ga$$
$$ah = (ahb)h = ahbb^{(1)} = ab^{(1)} = \big(ab^{(1)}a\big)b^{(1)} = ag.$$

Therefore, $h \in \tilde{\mathcal{G}}(a)$, so $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.

(ii) ⇒ (iii) is clear since $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(b)$, for each $b \in R$.

(iii) ⇒ (i): Suppose that (iii) is satisfied and let $a <^{\mathcal{G}} b$ where b is not maximal. For any $h \in \mathcal{G}(b)$ there exists $g \in \mathcal{G}(a)$ such that $ag = ah$ and $ga = ha$. Since $\mathcal{G}$ is semi-complete, we obtain $hah = gag \in \mathcal{G}(a)$, so the pair (a, b) satisfies the (T)-condition. □

In the next theorem we will show that under certain assumptions, the sufficient conditions given in Theorem 3.5 and Corollary 3.6, are also necessary for a $\mathcal{G}$-based relation to be a partial order on R. The equivalence of (i) and (ii) in the next theorem is a generalization of the result of Mitra (Theorem 2.6 in [20]) and Mitra, Bhimasankaram and Malik (Theorem 7.2.31 in [22]), who have considered the case $R = M_n$, where M_n is the algebra of all $n \times n$ complex matrices. Note that we cannot use the linear algebra techniques which are a dominant tool in matrix case.

Theorem 3.7. *Let R be an FD ring and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a semi-complete g-map. Then the following statements are equivalent:*

(i) *The relation $<^{\mathcal{G}}$ is a partial order on Ω_G.*
(ii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then the pair (a, b) satisfies the (T)-condition.*
(iii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*
(iv) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that R is an FD ring, $\mathcal{G}$ is semi-complete g-map and suppose that $<^{\mathcal{G}}$ is transitive. Let a, b be elements in R such that $a <^{\mathcal{G}} b$. Obviously, if $\mathcal{G}(b) = \emptyset$ then (a, b) satisfies the (T)-condition. Therefore, assume that $\mathcal{G}(b) \neq \emptyset$. Let $b^{(1)} \in \mathcal{G}(b)$. We need to prove that $b^{(1)}ab^{(1)} \in \mathcal{G}(a)$. Let $h = b^{(1)}bb^{(1)}$. Since G is semi-complete, we have $h \in \mathcal{G}(b) \subseteq b\{1\}$. It is easy to see that $hbh = h$. Also, from (3.1) we obtain

$$hah = b^{(1)}\big(bb^{(1)}a\big)b^{(1)}bb^{(1)} = b^{(1)}\big(ab^{(1)}b\big)b^{(1)} = b^{(1)}ab^{(1)}. \tag{3.5}$$

Let e_i and f_i, $i = 1, 2, 3$, be as in Theorem 3.1. Then a and b have representations (3.3). From Remark 2.3, we have that $bh \sim hb$. R is an FD ring, and consequently $e_3 = 1 - bh \sim 1 - hb = f_3$. It follows that there exist $x \in e_3Rf_3$ and $y \in f_3Re_3$ such that $e_3 = xy$ and $f_3 = yx$. Note that $hah = f_1h = he_1$ so $hah \in f_1Re_1$. Similarly $h(b - a)h \in f_2Re_2$. It follows that

$$h = hbh = hah + h(b - a)h = \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f\times e}. \tag{3.6}$$

Let $c = b + x$ and $z = h + y$. Therefore,

$$c = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & x \end{bmatrix}_{e\times f} \quad \text{and} \quad z = \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & y \end{bmatrix}_{f\times e}. \tag{3.7}$$

From (3.1) and (3.2) we conclude that

$$cz = \begin{bmatrix} ahah & 0 & 0 \\ 0 & (b-a)h(b-a)h & 0 \\ 0 & 0 & xy \end{bmatrix}_{e\times e} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_{e\times e} = 1.$$

Similarly, $zc = 1$, so c is invertible and $z = c^{-1}$. We next prove that $b <^{\mathcal{G}} c$. From (3.3), (3.6), (3.7) we obtain that

$$bh = ch = \begin{bmatrix} ahah & 0 & 0 \\ 0 & (b-a)h(b-a)h & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

