Reverse order law for the Moore-Penrose inverse of closed-range adjointable operators on Hilbert C^*-modules

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January 30, 2015

1 Introduction

Let $\mathcal{A}$ be a complex C^*-algebra with the norm $\|\cdot\|$, and let $\mathcal{M}$ be a complex linear space. $\mathcal{M}$ is a (right) $\mathcal{A}$-module, provided that there exists an exterior multiplication $\cdot : \mathcal{M} \times \mathcal{A} \to \mathcal{M}$, obeying the following properties, for all $x, y \in \mathcal{M}$, all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$:

$(x + y) \cdot a = x \cdot a + y \cdot a$; $\quad x \cdot (a + b) = x \cdot a + y \cdot b$;

$x \cdot (ab) = (x \cdot a) \cdot b$; $\quad \lambda(xa) = (\lambda x)a = x(\lambda a)$.

If $\mathcal{M}$ is an $\mathcal{A}$-module, then the $\mathcal{A}$-valued inner product is the function $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, satisfying the following conditions, for all $x, y \in \mathcal{M}$, all $a \in \mathcal{A}$:

$\langle x, x \rangle \geq 0$ in A; $\quad x = 0$ if and only if $\langle x, x \rangle = 0$;

$\langle x, y \rangle = \langle y, x \rangle^*$; $\quad \langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$;

$\langle x, y \cdot a \rangle = \langle x, y \rangle a$.

Thus, $\mathcal{M}$ becomes a pre-Hilbert $\mathcal{A}$-module.

The norm on a pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ is defined by $\|x\|_{\mathcal{M}} = \|\langle x, x \rangle\|^{1/2}$. This norm satisfies some nice properties, which are related to the Cauchy-Bunyakovsky-Schwarz inequality:

$\langle x, y \rangle \langle y, x \rangle \leq \|y\|^2_{\mathcal{M}} \langle x, x \rangle$, for all $x, y \in \mathcal{M}$;

$\|x \cdot a\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \|a\|$, for all $x \in \mathcal{M}$ and all $a \in A$;

$\|\langle x, y \rangle\| \leq \|x\|_{\mathcal{M}} \|y\|_{\mathcal{M}}$ for all $x, y \in \mathcal{M}$.

Finally, if $\mathcal{M}$ is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{M}}$, then $\mathcal{M}$ is a Hilbert $\mathcal{A}$-module. We also say that $\mathcal{M}$ is a Hilbert C^*-module (over $\mathcal{A}$). If H is a complex Hilbert space, then H is a Hilbert $\mathbb{C}$-module. Hence, Hilbert C^*-modules are between Hilbert spaces and Banach spaces.

Let $\mathcal{M}, \mathcal{N}$ be Hilbert $\mathcal{A}$-modules, and let $T : \mathcal{M} \to \mathcal{N}$ be a linear mapping. T is an operator, if T is bounded (as an operator between Banach spaces) and T is $\mathcal{A}$-linear, i.e. $T(x \cdot a) = T(x) \cdot a$ for all $x \in \mathcal{M}$ and all $a \in \mathcal{A}$.

If T is an operator from $\mathcal{M}$ to $\mathcal{N}$, and there exists an operator T^* from $\mathcal{N}$ to $\mathcal{M}$ satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{M}$ and all $y \in \mathcal{N}$, them T^* is the adjoint of T, and T is adjointable. Notice that there exist operators which are not adjointable. We use $\mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ to denote the set of all adjointable operators from $\mathcal{M}$ to $\mathcal{N}$. Recall that $\mathrm{End}^*(\mathcal{M}) = \mathrm{Hom}^*(\mathcal{M}, \mathcal{M})$ is a C^*-algebra.

If $T \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$, then $\mathcal{R}(T)$ denote the range of T, and $\mathcal{N}(T)$ denote the kernel of T. Notice that $\mathcal{N}(T)$ is always closed.

