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Group, Moore–Penrose, core and dual core inverse in rings with involution[☆]

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Let R be a ring with involution. The recently introduced notions of the core and dual core inverse are extended from matrix to an arbitrary $*$-ring case. It is shown that the group, Moore–Penrose, core and dual core inverse are closely related and they can be treated in the same manner using appropriate idempotents. The several characterizations of these inverses are given. Some new properties are obtained and some known results are generalized. A number of characterizations of EP elements in R are obtained. It is shown that core and dual core inverse belong to the class of inverses along an element and to the class of (b, c)-inverses.

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1. Introduction

Let M_n be the algebra of all $n \times n$ complex matrices. The Moore–Penrose inverse (MP inverse for short) of matrix A is the unique matrix $A^\dagger$ satisfying

$$(1)\ AA^\dagger A = A \qquad (2)\ A^\dagger AA^\dagger = A^\dagger \qquad (3)\ \left(AA^\dagger\right)^* = AA^\dagger \qquad (4)\ \left(A^\dagger A\right)^* = A^\dagger A.$$

The inverse was introduced by Moore [10] and latter rediscovered independently by Bjerhammar [4] and Penrose [12]. When $\operatorname{ind}(A) \leq 1$ i.e. $\operatorname{rank}(A) = \operatorname{rank}(A^2)$, the group inverse of A (see [2]) is unique matrix $A^\#$ defined by

$$(1)\ AA^\# A = A \qquad (2)\ A^\# AA^\# = A^\# \qquad (5)\ AA^\# = A^\# A.$$

Recently, Baksalary and Trenkler reintroduced in [1] the generalized inverse $A^-_{\rho^*,\chi}$, which is originally discussed by Rao and Mitra in [14] (as pointed out in [13]). They named it as core inverse and defined by the following definition.

Definition 1.1. (See [1].) A matrix $A^{\oplus} \in M_n$ is the core inverse of $A \in M_n$ if it satisfies

$$AA^{\oplus} = P_A \quad \text{and} \quad \mathcal{R}\left(A^{\oplus}\right) \subseteq \mathcal{R}(A). \tag{1}$$

Here P_A stands for the orthogonal projection on $\mathcal{R}(A)$. The core inverse exists if and only if $\operatorname{ind}(A) \leq 1$ in which case it is unique. Also, it is proved in [1] that the core inverse coincides with Bott–Duffin inverse $P_A[(A - I)P_A + I]^{-1}$. In the same paper authors defined one more inverse, $\tilde{A}$, which is closely related to core inverse. We call this inverse dual core inverse of A and denote it by $A_{\oplus}$. It is defined by [1]

$$A_{\oplus} A = P_{A^*} \quad \text{and} \quad \mathcal{R}(A_{\oplus}) \subseteq \mathcal{R}\left(A^*\right).$$

The core inverse is a special case of the core-EP inverse discussed by Manjunatha Prasad and Mohana in [13]. They rephrase the term core inverse as core-EP generalized inverse. Also, they used the term *core-EP generalized inverse instead of dual core inverse. They pointed out that it is the better terminology for the use.

From now on R denotes a ring with involution; we say $*$-ring for short. Our aim is to extend the definitions of these inverses to the case of $*$-ring. We will show that all four kinds of inverses can be treated in the similar way. The MP and group inverse of an element $a \in R$ are defined in the same way as in the matrix case; if they exist then they are unique. Some characterizations of the MP invertibility of an element of a ring are given in [8].

In Section 2 we will give an equivalent definition of the core inverse of matrix which serves us as a definition of core inverse of an element of a ring with involution: $x \in R$ is a core inverse of $a \in R$ if

$$axa = a, \qquad xR = aR \quad \text{and} \quad Rx = Ra^*.$$

The analogous alternative definitions for group, MP and dual core inverse of $a \in R$ are also given, see (3). In Theorems 2.11, 2.12, 2.14, 2.15, Corollary 2.13, we will characterize the existence of these inverses by the existence of idempotent $q \in R$ and self-adjoint idempotents $p, r \in R$ satisfying $aR = qR$, $Ra = Rq$, $pR = aR$ and $Rr = Ra$. Namely, $a \in R$ is group invertible if and only if idempotent q exists; a is MP invertible if and only if p and r exist; a is core invertible if and only if p and q exist; a is dual core invertible if and only if r and q exist. Using these idempotents we obtain appropriate matrix representations for a, $a^{\#}$, $a^{\dagger}$, $a^{\oplus}$ and $a_{\oplus}$. We will characterize the core and dual core inverse by the set of equations in Theorems 2.14 and 2.15. This result is new even in the case $R = M_n$. We will obtain a number of new properties and generalize most of the known properties of core inverse of complex matrix, that make sense in a $*$-ring. We note that in the matrix case, the study of generalized inverses uses mainly finite dimensional linear algebra methods. In our setting of arbitrary $*$-ring, we cannot use these methods.

In Section 3, the EP elements will be characterized.

In Section 4, we will show that considered inverses belong to the class of inverses along an element, introduced by Mary in [9] and to the class of outer generalized inverses introduced by Drazin in [6].

In a sequel we give some preliminaries. If $a \in R$ and there exists $x \in R$ such that $axa = a$ then we say that a is von Neumann regular (regular for short) and x is inner generalized inverse of a. If $y \in R$ and $yay = y$ then y is called outer generalized inverse of a. An element x is called reflexive generalized inverse of a if x is both inner and outer generalized inverse of a. If x satisfies equations $q_1, q_2, ..., q_n$ then x is called $\{q_1, q_2, ..., q_n\}$ inverse of a. The set of all such inverses is denoted by $a\{q_1, q_2, ..., q_n\}$. For example, $a\{1, 2, 5\} = \{a^{\#}\}$. We write $R^{(1)}$, $R^{\#}$, $R^{\dagger}$, $R^{\oplus}$, $R_{\oplus}$ for the set of all regular, group, MP, core, dual core invertible elements of a ring R respectively. An element $a \in R$ is EP if $a^{\#}$ and $a^{\dagger}$ exist and $a^{\#} = a^{\dagger}$. We will denote by aR and Ra the right and left ideal generated by a; $aR = \{ax : x \in R\}$ and $Ra = \{xa : x \in R\}$. Also $aRb = \{axb : x \in R\}$. The right annihilator of a is denoted by a° and is defined by $a^{\circ} = \{x \in R : ax = 0\}$. Similarly, the left annihilator of a is the set ${}^{\circ}a = \{x \in R : xa = 0\}$. Finally, if $p, q \in R$ are idempotents then arbitrary $x \in R$ can be written as

$$x = pxq + px(1 - q) + (1 - p)xq + (1 - p)x(1 - q)$$

or in the matrix form

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}_{p \times q},$$

where $x_{1,1} = pxq$, $x_{1,2} = px(1-q)$, $x_{2,1} = (1-p)xq$, $x_{2,2} = (1-p)x(1-q)$. If $x = (x_{i,j})_{p\times q}$ and $y = (y_{i,j})_{p\times q}$, then $x + y = (x_{i,j} + y_{i,j})_{p\times q}$. Moreover, if $r \in R$ is idempotent and $z = (z_{i,j})_{q\times r}$, then one can use usual matrix rules in order to multiply x and z.