Similarly, $hb = hc = f_1 + f_2$. By definition, we conclude that $b <^{\mathcal{G}} c$. Since $<^{\mathcal{G}}$ is transitive, we have $a <^{\mathcal{G}} c$. It follows that there exists $g \in \mathcal{G}(a)$ such that $ag = cg$ and $ga = gc$. We thus get $a = aga = cgc$. It follows that

$$g = c^{-1}ac^{-1} = \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & y \end{bmatrix}_{f\times e} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f} \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & y \end{bmatrix}_{f\times e}$$

$$= \begin{bmatrix} hahahah & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f\times e} = hah.$$

From (3.5), we conclude that $b^{(1)}ab^{(1)} = g \in \mathcal{G}(a)$.

(ii) $\Rightarrow$ (i) follows by Theorem 3.5.

The proofs of (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) proceed along the same lines as the proof of Corollary 3.6. $\square$

Corollary 3.8. *Let R be a ring and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map such that $\mathcal{G}(x) = \{g_x\}$ where g_x is a certain inner inverse of x. For $a, b \in R$ the following conditions are equivalent:*

(i) *If $a <^{\mathcal{G}} b$ then $g_b a g_b = g_a$.*
(ii) *If $a <^{\mathcal{G}} b$ then $ag_a = ag_b$ and $g_a a = g_b a$.*
(iii) *If $a <^{\mathcal{G}} b$ then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*

If one of the conditions (i)–(iii) is satisfied for each $a, b \in R$ then the relation $<^{\mathcal{G}}$ is a partial order on $\Omega_{\mathcal{G}}$.

Proof. The proof follows by Corollary 3.6. $\square$

Corollary 3.9. *Let R be an FD ring and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map such that, for $x \in R$, $\mathcal{G}(x) = \{g_x\}$ where g_x is a certain reflexive generalized inverse of x. Then the following conditions are equivalent:*

(i) *The order relation $<^{\mathcal{G}}$ is a partial order on $\Omega_{\mathcal{G}}$.*
(ii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $g_b a g_b = g_a$.*
(iii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $ag_a = ag_b$ and $g_a a = g_b a$.*
(iv) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*

Proof. Since for any $x \in \Omega_{\mathcal{G}}$, g_x is a reflexive generalized inverse of x, it follows that $\mathcal{G}$ is semi-complete. Now, the proof follows by Theorem 3.7. $\square$

We have already proved in Theorem 3.7 that under certain conditions $a <^{\mathcal{G}} b$ implies $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$. It is natural to ask if the converse is true.

Theorem 3.10. *Let R be a ring and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a semi-complete g-map. For $a, b \in \Omega_{\mathcal{G}}$, if $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$ and $a <^s b$ then $a <^{\mathcal{G}} b$.*

Proof. Suppose that $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$ and $a <^s b$ and fix $h \in \mathcal{G}(b)$. Then $h \in \tilde{\mathcal{G}}(a)$ so there exists a $g \in \mathcal{G}(a)$ such that $ha = ga$ and $ah = ag$. As $a <^s b$, we have $a = xb = by$ for some $x, y \in R$. We conclude that $bha = bhby = by = a$. Similarly, $a = ahb$. Since $\mathcal{G}$ is semi-complete, we have $f := gag \in \mathcal{G}(a)$. It is easy to see that $faf = f$ and $fa = gaga = ga = ha$. It follows that $af = bhaf = bfaf = bf$. Similarly, $af = ah$ and $fa = fb$. By definition, $a <^{\mathcal{G}} b$. $\square$

We proved in Theorem 3.7 that under certain conditions $a <^{\mathcal{G}} b$ implies that (a, b) satisfies (T)-condition, i.e. for each $h \in G(b)$ there exists a $g \in \mathcal{G}(a)$ such that $hah = g$. Also, $a <^{\mathcal{G}} b$ implies $a <^- b$. The converse result is proved in matrix case for some kind of matrix partial order. Namely, in Theorem 2.4 in [21] it is proved that for $A, B \in M_n$ if $A <^- B$ and $B^\# AB^\# <^s A^\#$ then $A <^\# B$. The same result is proved when the Moore–Penrose inverse and star order are considered instead of group inverse and sharp order. In the next theorem we will obtain the stronger result in the ring setting. Recall that $\mathcal{G}_r(a) = \mathcal{G}(a) \cap a\{1, 2\}$.