Among the situation that there exists non-adjointable operators between Hilbert $\mathcal{A}$-modules, there also is the following non-convenient situation. Let $\mathcal{K}$ be a closed submodule of $\mathcal{M}$. The orthogonal complement of $\mathcal{K}$ is defined as $K^\perp = \{x \in \mathcal{M} : \langle x, y\rangle = 0 \text{ for all } y \in \mathcal{K}\}$. Although $\mathcal{K}^\perp$ is a closed submodule of $\mathcal{M}$, we do not have in general $\mathcal{M} = \mathcal{K} \oplus \mathcal{K}^\perp$.

However, in the case which is the most important for this research, we have the following result.

Theorem 1.1. ([9], [10]) *Let $\mathcal{M}, \mathcal{N}$ be a Hilbert $\mathcal{A}$-modules, and let $T \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$. If $\mathcal{R}(T)$ is closed, then the following hold:*
$\mathcal{N}(T)$ is an orthogonally complemented submodule in $\mathcal{M}$ and $\mathcal{M} = \mathcal{R}(T^*) \oplus \mathcal{N}(T)$;
$\mathcal{R}(T)$ is an orthogonally complemented submodule in $\mathcal{N}$ and $\mathcal{N} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$.

Previous result allows us to investigate adjointable operators between Hilbert $\mathcal{A}$-modules in a similar way as on Hilbert spaces. For detailed treatment of Hilbert C^*-modules see [9] and [10].

Now, we have the usual definition of the Moore-Penrose inverse. Let $T \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$. The operator $T^\dagger \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ is the Moore-Penrose inverse of T, provided that the following holds:

$$TT^\dagger T = T,\ T^\dagger TT^\dagger = T^\dagger,\ (TT^\dagger)^* = TT^\dagger,\ (T^\dagger T)^* = T^\dagger T.$$

The Moore-Penrose inverse is unique in the case when it exists: this is standard for all standard structures that admits the existence of the Moore-Penrose inverse. Moreover, $T^\dagger$ exists if and only if $\mathcal{R}(T)$ is closed in $\mathcal{N}$ (see [14]).

In this paper we are interested in the reverse order law for the Moore-Penrose inverse. If a, b are invertible elements in an unital semigroup, then

$(ab)^{-1} = b^{-1}a^{-1}$ is the reverse order law for the ordinary inverse. However, the rule $(ab)^\dagger = b^\dagger a^\dagger$ does not hold in general for the Moore-Penrose inverse. If a, b are Moore-Penrose invertible, then it does not follows that ab is also Moore-Penrose invertible. Since we consider only Hilbert modules, we refer to the result which explain when the product of two closed-range adjointalbe operators also has a closed range. One equivalent condition is proved in [12].

In this paper we prove some equivalencies of the reverse order rule $(AB)^\dagger = B^\dagger A^\dagger$, where A, B, AB are adjointable operators between Hilbert modules, that have closed ranges. This result is known in the case of bounded Hilbert space operators, and in some parts in rings with involutions. We demostrate the usefulness of Theorem 1.1 for the geometric theory of generalized inverse.

Let $T \in \mathrm{Hom}^*(\mathcal{M}, N)$ has a closed range. Then $T^\dagger T$ is the orthogonal projection from $\mathcal{M}$ onto $\mathcal{R}(T^*)$, and $TT^\dagger$ is the orthogonal projection from $\mathcal{N}$ onto $\mathcal{R}(T)$. Using these projections, we see that T has the following matrix decomposition:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}.$$

The operator T_1 is invertible and adjointable, so

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T^*) \\ \mathcal{N}(T) \end{bmatrix}.$$

This decomposition allows us to reduce some properties of non-invertible T to invertible T_1.

Previous representation is derived from block representations of operators on Banach and Hilbert spaces, as well as Hilbert C^*-modules (see, for example, [4], [6], [12], [13]). This representation, and derived ones, are systematically used in the investigation of generalized inverses.