2. Equivalent definitions and properties of $a^{\#}$, $a^{\dagger}$, $a^{\oplus}$ and $a_{\oplus}$

In this section we will give several characterizations for group, MP, core and dual core inverse and obtain some properties. We note that the results stated in Theorems 2.7, 2.8, 2.11, 2.12, 2.14, 2.15, Lemmas 2.9, 2.10, Corollary 2.13 are new even in the case $R = M_n$.

First we show that considered inverses are reflexive generalized inverses with prescribed range and null space. It is known that $A^{\dagger}$ is reflexive generalized inverse of A with range $\mathcal{R}(A^*)$ and null space $\mathcal{N}(A^*)$ [2]. We write

$$A^{\dagger} = A^{(1,2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}.$$

Also [2],

$$A^{\#} = A^{(1,2)}_{\mathcal{R}(A),\mathcal{N}(A)}.$$

To find a similar expression for core inverse, recall that $A^{\oplus} = A^{\#}P_A$ [1]. This means

$$A^{\oplus} = A^{\#}AA^{\dagger}, \tag{2}$$

so we obtain

$$\begin{aligned}\mathcal{R}(A) = \mathcal{R}(A^{\#}) = \mathcal{R}(A^{\#}AA^{\dagger}AA^{\#}) \subseteq \mathcal{R}(A^{\#}AA^{\dagger}) = \mathcal{R}(A^{\oplus}) \subseteq \mathcal{R}(A^{\#})\\ \mathcal{N}(A^*) = \mathcal{N}(A^{\dagger}) = \mathcal{N}(A^{\dagger}AA^{\#}AA^{\dagger}) \supseteq \mathcal{N}(A^{\#}AA^{\dagger}) = \mathcal{N}(A^{\oplus}) \supseteq \mathcal{N}(A^{\dagger}).\end{aligned}$$

We see at once that

$$A^{\oplus} = A^{(1,2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}.$$

Similarly,

$$A_{\oplus} = A^{(1,2)}_{\mathcal{R}(A^*),\mathcal{N}(A)}.$$

The definition of $A^{\oplus}$ given in Definition 1.1 does not make sense in rings. So, we need an equivalent definition.

Lemma 2.1. *A matrix $X \in M_n$ is the core inverse of $A \in M_n$ if and only if $AXA = A$, $XM_n = AM_n$ and $M_nX = M_nA^*$.*

Proof. Suppose that X is the core inverse of A. It is clear that $XM_n \subseteq AM_n$ since $\mathcal{R}(X) \subseteq \mathcal{R}(A)$. By (2), we see that $AXA = A$ and $XA = A^{\#}A$, so $A = XA^2$, hence $AM_n \subseteq XM_n$. Also, $A^* = A^*(AX)^* = A^*AX$ so $M_nA^* \subseteq M_nX$. Finally, $X = A^{\#}AA^{\dagger} = A^{\#}(A^{\dagger})^*A^*$ implies $M_nX \subseteq M_nA^*$. Conversely, suppose that

$A = AXA$, $XM_n = AM_n$ and $M_nX = M_nA^*$. It follows that $\mathcal{R}(X) \subseteq \mathcal{R}(A)$ and there exist $V \in M_n$ such that $X = VA^*$. It is now clear that $(AX)^2 = AX$, and $X = VA^* = VA^*X^*A^* = XX^*A^*$. Therefore $AX = AX(AX)^*$ which is self-adjoint, so $AX = P_A$. □

Similarly, we can show the analogous result for dual core inverse.

Lemma 2.2. *A matrix $X \in M_n$ is the dual core inverse of $A \in M_n$ if and only if $AXA = A$, $XM_n = A^*M_n$ and $M_nX = M_nA$.*

Now, we can give the extensions of the concepts of the core and dual core inverse from M_n to R.

Definition 2.3. Let $a \in R$. An element $a^{\#\!\!\!\!\bigcirc} \in R$ satisfying

$$aa^{\#\!\!\!\!\bigcirc}a = a, \qquad a^{\#\!\!\!\!\bigcirc}R = aR \quad \text{and} \quad Ra^{\#\!\!\!\!\bigcirc} = Ra^*$$

is called core inverse of a.

Definition 2.4. Let $a \in R$. An element $a_{\#\!\!\!\!\bigcirc} \in R$ satisfying

$$aa_{\#\!\!\!\!\bigcirc}a = a, \qquad a_{\#\!\!\!\!\bigcirc}R = a^*R \quad \text{and} \quad Ra_{\#\!\!\!\!\bigcirc} = Ra$$

is called dual core inverse of a.

In the similar way we can give the characterizations of the group and MP inverse. First we need some auxiliary lemmas.

Lemma 2.5. *Let $a, b \in R$. Then:*

(i) *If $aR \subseteq bR$ then ${}^\circ b \subseteq {}^\circ a$.*
(ii) *If $b \in R^{(1)}$ and ${}^\circ b \subseteq {}^\circ a$ then $aR \subseteq bR$.*

Proof. (i): Suppose that $aR \subseteq bR$ and $ub = 0$ for some $u \in R$. There exists $x \in R$ such that $a = bx$ so $ua = ubx = 0$.

(ii): Suppose now that ${}^\circ b \subseteq {}^\circ a$ and $b^{(1)} \in b\{1\}$. Since $(1 - bb^{(1)})b = 0$ we have $(1 - bb^{(1)})a = 0$ so $a = bb^{(1)}a$. Therefore, $aR \subseteq bR$. □

Lemma 2.6. *Let $a, b \in R$. Then:*

(i) *If $Ra \subseteq Rb$ then $b^\circ \subseteq a^\circ$.*
(ii) *If $b \in R^{(1)}$ and $b^\circ \subseteq a^\circ$ then $Ra \subseteq Rb$.*

Theorem 2.7. *Let $a, x \in R$. The following statements are equivalent:*

(i) *a is group invertible and $x = a^{\#}$.*
(ii) *$axa = a$, $xR = aR$ and $Rx = Ra$.*
(iii) *$axa = a$, ${}^{\circ}x = {}^{\circ}a$ and $x^{\circ} = a^{\circ}$.*
(iv) *$axa = a$, $xR \subseteq aR$ and $Rx \subseteq Ra$.*
(v) *$axa = a$, ${}^{\circ}a \subseteq {}^{\circ}x$ and $a^{\circ} \subseteq x^{\circ}$.*

Proof. (i) ⇒ (ii): We have $a = axa = aax = xaa$ and $x = xax = xxa = axx$ so $xR = aR$ and $Rx = Ra$.

(ii) ⇒ (iii) ⇒ (iv) ⇒ (v) follows by Lemmas 2.5 and 2.6.