Theorem 3.11. *Let $a, b \in R$ and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map. Suppose that $a <^- b$ and suppose that there exist $h \in \mathcal{G}(b)$ and $g \in \mathcal{G}_r(a)$ such that $hah <^s g$. Then $a <^{\mathcal{G}} b$.*

Proof. Suppose that $a <^- b$ and suppose that $hah <^s g$ for some $h \in \mathcal{G}(b)$ and $g \in \mathcal{G}_r(a)$. Therefore, $hah = xg = gy$ for some $x, y \in R$. As $a <^- b$ we have $a = aha = ahb = bha$. We obtain

$$\begin{aligned} g &= gag = g(aha)g = gahahag = ga(hah)ag \\ &= ga(xg)ag = gax(gag) = gaxg = ga(xg) \\ &= ga(gy) = (gag)y = gy, \end{aligned}$$

so $hah = g$. Now

$$\begin{aligned} gb &= hahb = h(ahb) = ha = haha = (hah)a = ga \\ bg &= bhah = (bha)h = ah = ahah = a(hah) = ag. \end{aligned}$$

It follows that $a <^{\mathcal{G}} b$. $\square$

Theorem 3.12. *Let $a, b \in R$ and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map. Then the following hold:*

(i) *If $\mathcal{G}$ is semi-complete and $a <^{\mathcal{G}} b$ then $a <^- b$ and $a = bgb$ for some $g \in \mathcal{G}_r(a)$.*
(ii) *If $b \in \Omega_{\mathcal{G}}$, $a <^- b$ and if there exists $g \in \mathcal{G}_r(a)$ such that $bgb <^s a$ then $a <^{\mathcal{G}} b$.*

Proof. (i): Suppose that $\mathcal{G}$ is semi-complete and $a <^{\mathcal{G}} b$. Then $a <^- b$ and there exists $a^{(1)} \in \mathcal{G}(a)$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)}a = a^{(1)}b$. Set $g = a^{(1)}aa^{(1)}$. Since $\mathcal{G}$ is semi-complete, we obtain $g \in \mathcal{G}(a)$. It is easy to see that $bgb = a$.

(ii): Suppose that $a <^- b$ and suppose that there exists $g \in \mathcal{G}_r(a)$ such that $bgb <^s a$. Therefore, there exist $x, y \in R$ such that $bgb = xa = ay$. Fix $h \in \mathcal{G}(b)$. From $a <^- b$, we obtain $a = aha = bha = ahb$. We have

$$\begin{aligned} a &= aga = (ahb)g(bha) = ah(bgb)ha = ah(xa)ha \\ &= ahx(aha) = ahxa = ah(xa) = ah(ay) = (aha)y = ay, \end{aligned}$$

so $bgb = a$. It follows that

$$\begin{aligned} bg &= bgag = bg(bha)g = (bgb)hag = ahag = ag \\ gb &= gagb = g(ahb)gb = gah(bgb) = gaha = ga, \end{aligned}$$

so $a <^{\mathcal{G}} b$. $\square$

Theorem 3.12 is a generalization of Theorem 2.3 in [21] where Mitra considers the case $R = M_n$, $\mathcal{G}(A) = \{A^\dagger\}$ and $\mathcal{G}(A) = \{A^\#\}$. The following straightforward result is inspired by (and is much stronger than) Theorem 21 in [18].

Theorem 3.13. *Let $a, b \in R$. If $a <^s b$ and $b\{1\} \cap a\{1, 2\} \neq \emptyset$ then $a = b$.*

Proof. Let $g \in b\{1\} \cap a\{1, 2\}$, that is $bgb = b$, $aga = a$ and $gag = g$. Since $a <^s b$ there exist $x, y \in R$ such that $a = xb = by$. Therefore, $bga = bgby = by = a$ and $agb = xbgb = xb = a$. It follows that

$$b = bgb = bgagb = agb = a. \qquad \square$$

It is known that for minus, star and sharp matrix partial orders $A < B$ implies $B - A < B$, where $< \in \{<^-, <^\dagger, <^\#\}$. But, when $< \in \{<^{\textcircled{\#}}, <_{\textcircled{\#}}\}$, $A < B$ does not imply $B - A < B$. In the next theorem we give the answer to the question when this property is valid for $\mathcal{G}$-based order relation. See also Theorem 2.8 in [20].