Let $T \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ have a closed range. T is EP if and only if $TT^\dagger = T^\dagger T$. Equivalently, T is EP if and only if $\mathcal{R}(T) = \mathcal{R}(T^*)$ (see [12] for EP operators on Hilbert modules). Obviously, T is EP if and only if T^* is EP. Notice that selfadjoint and normal operators with closed range are EP operators.

We use $[T, S] = TS - ST$ to denote the commutator of operators T and S. In this paper we use the fact that if T and S are selfadjoint, then TS is selfadjoint if and only if $[T, S] = 0$.

2 Results

We prove the following main result of this paper.

Theorem 2.1. *Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{M}, \mathcal{N}, \mathcal{K}$ be Hilbert $\mathcal{A}$-modules. Suppose that $A \in \mathrm{Hom}^*(\mathcal{N}, \mathcal{K})$, $B \in \mathrm{Hom}^*(\mathcal{M}, \mathcal{N})$ be adjointable operators, such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(a) $(AB)^\dagger = B^\dagger A^\dagger$;

(b) $[A^\dagger A, BB^*] = 0$ *and* $[A^* A, BB^\dagger] = 0$;

(c) $\mathcal{R}(A^* AB) \subset \mathcal{R}(B)$ *and* $\mathcal{R}(BB^* A^*) \subset \mathcal{R}(A^*)$;

(d) $A^* ABB^*$ *is EP.*

Proof. Using previous ideas, we know that $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$, where A_1 is invertible, and consequently $A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Also, $B = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}$. Notice that $D = B_1^* B_1 + B_2^* B_2$ is positive and invertible in $\mathrm{End}^*(\mathcal{R}(B^*))$. Hence, $B^\dagger = (B^* B)^\dagger B^* = \begin{bmatrix} D^{-1} B_1^* & D^{-1} B_2^* \\ 0 & 0 \end{bmatrix}$.

We find equivalent forms of (a). Notice that $AB = \begin{bmatrix} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^\dagger A^\dagger = \begin{bmatrix} D^{-1} B_1^* A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $(A_1 B_1)^\dagger = D^{-1} B_1^* A_1^{-1}$. We have the following: $A_1 B_1 (D^{-1} B_1^* A_1^{-1}) A_1 B_1 = A_1 B_1$ if and only if

$$B_1 D^{-1} B_1^* B_1 = B_1. \tag{1}$$

Also, $D^{-1} B_1^* A_1^{-1} (A_1 B_1) D^{-1} B_1^* A_1^{-1} = D^{-1} B_1^* A_1^{-1}$ if and only if (1) holds. The operator $A_1 B_1 D^{-1} B_1^* A_1^{-1}$ is Hermitian if and only if

$$[A_1^* A_1, B_1 D^{-1} B_1^*] = 0. \tag{2}$$

Finally, $D^{-1} B_1^* A_1^{-1} A_1 B_1$ is Hermitian if and only if

$$[D, B_1^* B_1] = 0. \tag{3}$$

Now we find equivalent forms of (b). We have $A^\dagger A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $A^* A = \begin{bmatrix} A_1^* A_1 & 0 \\ 0 & 0 \end{bmatrix}$, $BB^* = \begin{bmatrix} B_1 B_1^* & B_1 B_2^* \\ B_2 B_1^* & B_2 B_2^* \end{bmatrix}$ and $BB^\dagger = \begin{bmatrix} B_1 D^{-1} B_1^* & B_1 D^{-1} B_2^* \\ B_2 D^{-1} B_1^* & B_2 D^{-1} B_2^* \end{bmatrix}$. Hence, $[A^\dagger A, BB^*] = 0$ if and only if