(v) ⇒ (i): From $axa = a$ it follows that $ax - 1 \in {}^{\circ}a \subseteq {}^{\circ}x$ and $1 - xa \in a^{\circ} \subseteq x^{\circ}$ so $(ax - 1)x = 0$ and $x(1 - xa) = 0$. Now, $x = ax^2 = x^2a$, hence $ax = ax^2a = xa$ and $xax = x^2a = x$. By the uniqueness of the group inverse, $x = a^{\#}$. □

Theorem 2.8. *Let $a, x \in R$. The following statements are equivalent:*

(i) *a is MP invertible and $x = a^{\dagger}$.*
(ii) *$axa = a$, $xR = a^*R$ and $Rx = Ra^*$.*
(iii) *$axa = a$, ${}^{\circ}x = {}^{\circ}(a^*)$ and $x^{\circ} = (a^*)^{\circ}$.*
(iv) *$axa = a$, $xR \subseteq a^*R$ and $Rx \subseteq Ra^*$.*
(v) *$axa = a$, ${}^{\circ}(a^*) \subseteq {}^{\circ}x$ and $(a^*)^{\circ} \subseteq x^{\circ}$.*

Proof. (i) ⇒ (ii): By the properties of MP inverse we easily obtain $a^* = xaa^* = a^*ax$ and $x = a^*x^*x = xx^*a^*$ so $xR = a^*R$ and $Rx = Ra^*$.

(ii) ⇒ (iii) ⇒ (iv) ⇒ (v) follows by Lemmas 2.5 and 2.6.

(v) ⇒ (i): Since $a^*x^*a^* = a^*$, we see that $(1 - x^*a^*) \in (a^*)^{\circ} \subseteq x^{\circ}$ and $(1 - a^*x^*) \in {}^{\circ}(a^*) \subseteq {}^{\circ}x$. Therefore, $x = xx^*a^*$ and $x = a^*x^*x$. This yields $ax = ax(ax)^*$ and $xa = (xa)^*xa$; hence ax and xa are self-adjoint. Finally, $xax = x(ax)^* = xx^*a^* = x$. It follows that $x = a^{\dagger}$. □

Definitions 2.3, 2.4 and Theorems 2.7 (ii), 2.8 (ii) show that group, MP, core and dual core inverses can be defined analogously:

$$x \in R \text{ is group inverse of } a \text{ if and only if } axa = a,\ xR = aR,\ Rx = Ra,$$

$$x \in R \text{ is MP inverse of } a \text{ if and only if } axa = a,\ xR = a^*R,\ Rx = Ra^*,$$

$$x \in R \text{ is core inverse of } a \text{ if and only if } axa = a,\ xR = aR,\ Rx = Ra^*,$$

$$x \in R \text{ is dual core inverse of } a \text{ if and only if } axa = a,\ xR = a^*R,\ Rx = Ra. \tag{3}$$

As we can see, the four inverses are closely related and it can be said that they form a certain subclass of the class of all inner inverses. Moreover, we can conclude that core and dual core inverse are between group and MP inverse.

We will now show that the existence of considered inverses is closely related with existence of some idempotents. First, we give some auxiliary results.

Lemma 2.9. *If q_1 and q_2 are idempotents such that $Rq_1 \subseteq Rq_2$ and $q_2R \subseteq q_1R$ then $q_1 = q_2$.*

Proof. If $Rq_1 \subseteq Rq_2$ then $q_1 = uq_2$ for some $u \in R$ so $q_1q_2 = uq_2^2 = uq_2 = q_1$. Similarly, $q_2R \subseteq q_1R$ implies $q_1q_2 = q_2$. □

Lemma 2.10. *If p_1 and p_2 are self-adjoint idempotents such that $Rp_1 = Rp_2$ or $p_1R = p_2R$ then $p_1 = p_2$.*

Proof. If $Rp_1 = Rp_2$ then, like in previous lemma, $p_1 = p_1p_2$ and $p_2 = p_2p_1$. But $p_2 = p_2^* = p_1^*p_2^* = p_1p_2 = p_1$. Similarly, $p_1R = p_2R$ implies $p_1 = p_2$. □

Theorem 2.11. *Let $a \in R$. The following assertions are equivalent:*

(i) *a is group invertible.*
(ii) *There exists an idempotent $q \in R$ such that $qR = aR$ and $Rq = Ra$.*
(iii) *$a \in R^{(1)}$ and there exists idempotent $q \in R$ such that ${}^\circ a = {}^\circ q$ and $a^\circ = q^\circ$.*

If the previous assertions are valid then the assertions (ii) and (iii) deal with the same unique idempotent q. Moreover, $qa^{(1)}q$ is invariant under the choice of $a^{(1)} \in a\{1\}$ and

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times q}, \qquad a^\# = \begin{bmatrix} qa^{(1)}q & 0 \\ 0 & 0 \end{bmatrix}_{q\times q}. \tag{4}$$

Proof. (i) ⇒ (ii): Suppose that a is group invertible and set $q = aa^\# = a^\#a$. Then $a = qa = aq$ so $qR = aR$, $Rq = Ra$.

(ii) ⇒ (iii): From $qR = aR$ we have $q = ax$ and $a = qz$ for some $x, z \in R$. Therefore, $qa = q^2z = qz = a$ and $axa = qa = a$, so $a \in R^{(1)}$. The rest of the proof follows by Lemma 2.5 (i) and Lemma 2.6 (i).

(iii) ⇒ (i): Suppose that $a \in R^{(1)}$ and suppose that there exists an idempotent q such that $a^\circ = q^\circ$ and ${}^\circ a = {}^\circ q$. Let $a^{(1)} \in a\{1\}$ be arbitrary. Since $1 - a^{(1)}a \in a^\circ \subseteq q^\circ$ we obtain $q = qa^{(1)}a$. Also, $1 - q \in q^\circ \subseteq a^\circ$, so $a = aq$. Similarly, $q = aa^{(1)}q$ and $a = qa$. Set $x = qa^{(1)}q$. We have $x = a^\#$, because

$$ax = aqa^{(1)}q = aa^{(1)}q = q, \qquad xa = qa^{(1)}qa = qa^{(1)}a = q,$$
$$axa = qa = a, \qquad xax = qx = x.$$

Now the invariance of $qa^{(1)}q$ under the choice of $a^{(1)} \in a\{1\}$ follows. Note that we have also proved representations (4) since $a = qaq$ and $a^\# = qa^{(1)}q$. The uniqueness of q follows by Lemma 2.9. □

Theorem 2.12. *Let $a \in R$. The following assertions are equivalent:*

(i) *a is MP invertible.*
(ii) *There exist self-adjoint idempotents $p, r \in R$ such that $pR = aR$ and $Rr = Ra$.*
(iii) *$a \in R^{(1)}$ and there exist self-adjoint idempotents $p, r \in R$ such that ${}^\circ a = {}^\circ p$ and $a^\circ = r^\circ$.*

If the previous assertions are valid then the assertions (ii) and (iii) deal with the same pair of unique self-adjoint idempotents p and r. Moreover, $ra^{(1)}p$ is invariant under the choice of $a^{(1)} \in a\{1\}$ and

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times r}, \qquad a^\dagger = \begin{bmatrix} ra^{(1)}p & 0 \\ 0 & 0 \end{bmatrix}_{r \times p}. \tag{5}$$

Proof. (i) $\Rightarrow$ (ii): Suppose that a is MP invertible and set $p = aa^\dagger$ and $r = a^\dagger a$. It is clear that p and r are self-adjoint idempotents. Since $a = pa = ar$ we conclude that $pR = aR$ and $Rr = Ra$.