Theorem 3.14. *Let R be an FD ring and let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map such that $\mathcal{G}(x) = \{g_x\}$ where g_x is a certain reflexive generalized inverse of x. Suppose that $<^{\mathcal{G}}$ is a partial order on $\Omega_{\mathcal{G}}$ and suppose that $a <^{\mathcal{G}} b$. Then $b - a <^{\mathcal{G}} b$ if and only if $g_{b-a} = g_b - g_a$.*

Proof. Suppose that $a <^{\mathcal{G}} b$ and $b - a <^{\mathcal{G}} b$. From Corollary 3.9 it follows that $g_b a g_b = g_a$ and $g_b(b - a)g_b = g_{b-a}$. Thus, $g_{b-a} = g_b b g_b - g_b a g_b = g_b - g_a$. On the other hand suppose that $a <^{\mathcal{G}} b$ and $g_{b-a} = g_b - g_a$. From Corollary 3.9, it follows that $ag_a = ag_b$ and $g_a a = g_b a$. Thus, we obtain

$$(b-a)g_{b-a} = (b-a)(g_b - g_a) = b(g_b - g_a) - ag_b + ag_a = bg_{b-a}.$$

In the same manner we obtain $g_{b-a}(b-a) = g_{b-a}b$ so $b - a <^{\mathcal{G}} b$. $\square$

In the sequel we will consider specific known partial orders and their properties. It is known that minus, star, sharp, core and dual core partial orders are indeed partial order relations. For the proof in the matrix case see [22] and [1]. For the proof in the ring case see [10] and [29]. In contrast to existing proofs we will prove these facts using Corollary 3.8.

G1. The minus partial order is a partial order relation on the set $\Omega_{\mathcal{G}} := \{a \in R : a\{1\} \neq \emptyset\}$.
By Theorem 3.5 it is sufficient to show that $b^{(1)}ab^{(1)} \in a\{1\}$ for each $b^{(1)} \in b\{1\}$ whenever $a <^{-} b$. Suppose that $a, b \in \Omega_{\mathcal{G}}$, $a <^{-} b$ and $b^{(1)} \in b\{1\}$. Thus, there exists $g \in a\{1\}$ such that $ag = bg$ and $ga = gb$. We have

$$\begin{aligned} ab^{(1)}ab^{(1)}a &= agab^{(1)}agab^{(1)}aga = agbb^{(1)}bgbb^{(1)}bga \\ &= agbgbga = agagaga = a, \end{aligned}$$

so $b^{(1)}ab^{(1)} \in a\{1\}$.

G2. The sharp partial order is a partial order relation on the set $\Omega_{\mathcal{G}} := \{a \in R : a^{\#} \text{ exists}\}$.
By Corollary 3.8 it is sufficient to show that $b^{\#}ab^{\#} = a^{\#}$ when $a <^{\#} b$. So, suppose that $a^{\#}$ and $b^{\#}$ exist and suppose that $a <^{\#} b$. This means that $aa^{\#} = ba^{\#}$ and $a^{\#}a = a^{\#}b$. We have

$$\begin{aligned} b^{\#}ab^{\#} &= b^{\#}aa^{\#}ab^{\#} = b^{\#}ba^{\#}bb^{\#} = b^{\#}ba\left(a^{\#}\right)^3 abb^{\#} \\ &= b^{\#}bb\left(a^{\#}\right)^3 bbb^{\#} = b\left(a^{\#}\right)^3 b = a\left(a^{\#}\right)^3 a = a^{\#}. \end{aligned}$$