$$B_1 B_2^* = 0. \tag{4}$$

Also, $[A^*A, BB^\dagger] = 0$ if and olny if

$$[A_1^*A_1, B_1D^{-1}B_1^*] = 0 \tag{5}$$

and

$$B_2D^{-1}B_1^* = 0. \tag{6}$$

We find equivalent conditions for (c). Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ holds if and only if $BB^\dagger A^*AB = A^*AB$. Also, $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$. From previous decompositions of operators we see that $A^\dagger ABB^*A^* = BB^*A^*$ if and only if

$$B_2B_1^* = 0, \tag{7}$$

which the same as (4). We have $BB^\dagger A^*AB = A^*AB$ if and only if

$$B_1D^{-1}B_1^*A_1^*A_1B_1 = A_1^*A_1B_1 \tag{8}$$

and

$$B_2D^{-1}B_1^*A_1^*A_1B_1 = 0. \tag{9}$$

Thus, (c) is equivalent to (7), (8) i (9).

Finally, (d) is equivalent to

$$\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A), \tag{10}$$

assuming that this submodule is closed.

(b) $\Longrightarrow$ (a): We prove the following:

$$\Big((4) \wedge (5) \wedge (6)\Big) \implies \Big((1) \wedge (2) \wedge (3)\Big).$$

Suppose that (4), (5) and (6) hold. Obviously, (2) holds. Also,

$$B_1^* = DD^{-1}B_1^* = (B_1^*B_1 + B_2^*B_2)D^{-1}B_1^* = B_1^*B_1D^{-1}B_1^*.$$

Thus, (1) holds. We see that $B_1^*B_1D^{-1}B_1^*B_1 = B_1^*B_1$ is satisfied, so $\mathcal{R}(B_1^*B_1)$ is closed. We have the following matrix form of $B_1^*B_1$: $B_1^*B_1 = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B_1^*B_1) \\ \mathcal{N}(B_1^*B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B_1^*B_1) \\ \mathcal{N}(B_1^*B_1) \end{bmatrix}$. Since $\mathcal{R}(B_2^*B_2) \subset \mathcal{N}(B_1^*B_1)$ we have $B_2^*B_2 = \begin{bmatrix} 0 & 0 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B_1^*B_1) \\ \mathcal{N}(B_1^*B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B_1^*B_1) \\ \mathcal{N}(B_1^*B_1) \end{bmatrix}$. However, $B_2^*B_2$ is Hermitian, so

$C_3 = 0$. Thus, $D = \begin{bmatrix} C_1 & 0 \\ 0 & C_4 \end{bmatrix}$ and it obviously commutes with $B_1^* B_1$. Thus, (3) holds.

(a) $\Longrightarrow$ (b): We prove

$$\Big((1) \wedge (2) \wedge (3)\Big) \implies \Big((4) \wedge (5) \wedge (6)\Big).$$

Suppose that (1), (2) and (3) hold. Since D commutes with $B_1^* B_1$, we get that D^{-1} commute with $B_1^* B_1$. Hence, we get

$$B_1 = B_1 D^{-1} B_1^* B_1 = B_1 (D - B_2^* B_2) D^{-1} = B_1 - B_2^* B_2 D^{-1}.$$

It follows that $B_1 B_2^* B_2 = 0$. Since $\overline{\mathcal{R}(B_2^*)} = \overline{\mathcal{R}(B_2^* B_2)}$ and $\mathcal{R}(B_2^* B_2) \subset \mathcal{N}(B_1)$, we get $\mathcal{R}(B_2^*) \subset \mathcal{N}(B_1)$, so $B_1 B_2^* = 0$. Thus, (4) is proved. Also, (5) is obvious. From $B_1 B_2^* = 0$ we get $B_1^* B_1 B_2^* = 0$ and $B_1^* B_1 D^{-1} B_2^* = 0$. Hence, $B_2 D^{-1} B_1^* B_1 = 0$. In the same manner as before, we conclude that $B_2 D^{-1} B_1^* = 0$, so (6) holds.