(ii) $\Rightarrow$ (iii): If we use p instead of q then the proof proceeds along the same lines as the proof of Theorem 2.11 (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i): As in the proof of Theorem 2.11 we can show that $a = pa = ar$, $p = aa^{(1)}p$ and $r = ra^{(1)}a$. Set $x = ra^{(1)}p$. We have $x = a^\dagger$ because

$$\begin{aligned} ax &= ara^{(1)}p = aa^{(1)}p = p = p^* \\ xa &= ra^{(1)}pa = ra^{(1)}a = r = r^* \\ axa &= pa = a \\ xax &= rx = x. \end{aligned}$$

Now the invariance of $ra^{(1)}p$ under the choice of $a^{(1)} \in a\{1\}$ follows because it is known that MP inverse is unique when it exists. Note that we have also proved representations (5) since $a = par$ and $a^\dagger = x = ra^{(1)}p$. The uniqueness of p and r follows by Lemma 2.10. □

Recall that a $*$-ring R is Rickart $*$-ring if for every $a \in R$ there exists self-adjoint idempotent p such that ${}^\circ a = Rp$ [3]. The analogous property for right annihilators is automatically fulfilled in this case. Note that $Rp = {}^\circ(1 - p)$.

Corollary 2.13. *Let $a \in R$ where R is Rickart $*$-ring. Then a is MP invertible if and only if a is regular.*

The analogous characterizations of core and dual core inverses using idempotents and annihilators are given in the next two theorems. Furthermore, we characterize these inverses by the set of equations.

Theorem 2.14. *Let $a \in R$. The following assertions are equivalent:*

(i) *a is core invertible.*
(ii) *There exists $x \in R$ such that $axa = a$, ${}^\circ x = {}^\circ a$ and $x^\circ = (a^*)^\circ$.*
(iii) *There exists $x \in R$ such that*

$$(1)\ axa = a \qquad (2)\ xax = x \qquad (3)\ (ax)^* = ax \qquad (6)\ xa^2 = a \qquad (7)\ ax^2 = x.$$

(iv) *There exist self-adjoint idempotent $p \in R$ and idempotent $q \in R$ such that $pR = aR$, $qR = aR$ and $Rq = Ra$.*
(v) *$a \in R^{(1)}$ and there exist self-adjoint idempotent $p \in R$ and idempotent $q \in R$ such that ${}^\circ a = {}^\circ p$, ${}^\circ a = {}^\circ q$ and $a^\circ = q^\circ$.*

If the previous assertions are valid then $x = a^{\oplus}$, $a^{\oplus}$ is unique and the assertions (iv) and (v) deal with the same pair of unique idempotents p and q. Moreover, $qa^{(1)}p$ is invariant under the choice of $a^{(1)} \in a\{1\}$ and

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}, \qquad a^{\oplus} = \begin{bmatrix} qa^{(1)}p & 0 \\ 0 & 0 \end{bmatrix}_{q\times p}. \tag{6}$$

Proof. (i) $\Rightarrow$ (ii): Suppose that a is core invertible and let $x = a^{\oplus}$. By definition, $axa = a$, $xR = aR$ and $Rx = Ra^*$. By Lemmas 2.5 and 2.6, it follows that ${}^\circ x = {}^\circ a$ and $x^\circ = (a^*)^\circ$.

(ii) $\Rightarrow$ (iii): Suppose that there exists $x \in R$ such that $axa = a$, ${}^\circ x = {}^\circ a$ and $x^\circ = (a^*)^\circ$. We can follow the proofs of Theorems 2.7 and 2.8 to obtain that

$$x = ax^2, \qquad ax = (ax)^* \quad \text{and} \quad xax = x.$$

From $xa - 1 \in {}^\circ x \subseteq {}^\circ a$ we have

$$a = xa^2.$$

(iii) $\Rightarrow$ (iv): Set $p = ax$ and $q = xa$. From $axa = a$ it follows that p and q are idempotents such that $pR = aR$ and $Rq = Ra$. Eq. (3) shows that p is self-adjoint. From $a = xa^2 = qa$ and $q = xa = ax^2a$ we conclude that $qR = aR$.

(iv) $\Rightarrow$ (v): The proof is similar to the proof of Theorem 2.11 (ii) $\Rightarrow$ (iii).

(v) $\Rightarrow$ (i): Suppose that $a \in R^{(1)}$ and suppose that there exist self-adjoint idempotent $p \in R$ and idempotent $q \in R$ such that ${}^\circ a = {}^\circ p$, ${}^\circ a = {}^\circ q$ and $a^\circ = q^\circ$. Fix $a^{(1)} \in a\{1\}$. In the proof of Theorem 2.11 we showed that $a = qa = aq$ and $q = qa^{(1)}a = aa^{(1)}q$. In the proof of Theorem 2.12 we showed that $a = pa$ and $p = aa^{(1)}p$. Let $a^- \in a\{1\}$ be arbitrary. Then $qa^-p = qa^{(1)}aa^-aa^{(1)}p = qa^{(1)}aa^{(1)}p = qa^{(1)}p$, so qa^-p is invariant under the choice of $a^- \in a\{1\}$. Set $x = qa^{(1)}p$. We have $axa = aqa^{(1)}pa = aa^{(1)}a = a$.

Also, $x = qa^{(1)}p = aa^{(1)}qa^{(1)}p$ and $xa^2 = qa^{(1)}pa^2 = qa^{(1)}aa = qa = a$, so $xR = aR$. Moreover,

$$x = qa^{(1)}p^* = qa^{(1)}\left(aa^{(1)}p\right)^* = qa^{(1)}p\left(a^{(1)}\right)^* a^* \quad \text{and}$$
$$a^*ax = a^*aqa^{(1)}p = a^*aa^{(1)}p = a^*p = (pa)^* = a^*,$$

so $Rx = Ra^*$. It follows that $x = a^{\oplus}$, i.e. a is core invertible.

The uniqueness of p and q follows by Lemmas 2.10 and 2.9. If x is core inverse of a then we showed that x has properties given in (ii) and (iii). Suppose that there exist two elements x and y satisfying equations in (iii). By the proof of (iii) $\Rightarrow$ (iv) and the uniqueness of p and q we conclude that $p = ax = ay$ and $q = xa = ya$. Therefore, $x = xax = yay = y$. We also proved that if there exists some x satisfying equations in (iii) then a is core invertible but its core inverse must satisfies equations in (iii) which uniquely determine x. It follows that x appearing in (ii) and x appearing in (iii) are both equal to $a^{\oplus}$ and that core inverse of a is unique. Representations (6) follow by $a = paq$ and $a^{\oplus} = x = qa^{(1)}p$. $\square$

The theorem concerning the dual core inverse can be proved similarly.