G3. The star partial order is a partial order relation on the set $\Omega_{\mathcal{G}} := \{a \in R : a^{\dagger} \text{ exists}\}$.
By Corollary 3.8 we need to show that $b^{\dagger}ab^{\dagger} = a^{\dagger}$ whenever $a <^{*} b$. Suppose that $a^{\dagger}$ and $b^{\dagger}$ exist and suppose that $a <^{\dagger} b$. Thus, $aa^{\dagger} = ba^{\dagger}$ and $a^{\dagger}a = a^{\dagger}b$. We have

$$\begin{aligned} b^{\dagger}ab^{\dagger} &= b^{\dagger}aa^{\dagger}ab^{\dagger} = b^{\dagger}ba^{\dagger}bb^{\dagger} = b^{\dagger}ba^{\dagger}aa^{\dagger}aa^{\dagger}bb^{\dagger} \\ &= \left(b^{\dagger}b\right)^*\left(a^{\dagger}a\right)^*a^{\dagger}\left(aa^{\dagger}\right)^*\left(bb^{\dagger}\right)^* = \left(a^{\dagger}ab^{\dagger}b\right)^*a^{\dagger}\left(bb^{\dagger}aa^{\dagger}\right)^* \\ &= \left(a^{\dagger}bb^{\dagger}b\right)^*a^{\dagger}\left(bb^{\dagger}ba^{\dagger}\right)^* = \left(a^{\dagger}b\right)^*a^{\dagger}\left(ba^{\dagger}\right)^* = \left(a^{\dagger}a\right)^*a^{\dagger}\left(aa^{\dagger}\right)^* = a^{\dagger}. \end{aligned}$$

G4. The core partial order is a partial order relation on the set $\Omega_{\mathcal{G}} := \{a \in R : a^{\#\!\!\!\!\bigcirc} \text{ exists}\}$.

Suppose that $a^{\oplus}$ and $b^{\oplus}$ exist and suppose that $a <^{\oplus} b$. That is, $aa^{\oplus} = ba^{\oplus}$ and $a^{\oplus}a = a^{\oplus}b$. Of course, we want to show that $b^{\oplus}ab^{\oplus} = a^{\oplus}$. Using the equations which define the core inverse

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (6)\ xa^2 = a, \quad (7)\ ax^2 = x,$$

we obtain

$$\begin{aligned} b^{\oplus}ab^{\oplus} &= b^{\oplus}aa^{\oplus}ab^{\oplus} = b^{\oplus}ba^{\oplus}bb^{\oplus} = b^{\oplus}ba\big(a^{\oplus}\big)^2bb^{\oplus} \\ &= b^{\oplus}bb\big(a^{\oplus}\big)^2bb^{\oplus} = ba^{\oplus}a^{\oplus}aa^{\oplus}bb^{\oplus} = a\big(a^{\oplus}\big)^2\big(aa^{\oplus}\big)^*\big(bb^{\oplus}\big)^* \\ &= a^{\oplus}\big(bb^{\oplus}aa^{\oplus}\big)^* = a^{\oplus}\big(bb^{\oplus}ba^{\oplus}\big)^* = a^{\oplus}\big(ba^{\oplus}\big)^* = a^{\oplus}\big(aa^{\oplus}\big)^* = a^{\oplus}. \end{aligned}$$

G5. The dual core partial order is a partial order relation on the set $\Omega_{\mathcal{G}} := \{a \in R : a_{\oplus}$ exists$\}$.
In the same manner as we do for the core partial order, one can show that when $a_{\oplus}$ and $b_{\oplus}$ exist, $a <_{\oplus} b$ implies $b_{\oplus}ab_{\oplus} = a_{\oplus}$.

By Corollaries 3.6 and 3.8 and by the items G1–G5, it follows that $a < b$ implies $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$, where $< \in \{<^-, <^\#, <^\dagger, <^{\oplus}, <_{\oplus}\}$ and $\mathcal{G}$ is an appropriate g-map. We close this section characterizing the sets $\tilde{\mathcal{G}}(a)$ for these g-maps.