(a)$\wedge$(b) $\Longrightarrow$ (c): It is enough to observe the following elementary implications:

$$(5) \wedge (1) \implies (8), \ (4) \iff (7), \ (6) \implies (9).$$

(c) $\Longrightarrow$ (b): We prove the implication:

$$\Big((7) \wedge (8) \wedge (9)\Big) \implies \Big((4) \wedge (5) \wedge (6)\Big).$$

Obviously, (7) $\iff$ (4). From (9) we get $\mathcal{R}(B_1^* A_1^*) = \mathcal{R}(B_1^* A_1^* A_1 B_1) \subset \mathcal{N}(B_2 D^{-1})$, implying that $B_2 D^{-1} B_1^* A_1^* = 0$, so (6) follows. We multiply (8) by $(A_1 B_1)^\dagger$ and use the equality $G^* G G^\dagger = G^*$ whenever G is Moore-Penrose invertible. Hence, we get $B_1 D^{-1} B_1^* A_1^* = A_1^* A_1 B_1 (A_1 B_1)^\dagger$, implying that $B_1 D^{-1} B_1^* A_1^* A_1 = A_1^* (A_1 B_1 (A_1 B_1)^\dagger) A_1$. We know that $A_1 B_1 (A_1 B_1)^\dagger$ is selfadjoint, so $A_1^* (A_1 B_1 (A_1 B_1)^\dagger) A_1$ is selfadjoint. Now, $B_1 D^{-1} B_1^* A_1^* A_1$ is selfadjoint. Since both $B_1 D^{-1} B_1^*$ and $A_1^* A_1$ are selfadjoint, we get $[B_1 D^{-1} B_1^*, A_1^* A_1] = 0$, so (5) follows.

(d) $\Longrightarrow$ (c): Let $A^* A B B^*$ be EP. Then we have

$$\mathcal{R}(A^* A B) = \mathcal{R}(A^* A B B^*) = \mathcal{R}(B B^* A^* A) \subset \mathcal{R}(B)$$

and

$$\mathcal{R}(B B^* A^*) = \mathcal{R}(B B^* A^* A) = \mathcal{R}(A^* A B B^*) \subset \mathcal{R}(A^*).$$

Hence, (c) holds.

(c) $\Longrightarrow$ (d): Suppose that all conditions (7),(8),(9) hold. We find the equivalent form of (10). Under these assumptions, we have that (10) is equivalent to

$$\mathcal{R}\left(\begin{bmatrix} A_1^*A_1B_1B_1^* & A_1^*A_1B_1^*B_2 \\ 0 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} B_1B_1^*A_1^* & 0 \\ B_2B_1^*A_1^*A_1 & 0 \end{bmatrix}\right).$$

Since (7) holds, we see that (1) is equivalent to

$$\mathcal{R}(A_1^*A_1B_1B_1^*) = \mathcal{R}(B_1B_1^*A_1^*A_1).$$

The operator A_1 is invertible, so the last equality is equivalent to

$$\mathcal{R}(A_1^*A_1B_1B_1^*) = \mathcal{R}(B_1B_1^*).$$

Using the closed-range assumptions, the last one is equivalent to

$$\mathcal{R}(A_1^*A_1B_1) = \mathcal{R}(B_1),$$

which is the same as

$$B_1B_1^\dagger A_1^*A_1B_1 = A_1^*A_1B_1. \tag{11}$$

Now we start from (8) and obtain the following:

$$B_1B_1^\dagger A_1^*A_1B_1 = B_1B_1^\dagger B_1D^{-1}B_1^*A_1^*A_1B_1 = B_1D^{-1}B_1^*A_1^*A_1B_1 = A_1^*A_1B_1.$$

Thus, (8) implies (11). Hence, we have just proved that (c) implies (d). $\square$

This theorem represents an extension of well-know results for matrices and operators on Hilbert spaces (see [1], [2], [3], [7], [8]) to the more general settings: we considered the Moore-Penrose inverse of a product of closed-range adjointable operators on Hilbert C^*-modules. See also [5] and [11] for some algebraic aspects.

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