Theorem 2.15. *Let $a \in R$. The following assertions are equivalent:*

(i) *a is dual core invertible.*
(ii) *There exists $x \in R$ such that $axa = a$, ${}^\circ x = {}^\circ(a^*)$ and $x^\circ = a^\circ$.*
(iii) *There exists $x \in R$ such that*

$$(1)\ axa = a \qquad (2)\ xax = x \qquad (4)\ (xa)^* = xa \qquad (8)\ a^2x = a \qquad (9)\ x^2a = x.$$

(iv) *There exist self-adjoint idempotent $r \in R$ and idempotent $q \in R$ such that $Rr = Ra$, $qR = aR$ and $Rq = Ra$.*
(v) *$a \in R^{(1)}$ and there exist self-adjoint idempotent $r \in R$ and idempotent $q \in R$ such that $a^\circ = r^\circ$, ${}^\circ a = {}^\circ q$ and $a^\circ = q^\circ$.*

If the previous assertions are valid then $x = a_{\oplus}$, $a_{\oplus}$ is unique and the assertions (iv) and (v) deal with the same pair of unique idempotents r and q. Moreover, $ra^{(1)}q$ is invariant under the choice of $a^{(1)} \in a\{1\}$ and

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times r}, \qquad a_{\oplus} = \begin{bmatrix} ra^{(1)}q & 0 \\ 0 & 0 \end{bmatrix}_{r\times q}.$$

We will use q_a, p_a and r_a for the idempotents associated with $a \in R$, given in Theorems 2.11, 2.12, 2.14, 2.15. We will write q, p and r instead of q_a, p_a and r_a when no confusion can arise.

Let us look at the equations in Theorems 2.14 (iii) and 2.15 (iii) that characterize core and dual core inverse respectively. Note that these equations are combinations of equations that characterize group inverse and equations that characterize MP inverse. To see that it is enough to check that the following sets of equations are equivalent:

(i) $axa = a$, $xax = x$, $ax = xa$;
(ii) $axa = a$, $xax = x$, $xa^2 = a$, $a^2x = a$;
(iii) $axa = a$, $xax = x$, $x^2a = x$, $ax^2 = x$.

It is clear that (i) implies (ii) and (iii). If $axa = a$, $xax = x$, $xa^2 = a$, $a^2x = a$ then $ax = xa^2x = xa$, so (ii) implies (i). Similarly, (iii) implies (i).

Remark 2.16. In Theorems 2.11, 2.12, 2.14, 2.15 the connection between the generalized invertibility of a and the existence of idempotents $p = p_a$, $q = q_a$ and $r = r_a$ with given properties is provided. It follows that a is both group and MP invertible if and only if a is both core and dual core invertible. If a is core or dual core invertible then a is group invertible. In other words, $R^{\#} \cap R^{\dagger} = R^{\textcircled{\#}} \cap R_{\textcircled{\#}}$ and $R^{\textcircled{\#}} \cup R_{\textcircled{\#}} \subseteq R^{\#}$. It should be noted that inner invertibility of a (there exists x such that $axa = a$) implies MP invertibility of a in the case when R is C^*-algebra (see [7]) or Rickart $*$-ring (Corollary 2.13).

Remark 2.17. The statements (ii) and (iii) in Theorem 2.14 and the statements (ii) and (iii) of Theorem 2.15 can be used as equivalent definitions of core inverse and dual core inverse, respectively.

Suppose that $a \in R^{\#} \cap R^{\dagger}$. By Theorems 2.11 and 2.12, it follows that there exist unique idempotent $q = q_a$ and unique self-adjoint idempotents $p = p_a$ and $r = r_a$ with given properties. By the uniqueness, we conclude that these idempotents are the same as idempotents in Theorems 2.14 and 2.15. Therefore,

$$\begin{aligned} q &= aa^{\#} = a^{\#}a = a^{\textcircled{\#}}a = aa_{\textcircled{\#}} \\ p &= aa^{\dagger} = aa^{\textcircled{\#}} \\ r &= a^{\dagger}a = a_{\textcircled{\#}}a. \end{aligned} \tag{7}$$

Now, it is easy to show that

$$pq = q, \qquad qp = p, \qquad rq = r, \qquad qr = q. \tag{8}$$

Moreover,

$$q^*p = (pq)^* = q^*, \qquad pq^* = (qp)^* = p, \qquad q^*r = (rq)^* = r, \qquad rq^* = (qr)^* = q^*. \tag{9}$$

We also proved in Theorems 2.11, 2.12, 2.14, 2.15 that

$$\begin{aligned} a = qaq = paq = qar = par, \qquad a^{\#} = qa^{(1)}q, \\ a^{\dagger} = ra^{(1)}p, \qquad a^{\oplus} = qa^{(1)}p, \qquad a_{\oplus} = ra^{(1)}q, \end{aligned} \tag{10}$$

where $a^{(1)} \in a\{1\}$ is arbitrary. By (8)–(10), it follows that

$$\begin{aligned} a &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times r} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times r} \\ a^{\#} &= \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} = \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{q\times r} = \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{p\times r} \\ a^{\dagger} &= \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{r\times p} = \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times p} = \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{r\times q^*} = \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times q^*} \\ a^{\oplus} &= \begin{bmatrix} a^{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{q\times p} = \begin{bmatrix} a^{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} a^{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{q\times q^*} = \begin{bmatrix} a^{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{p\times q^*} \\ a_{\oplus} &= \begin{bmatrix} a_{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{r\times q} = \begin{bmatrix} a_{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times q} = \begin{bmatrix} a_{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{r\times r} = \begin{bmatrix} a_{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times r}. \end{aligned} \tag{11}$$

The elements in upper left corners in (11) belong to the sets of the forms p_1Rp_2, where p_1 and p_2 are idempotents. When $p_1 \neq p_2$ we cannot consider the invertibility of the corner element in p_1Rp_2, but it has some similar property. Let us look, for example, the representation $a^{\oplus} = \begin{bmatrix} a^{\oplus} & 0 \\ 0 & 0 \end{bmatrix}_{q\times q^*} \in qRq^*$. There exists unique element $x \in q^*Rq$ such that $xa^{\oplus} = q^*$ and $a^{\oplus}x = q$. Namely, $x = q^*aq$. The analogous property can be shown for all corner elements in (11). The proof is left to the reader.

It is clear that $(a^{\#})^{\#} = a$ and $(a^{\dagger})^{\dagger} = a$. The expressions for $(A^{\oplus})^{\dagger}$ and $(A^{\oplus})^{\oplus}$, where $A \in M_n$, are given in [1]. We give expressions for other "double" inverses.