- It is clear that $\tilde{\mathcal{G}}(a) = a\{1\}$ when $\mathcal{G}(a) = a\{1\} \neq \emptyset$.
- Suppose that $a^\#$ exists and $\mathcal{G}(a) = \{a^\#\}$. Then $\tilde{\mathcal{G}}(a) = a\{1,5\}$.
Suppose that $g \in \tilde{\mathcal{G}}(a)$, that is $ga = a^\#a$ and $ag = aa^\#$. Then $aga = aa^\#a = a$ and $ag = aa^\# = a^\#a = ga$ so $g \in a\{1,5\}$. Conversely, suppose that $g \in a\{1,5\}$, i.e. $aga = a$ and $ag = ga$. Then
$$\begin{aligned} ga &= gaa^\#a = gaaa^\# = agaa^\# = aa^\# = a^\#a \\ ag &= aa^\#ag = a^\#aag = a^\#aga = a^\#a = aa^\#, \end{aligned}$$
so $g \in \tilde{\mathcal{G}}(a)$.
- Suppose that $a^\dagger$ exists and $\mathcal{G}(a) = \{a^\dagger\}$. Then $\tilde{\mathcal{G}}(a) = a\{1,3,4\}$.
Suppose that $g \in \tilde{\mathcal{G}}(a)$. Then $ga = a^\dagger a$ and $ag = aa^\dagger$. We have $aga = aa^\dagger a = a$ and $ag = aa^\dagger$, $ga = a^\dagger a$ which are self-adjoint. Therefore, $a \in a\{1,3,4\}$. Suppose now that $a \in a\{1,3,4\}$. We obtain
$$aa^\dagger = agaa^\dagger = (ag)^*\big(aa^\dagger\big)^* = \big(aa^\dagger ag\big)^* = (ag)^* = ag$$
and similarly $a^\dagger a = ga$. Thus, $a \in \tilde{\mathcal{G}}(a)$.
- Suppose that $a^{\oplus}$ exists and $\mathcal{G}(a) = \{a^{\oplus}\}$. Then $\tilde{\mathcal{G}}(a) = a\{3,6\}$.

Suppose that $g \in \tilde{\mathcal{G}}(a)$, that is $ga = a^{\oplus}a$ and $ag = aa^{\oplus}$. Then $ga^2 = a^{\oplus}a^2 = a$ and $ag = aa^{\oplus}$ which is self-adjoint. Therefore, $g \in a\{3, 6\}$. Conversely, suppose that $ag = (ag)^*$ and $ga^2 = a$. We obtain

$$ga = gaa^{\oplus}a = ga\left(a\left(a^{\oplus}\right)^2\right)a = a\left(a^{\oplus}\right)^2 a = a^{\oplus}a.$$

This gives $aga = aa^{\oplus}a = a$. Also

$$ag = aa^{\oplus}ag = \left(aa^{\oplus}\right)^*(ag)^* = \left(agaa^{\oplus}\right)^* = \left(aa^{\oplus}\right)^* = aa^{\oplus}.$$

It follows that $g \in \tilde{\mathcal{G}}(a)$.

- Suppose that $a_{\oplus}$ exists and $\mathcal{G}(a) = \{a_{\oplus}\}$. Then $\tilde{\mathcal{G}}(a) = a\{4, 8\}$.
  We can show this statement in the same manner as in the case of core inverse.

Remark 3.15. Let $\mathcal{G} : R \longrightarrow \mathcal{P}(R)$ be a g-map such that $\mathcal{G}(a) = \{g_x\}$ where g_x is a certain reflexive generalized inverse of x. Suppose that $a \in \Omega_{\mathcal{G}}$. Then

$$\tilde{\mathcal{G}}(a) = \left\{ \begin{bmatrix} g_a & 0 \\ 0 & g_4 \end{bmatrix}_{q \times p} : g_4 \in (1-q)R(1-p) \right\},$$

where $p = ag_a$ and $q = g_a a$.

Indeed, suppose that g belongs to the set of the right hand side. Since $a = paq$ and $g_a = qg_a p$, we obtain

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} \quad \text{and} \quad g_a = \begin{bmatrix} g_a & 0 \\ 0 & 0 \end{bmatrix}_{q \times p}.$$

Now it is easy to see that $ga = g_a a$ and $ag = ag_a$, i.e. $g \in \tilde{\mathcal{G}}(a)$. Conversely, suppose that $g \in \tilde{\mathcal{G}}(a)$ and let $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right]_{q \times p}$. From $ag = ag_a$ it follows that

$$\begin{bmatrix} ag_1 & ag_2 \\ 0 & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} ag_a & 0 \\ 0 & 0 \end{bmatrix}_{p \times p}.$$

Therefore, $ag_1 = ag_a$ and $ag_2 = 0$. Multiplying these equations by g_a from the left side we obtain $qg_1 = g_a$ and $qg_2 = 0$, respectively. Since $g_1 \in qRp$ and $g_2 \in qR(1-p)$ we conclude that $g_1 = g_a$ and $g_2 = 0$. Similarly, from $ga = g_a a$ we obtain $g_3 = 0$.