Theorem 2.18. *Let $a \in R^{\#} \cap R^{\dagger}$. Then:*

(i) $p_{a^{\#}} = p_a$, $q_{a^{\#}} = q_a$, $r_{a^{\#}} = r_a$ *and*

$$\left(a^{\#}\right)^{\#} = a, \qquad \left(a^{\#}\right)^{\dagger} = r_a a p_a, \qquad \left(a^{\#}\right)^{\oplus} = a p_a, \qquad \left(a^{\#}\right)_{\oplus} = r_a a.$$

(ii) $p_{a^{\dagger}} = r_a$, $q_{a^{\dagger}} = q_a^*$, $r_{a^{\dagger}} = p_a$ *and*

$$\left(a^{\dagger}\right)^{\#} = q_a^* a q_a^*, \qquad \left(a^{\dagger}\right)^{\dagger} = a, \qquad \left(a^{\dagger}\right)^{\oplus} = q_a^* a, \qquad \left(a^{\dagger}\right)_{\oplus} = a q_a^*.$$

(iii) $p_{a^{\oplus}} = q_{a^{\oplus}} = r_{a^{\oplus}} = p_a$ *and*

$$\left(a^{\oplus}\right)^{\#} = \left(a^{\oplus}\right)^{\dagger} = \left(a^{\oplus}\right)^{\oplus} = \left(a^{\oplus}\right)_{\oplus} = a p_a.$$

(iv) $p_{a_{\textcircled{\#}}} = q_{a_{\textcircled{\#}}} = r_{a_{\textcircled{\#}}} = r_a$ *and*

$$(a_{\textcircled{\#}})^{\#} = (a_{\textcircled{\#}})^{\dagger} = (a_{\textcircled{\#}})^{\textcircled{\#}} = (a_{\textcircled{\#}})_{\textcircled{\#}} = r_a a.$$

Proof. We give the proof only for the statement (iii); the other statements may be proved in the same manner. Since $a^{\textcircled{\#}}R = aR = p_aR$ and $Ra^{\textcircled{\#}} = Ra^* = (aR)^* = (p_aR)^* = Rp_a$, we conclude that

$$p_{a^{\textcircled{\#}}} = q_{a^{\textcircled{\#}}} = r_{a^{\textcircled{\#}}} = p_a.$$

By (10), we obtain

$$\begin{aligned}\left(a^{\textcircled{\#}}\right)^{\#} = \left(a^{\textcircled{\#}}\right)^{\dagger} = \left(a^{\textcircled{\#}}\right)^{\textcircled{\#}} = \left(a^{\textcircled{\#}}\right)_{\textcircled{\#}} &= \begin{bmatrix} p_a(a^{\textcircled{\#}})^{(1)}p_a & 0 \\ 0 & 0 \end{bmatrix}_{p_a \times p_a} \\ &= \begin{bmatrix} p_a a p_a & 0 \\ 0 & 0 \end{bmatrix}_{p_a \times p_a} = \begin{bmatrix} a p_a & 0 \\ 0 & 0 \end{bmatrix}_{p_a \times p_a}. \qquad \square\end{aligned}$$

From Theorem 2.18 it follows that $a^{\textcircled{\#}}$ and $a_{\textcircled{\#}}$ are EP. The properties of core inverse given in the following theorem are a generalization of the case $R = M_n$ (see [1]) to the case of arbitrary $*$-ring.

Theorem 2.19. *Let $a \in R^{\textcircled{\#}}$ and $n \in \mathbb{N}$. Then:*

(i) $a^{\textcircled{\#}} = a^{\#}p_a$;
(ii) $(a^{\textcircled{\#}})^2 a = a^{\#}$;
(iii) $(a^{\textcircled{\#}})^n = (a^n)^{\textcircled{\#}}$;
(iv) $((a^{\textcircled{\#}})^{\textcircled{\#}})^{\textcircled{\#}} = a^{\textcircled{\#}}$;
(v) *If $a \in R^{\dagger}$ then*

$$a^{\#} = a^{\textcircled{\#}} a a_{\textcircled{\#}}, \qquad a^{\dagger} = a_{\textcircled{\#}} a a^{\textcircled{\#}}, \qquad a^{\textcircled{\#}} = a^{\#} a a^{\dagger}, \qquad a_{\textcircled{\#}} = a^{\dagger} a a^{\#}.$$

Proof. Since $a \in R^{\textcircled{\#}}$ we have the existence of $a^{\#}$, q_a and p_a.

(i): By Theorem 2.14 (iii) and (7), we have $a^{\textcircled{\#}} = a^{\textcircled{\#}} a a^{\textcircled{\#}} = a^{\#} a a^{\textcircled{\#}} = a^{\#}p_a$.

(ii): $(a^{\textcircled{\#}})^2 a = a^{\textcircled{\#}} q_a \overset{(i)}{=} a^{\#} p_a q_a \overset{(8)}{=} a^{\#} q_a = a^{\#}$.

(iii): Since $a = a^n(a^{\#})^{n-1} = (a^{\#})^{n-1}a^n$ we conclude that $Ra^n = Ra = Rq_a$ and $a^nR = aR = q_aR = p_aR$. Using $a(a^{\textcircled{\#}})^2 = a^{\textcircled{\#}}$ we see that $a^n(a^{\textcircled{\#}})^n = aa^{\textcircled{\#}}$ so $a^n(a^{\textcircled{\#}})^n a^n = aa^{\textcircled{\#}}a^n = a^n$, i.e. $(a^{\textcircled{\#}})^n \in a^n\{1\}$. By Theorem 2.14, we obtain

$$\left(a^n\right)^{\textcircled{\#}} = q_a\left(a^n\right)^{(1)} p_a = q_a\left(a^{\textcircled{\#}}\right)^n p_a = \left(a^{\textcircled{\#}}\right)^n.$$

(iv): By the proof of (iii) of Theorem 2.18, we obtain

$$\left(\left(a^{\textcircled{\#}}\right)^{\textcircled{\#}}\right)^{\textcircled{\#}} = a^{\textcircled{\#}} p_{a^{\textcircled{\#}}} = a^{\textcircled{\#}} p_a = a^{\textcircled{\#}}.$$

(v): If $a \in R^{\dagger}$ then, by (7), $a^{\#} = a^{\#}aa^{\#} = a^{\oplus}aa_{\oplus}$, $a^{\dagger} = a^{\dagger}aa^{\dagger} = a_{\oplus}aa^{\oplus}$, $a^{\oplus} = a^{\oplus}aa^{\oplus} = a^{\#}aa^{\dagger}$ and $a_{\oplus} = a_{\oplus}aa_{\oplus} = a^{\dagger}aa^{\#}$. □

The analogous result for dual core inverse of $a \in R_{\oplus}$ is valid. The expressions in (v) in Theorem 2.19 perhaps best illustrate the connection between the group, MP, core and dual core inverse. Once again, we see that the core and dual core inverse are between group and MP inverse and vice versa.

3. Characterizations of EP elements

In this section, we consider the equivalent conditions for EP-ness of $a \in R$. Recall that $a \in R$ is EP if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$. EP matrices and EP operators have been extensively studied. Recently, the EP elements are investigated in the context of rings with involution. For a recent account of the theory see, for example, [5,11] and the references given there.

Theorem 3.1. *Let $a \in R$. The following assertions are equivalent:*

(i) *a is EP, i.e. $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$.*
(ii) *$a \in R^{\dagger}$ and $p_a = r_a$.*
(iii) *$a \in R^{\oplus}$ and $p_a = q_a$.*
(iv) *$a \in R_{\oplus}$ and $r_a = q_a$.*
(v) *$a \in R^{\oplus}$ and $a^{\#} = a^{\oplus}$.*
(vi) *$a \in R_{\oplus}$ and $a^{\#} = a_{\oplus}$.*
(vii) *$a \in R^{\#} \cap R^{\dagger}$ and $a^{\dagger} = a^{\oplus}$.*
(viii) *$a \in R^{\#} \cap R^{\dagger}$ and $a^{\dagger} = a_{\oplus}$.*
(ix) *$a \in R^{\#} \cap R^{\dagger}$ and $a^{\oplus} = a_{\oplus}$.*

Proof. (i) ⇒ (ii)–(ix): If a is EP then

$$p_a = aa^{\dagger} = aa^{\#} = q_a = a^{\#}a = a^{\dagger}a = r_a.$$

By (10),

$$a^{\#} = a^{\dagger} = a^{\oplus} = a_{\oplus} = q_a a^{(1)} q_a.$$

(ii) or (iii) or (iv) ⇒ (i): Suppose that $p_a = r_a$. We have $p_a R = aR$ and $Rp_a = Rr_a = Ra$ so there exists q_a and $p_a = q_a = r_a$. Hence, $a \in R^{\#}$ and $aa^{\dagger} = p_a = q_a = aa^{\#}$. Hence, $aa^{\#} = a^{\#}a$ is self-adjoint, so $a^{\dagger} = a^{\#}$. Similarly, $p_a = q_a$ or $r_a = q_a$ imply $p_a = r_a = q_a$ and we can proceed as before.

(v) ⇒ (i): Suppose that $a \in R^{\oplus}$ and $a^{\oplus} = a^{\#}$. Multiplying both sides by a from the left, we obtain $p_a = aa^{\oplus} = aa^{\#} = q_a$. From the previous part of the proof, it follows that a is EP.

The remaining implications may be shown similarly. □

Thus, a is EP if and only if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger} = a^{\textcircled{\#}} = a_{\textcircled{\#}}$. Some characterizations in the next theorem involve only group and MP inverse. Note that some of these characterizations are known. We give them for completeness. For $x, y \in R$ we write $[x, y] = xy - yx$.

Theorem 3.2. *Let $a \in R^{\dagger} \cap R^{\#}$. Then the following assertions are equivalent:*

(i) *a is EP.*

(ii) *At least one (any) element of the set*

$$\{[a, a^{\dagger}], [a, a^{\textcircled{\#}}], [a, a_{\textcircled{\#}}], [a^{\#}, a^{\dagger}], [a^{\#}, a^{\textcircled{\#}}], [a^{\#}, a_{\textcircled{\#}}]\}$$

is equal to zero.

(iii) *At least one (any) element of the set*

$$\{ap_a, r_a a, r_a a p_a, q_a^* a, a q_a^*, q_a^* a q_a^*\}$$

is equal to a.

(iv) *$ap_a = r_a a$.*

(v) *$r_a a p_a = r_a a$.*

(vi) *$r_a a p_a = a p_a$.*

(vii) *$q_a^* a = a p_a$.*

(viii) *$a q_a^* = r_a a$.*

Proof. Write $p = p_a$, $q = q_a$ and $r = r_a$.

(i) $\Rightarrow$ (ii)–(ix): If a is EP then by Theorem 3.1, $a^{\#} = a^{\dagger} = a^{\textcircled{\#}} = a_{\textcircled{\#}}$ and $q = p = r = q^*$. Now, the proofs easily follow.

For the proofs of converse implications we use (7)–(10) and Theorem 3.1 or one of the preceding already establish conditions.

(ii) $\Rightarrow$ (i): We need to show that if there exists some element from the set which is equal to zero then a is EP. If $aa^{\textcircled{\#}} = a^{\textcircled{\#}}a$ then $p = q$. By Theorem 3.1, a is EP. Suppose that $[a^{\#}, a^{\textcircled{\#}}] = 0$, that is $a^{\#}a^{\textcircled{\#}} = a^{\textcircled{\#}}a^{\#}$. Multiplying both sides from the left by a we obtain $qa^{\textcircled{\#}} = pa^{\#}$. By (10), $a^{\textcircled{\#}} = a^{\#}$, so a is EP. Suppose that $a^{\#}a^{\dagger} = a^{\dagger}a^{\#}$. Multiplying both sides from the left by a we obtain $a^{\textcircled{\#}} = pa^{\#} = a^{\#}$, so a is EP. Other cases ($[a, a^{\dagger}] = 0$, $[a, a_{\textcircled{\#}}] = 0$, $[a^{\#}, a_{\textcircled{\#}}] = 0$) may be proved similarly.

(iii) $\Rightarrow$ (i): If $ap = a$ then, multiplying both sides from the left by $a^{\#}$, we obtain $qp = q$, hence, $p = q$. Therefore, a is EP. If $ra = a$ then $q = aa^{\#} = raa^{\#} = rq = r$, thus a is EP. If $rap = a$ then $a = qa = qrap = qap = ap$. If $q^*a = a$ then $a = q^*a = rq^*a = ra$. If $aq^* = a$ then $a = aq^* = aq^*p = ap$. Finally, if $q^*aq^* = a$ then $a = q^*aq^*p = ap$.

(iv) $\Rightarrow$ (i): Suppose that $ap = ra$. Since $qr = q$ we have $a = qa = qra = qap = ap$. By the previous part of the proof, we conclude that a is EP.

(v) $\Rightarrow$ (i): If $rap = ra$ then $a = qa = qra = qrap = qap = ap$.

(vi) $\Rightarrow$ (i): If $rap = ap$ then $a = aq = apq = rapq = raq = ra$.

(vii) $\Rightarrow$ (i): Suppose that $q^*a = ap$. Since $pq^* = p$ we obtain $a = pa = pq^*a = pap = ap$. By the part (iii) $\Rightarrow$ (i) it follows that a is EP.

(viii) $\Rightarrow$ (i): As $q^*r = r$ we conclude that $aq^* = ra$ implies $a = ar = aq^*r = rar = ra$, so a is EP. □

Combining Theorem 2.18 with Theorem 3.2, we can generalize results from [1] and obtain a large number of new characterizations of EP-ness of a.

4. Connection with some classes of generalized inverses

In this section we will show that group, MP, core and dual core inverse belong to some specific classes of generalized inverses. Recently, Mary introduced in [9] a new generalized inverse in semigroup S called the inverse along an element. We consider the case when S is a $*$-ring R. For $a, b \in R$, pre-order relation $\mathcal{H}$ is defined in [9] by

$$a \leq_{\mathcal{H}} b \quad \Longleftrightarrow \quad Ra \subseteq Rb \quad \text{and} \quad aR \subseteq bR.$$

Definition 4.1. (See [9].) Let $a, d \in R$. We say that $x \in R$ is an inverse of a along d if it satisfies

$$xad = d = dax \quad \text{and} \quad x \leq_{\mathcal{H}} d.$$

It is proved that if an inverse of a along d exists, it is unique and it is outer generalized inverse of a. Mary proved in Theorem 11 in [9] that $a \in R$ is group invertible if and only if it is invertible along a in which case the inverse of a along a coincides with the group inverse of a. Also, $a \in R$ is MP invertible if and only if it is invertible along a^* in which case the inverse of a along a^* coincides with the MP inverse of a.

Recently, Drazin independently defined in [6] a new outer generalized inverse in semigroup S that is actually similar to the inverse along an element. We consider the case when S is a $*$-ring R.