Acknowledgements

The authors would like to thank the reviewers for their constructive comments that improved the presentation of the paper.

References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58 (6) (2010) 681–697.
[2] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, 2nd edition, Springer, New York, 2003.
[3] A.H. Clifford, G.B. Preston, The Algebraic Theory of Semigroups, Amer. Math. Soc., 1961.
[4] M.P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012) 1909–1923.
[5] M.P. Drazin, Natural structures on semigroups with involution, Bull. Amer. Math. Soc. 84 (1978) 139–141.
[6] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506–514.
[7] K.R. Goodearl, Ring Theory: Nonsingular Rings and Modules, Marcel Dekker, 1976.
[8] A. Hernández, M. Lattanzi, N. Thome, F. Urquiza, The star partial order and the eigenprojection at 0 on EP matrices, Appl. Math. Comput. 218 (2012) 10669–10678.
[9] A. Hernández, M. Lattanzi, N. Thome, On a partial order defined by the weighted Moore–Penrose inverse, Appl. Math. Comput. 219 (2013) 7310–7318.
[10] R.E. Hartwig, How to partially order regular elements, Math. Japon. 25 (1980) 1–13.
[11] N. Jacobson, Structure of Rings, Amer. Math. Soc., 1968.
[12] I. Kaplansky, Rings of Operators, W.A. Benjamin, 1968.
[13] T.Y. Lam, A First Course in Noncommutative Rings, Springer-Verlag, 1991.
[14] L. Lebtahi, P. Patrício, N. Thome, The diamond partial order in rings, Linear Multilinear Algebra 62 (3) (2014) 386–395.
[15] L. Lebtahi, N. Thome, A note on k-generalized projections, Linear Algebra Appl. 420 (2007) 572–575.
[16] E.S. Ljapin, Semigroups, Fizmatgiz, Moscow, 1960 (in Russian).
[17] S.B. Malik, Some more properties of core partial order, Appl. Math. Comput. 221 (2013) 192–201.
[18] S.B. Malik, L. Rueda, N. Thome, Further properties on the core partial order and other matrix partial orders, Linear Multilinear Algebra 62 (12) (2014) 1629–1648.
[19] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434 (2011) 1836–1844.
[20] S.K. Mitra, Matrix partial orders through generalized inverses: unified theory, Linear Algebra Appl. 148 (1991) 237–263.
[21] S.K. Mitra, On group inverses and the sharp order, Linear Algebra Appl. 92 (1987) 17–37.
[22] S.K. Mitra, P. Bhimasankaram, S.B. Malik, Matrix Partial Orders, Shorted Operators and Applications, World Scientific, 2010.
[23] H. Mitsch, A natural partial order for semigroups, Proc. Amer. Math. Soc. 97 (3) (1986) 384–388.
[24] E.H. Moore, On the reciprocal of the general algebraic matrix, Bull. Amer. Math. Soc. 26 (1920) 394–395.
[25] K.S.S. Nambooripad, The natural partial order on a regular semigroup, Proc. Edinb. Math. Soc. 23 (1980) 249–260.
[26] R. Penrose, A generalized inverse for matrices, Math. Proc. Cambridge Philos. Soc. 51 (1955) 406–413.
[27] D.S. Rakić, Decomposition of a ring induced by minus partial order, Electron. J. Linear Algebra 23 (2012) 1040–1059.
[28] D.S. Rakić, N.Č. Dinčić, D.S. Djordjević, Group, Moore–Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115–133.
[29] D.S. Rakić, D.S. Djordjević, Star, sharp, core and dual core partial oder on rings with involution, preprint.
[30] V.V. Wagner, Generalised groups, Dokl. Akad. Nauk SSSR 84 (1952) 1119–1122 (in Russian); English translation.