Definition 4.2. (See [6].) Let $a, b, c, x \in R$. Then we shall call x a (b, c)-inverse of a if both:

(1) $x \in (bRx) \cap (xRc)$ and
(2) $xab = b$, $cax = c$.

It is proved that there can be at most one (b, c)-inverse x of a and $xax = x$. Drazin proved in [6] that $a^{\#}$ is (a, a)-inverse of a and that $a^\dagger$ is (a^*, a^*) inverse of a.

Our aim is to connect the core and dual core inverse of a with generalized inverses given in Definitions 4.1 and 4.2.

Theorem 4.3. *Let $a \in R^\dagger$. Then:*

(i) *a is core invertible if and only if it is invertible along aa^*. In this case the inverse along aa^* coincides with core inverse of a.*

(ii) *a is dual core invertible if and only if it is invertible along a^*a. In this case the inverse along a^*a coincides with dual core inverse of a.*

Proof. (i): Suppose that $a \in R^\dagger \cap R^\#$ and let us prove that $x = a^{\oplus}$ is inverse of a along $d = aa^*$. Recall that $xa^2 = a$ and $(ax)^* = ax$, by Theorem 2.14. We see at once that

$$xad = xaaa^* = aa^* = d \quad \text{and}$$
$$dax = aa^*ax = aa^*(ax)^* = a(axa)^* = aa^* = d.$$

We proceed with following observation. Let $z = (a^\dagger)^* a^\dagger$. Then

$$aa^*z = aa^*\big(a^\dagger\big)^*a^\dagger = a\big(a^\dagger a\big)^*a^\dagger = aa^\dagger aa^\dagger = aa^\dagger \quad \text{and}$$
$$zaa^* = \big(a^\dagger\big)^*a^\dagger aa^* = \big(a^\dagger\big)^*\big(a^\dagger a\big)^*a^* = \big(aa^\dagger aa^\dagger\big)^* = \big(aa^\dagger\big)^* = aa^\dagger. \tag{12}$$

Since $ax^2 = x$, $xax = x$ and $ax = aa^\dagger$ we have

$$x = ax^2 = aa^\dagger x = aa^*zx = dzx \quad \text{and}$$
$$x = xax = xaa^\dagger = xzaa^* = xzd.$$

It follows that $x \in dR$ and $x \in Rd$ so $xR \subseteq dR$ and $Rx \subseteq Rd$; hence $x \leq_{\mathcal{H}} d$. By Definition 4.1, we conclude that $a^{\oplus}$ is inverse of a along aa^*.

Conversely, suppose that there exists inverse of $a \in R^\dagger$ along aa^*, denote it by x, and let us show that $a \in R^{\oplus}$ and $x = a^{\oplus}$. By Definition 4.1 we have that

$$xa^2a^* = aa^* = aa^*ax \tag{13}$$

and there exists $w \in R$ such that

$$x = aa^*w. \tag{14}$$

It is sufficient to show that x satisfies the equations given in Theorem 2.14 (iii). By (12), we have

$$ax = aa^\dagger ax = zaa^*ax = zaa^* = aa^\dagger,$$

so $(ax)^* = ax$. Now, $axa = aa^\dagger a = a$. Also,

$$xa^2 = xaaa^\dagger a = xa^2\big(a^\dagger a\big)^* = xa^2a^*\big(a^\dagger\big)^* \overset{(13)}{=} aa^*\big(a^\dagger\big)^* = a\big(a^\dagger a\big)^* = a \tag{15}$$
$$ax^2 \overset{(14)}{=} axaa^*w = aa^*w = x$$
$$xax = xaaa^*w \overset{(15)}{=} aa^*w \overset{(14)}{=} x.$$

The proof is complete.

(ii): This statement may be proved in the same manner as (i). □

Theorem 4.4. *Let $a \in R$. Then:*

(i) *a is core invertible if and only if (a, a^*)-inverse of a exists. In this case (a, a^*)-inverse of a coincides with core inverse of a.*
(ii) *a is dual core invertible if and only if (a^*, a)-inverse of a exists. In this case (a^*, a)-inverse of a coincides with dual core inverse of a.*

Proof. We will only show the statement (i) because the statement (ii) may be proved similarly. Suppose that $a \in R^{\oplus}$ and let $x = a^{\oplus}$, $b = a$ and $c = a^*$. By the properties of core inverse we obtain

$$\begin{aligned} xab &= xa^2 = a = b \\ cax &= a^*ax = a^*(ax)^* = (axa)^* = a^* = c \\ x &= xax = ax^2ax = bx^2ax \in bRx \\ x &= xax = x(ax)^* = xx^*a^* = xx^*c \in xRc. \end{aligned}$$

Therefore, by Definition 4.2, $x = a^{\oplus}$ is (a, a^*)-inverse of a.

Conversely, suppose that (a, a^*)-inverse of a exists, denote it by x, and let us show that x satisfies equations given in Theorem 2.14 (iii). By Definition 4.2,

$$xa^2 = a, \qquad a^*ax = a^* \tag{16}$$

and there exists $w \in R$ such that

$$x = awx. \tag{17}$$

We obtain

$$ax \overset{(16)}{=} \left(a^*ax\right)^*x = (ax)^*ax,$$

so $(ax)^* = ax$. Also,

$$axa = (ax)^*a = \left(a^*ax\right)^* \overset{(16)}{=} \left(a^*\right)^* = a \tag{18}$$

$$\begin{aligned} ax^2 &\overset{(17)}{=} axawx \overset{(18)}{=} awx = x \\ xax &\overset{(17)}{=} xaawx \overset{(16)}{=} awx = x. \end{aligned}$$

The proof of the theorem is complete. □

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References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58 (6) (2010) 681–697.

[2] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, 2nd edition, Springer, New York, 2003.

[3] S.K. Berberian, Baer $*$-Rings, Springer-Verlag, New York, 1972.

[4] A. Bjerhammar, Application of calculus of matrices to method of least squares; with special reference to geodetic calculations, Trans. Roy. Inst. Tech. Stockholm, vol. 49, 1951.

[5] W. Chen, On EP elements, normal elements and partial isometries in rings with involution, Electron. J. Linear Algebra 23 (2012) 553–561.

[6] M.P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012) 1909–1923.

[7] R. Harte, M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math. 103 (1) (1992) 71–77.

[8] J.J. Koliha, D. Djordjević, D. Cvetković, Moore–Penrose inverse in rings with involution, Linear Algebra Appl. 426 (2007) 371–381.

[9] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434 (2011) 1836–1844.

[10] E.H. Moore, On the reciprocal of the general algebraic matrix, Bull. Amer. Math. Soc. 26 (1920) 394–395.

[11] D. Mosić, D.S. Djordjević, J.J. Koliha, EP elements in rings, Linear Algebra Appl. 431 (2009) 527–535.

[12] R. Penrose, A generalized inverse for matrices, Math. Proc. Cambridge Philos. Soc. 51 (1955) 406–413.

[13] K. Manjunatha Prasad, K.S. Mohana, Core-EP inverse, Linear Multilinear Algebra 62 (6) (2014) 792–802.

[14] C.R. Rao, S.K. Mitra, Generalized Inverse of a Matrices and Its Application, Wiley, New York, 1971.