Factorization of weighted–EP elements in $C^*$-algebras

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Abstract

We present characterizations of weighted–EP elements in $C^*$-algebras using different kinds of factorizations.

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1 Introduction

The weighted–EP matrices are characterized by commutativity with their weighted Moore–Penrose inverse. They were introduced and investigated by Tian and Wang in [26]. The notion of weighted–EP matrices was extended to elements of $C^*$-algebras in [23].

Generalized inverses have lots of applications in numerical linear algebra, as well as in approximation methods in general Hilbert spaces. Hence, we characterize weighted–EP elements of $C^*$-algebras through various factorizations.

Let $\mathcal{A}$ be a unital $C^*$–algebra with the unit 1. An element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. The set of all regular elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^r$. An element $a \in \mathcal{A}$ satisfying $a^* = a$ is called symmetric (or Hermitian). An element $x \in \mathcal{A}$ is positive if $x = y^*y$ for some $y \in \mathcal{A}$. Notice that positive elements are self-adjoint.

An element $a^\dagger \in \mathcal{A}$ is the Moore–Penrose inverse (or MP-inverse) of $a \in \mathcal{A}$, if the following hold [25]:

$$\begin{align*}
a a^\dagger a &= a, & a^\dagger a a^\dagger &= a^\dagger, & (aa^\dagger)^* &= aa^\dagger, & (a^\dagger a)^* &= a^\dagger a.
\end{align*}$$

There is at most one $a^\dagger$ such that above conditions hold (see [13, 15]).

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Theorem 1.1. [13] In a unital $C^*$-algebra $A$, $a \in A$ is MP-invertible if and only if $a$ is regular.

Let $e, f$ be invertible positive elements in $A$. The element $a \in A$ has the weighted MP-inverse with weights $e, f$, if there exists $b \in A$ such that

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$  

The unique weighted MP-inverse with weights $e, f$, will be denoted by $a_{e,f}^\dagger$ if it exists [7].

Theorem 1.2. [7] Let $A$ be a unital $C^*$-algebra, and let $e, f$ be positive invertible elements of $A$. If $a \in A$ is regular, then the unique weighted MP-inverse $a_{e,f}^\dagger$ exists.

Define the mapping $(*, e, f) : x \mapsto x^{*e,f} = e^{-1}x^*f$, for all $x \in A$. Notice that $(*, e, f) : A \to A$ is not an involution, because in general $(xy)^{*e,f} \neq y^{*e,f}x^{*e,f}$. The following result is frequently used in the rest of the paper.

Theorem 1.3. [23] Let $A$ be a unital $C^*$-algebra, and let $e, f$ be positive invertible elements of $A$. For any $a \in A^-$, the following is satisfied:

(a) $(a_{e,f}^\dagger)_{f,e}^\dagger = a$;

(b) $(a_{e,f}^\dagger)_{f,e} = (a_{e,f}^\dagger)^{*e,f}$;

(c) $a_{e,f}^\dagger = a_{e,f}^\dagger a_{e,f}^\dagger = a_{e,f}^\dagger a_{e,f}^\dagger a_{e,f}^\dagger$;

(d) $a_{e,f}^\dagger = a_{e,f}^\dagger a_{e,f}^\dagger a_{e,f}^\dagger$;

(e) $(a_{e,f}^\dagger)^{*e,f} a_{e,f}^\dagger = a_{e,f}^\dagger$;

(f) $(a_{e,f}^\dagger a_{e,f}^\dagger)_{f,e} = a_{e,f}^\dagger (a_{e,f}^\dagger)^{*e,f}$;

(g) $(a_{e,f}^\dagger)^{*e,f} a_{e,f}^\dagger = a_{e,f}^\dagger$;

(h) $a_{e,f}^\dagger = (a_{e,f}^\dagger a_{e,f}^\dagger)^{*e,f} (a_{e,f}^\dagger)^{*e,f}$;

(i) $(a_{e,f}^\dagger a_{e,f}^\dagger)_{f,e} = a(a_{e,f}^\dagger a_{e,f}^\dagger)_{f,f} = (a_{e,f}^\dagger a_{e,f}^\dagger)_{f,e} a_{e,f}^\dagger$.

For $a \in A$ consider two annihilators

$$a^\circ = \{x \in A : ax = 0\}, \quad a^\circ = \{x \in A : xa = 0\}.$$  

Observe that,

$$(a^*)^\circ = a^\circ \Leftrightarrow a^\circ (a^*) = a^\circ, \quad aA = a^*A \Leftrightarrow AA^* = A.$$
Lemma 1.1. [11] The following hold for $a \in A$.

(i) $a \in A^{-1} \iff aA = A$ and $a^\circ = \{0\}$.

(ii) $a \in A^{-1} \iff A = (a^*A) \oplus a^\circ$.

(iii) $a^*A = A \iff a \in A^{-1}$ and $a^\circ = \{0\}$.

The following lemmas related to weighted MP-inverse are very useful.

Lemma 1.2. [23] Let $a \in A^{-1}$ and let $e, f$ be invertible positive elements in $A$. Then

(i) $a_{e,f}^\dagger A = a_{e,f}^\dagger aA = f^{-1}a^*A = a^{sf,e}A$;

(ii) $(a_{e,f}^\dagger)^*A = (aa_{e,f}^\dagger)^*A = eaA = (a^{sf,e})^*A$;

(iii) $a^\circ = (ea)^\circ$;

(iv) $(a^*)^\circ = (f^{-1}a^*)^\circ$;

(v) $(a_{e,f}^\dagger)^\circ = [(ea)^*]^\circ = (a^{sf,e})^\circ$;

(vi) $[(a_{e,f}^\dagger)^*]^\circ = (af^{-1})^\circ$.

Lemma 1.3. [23] Let $a \in A^-$, and let $e, f$ be invertible positive elements in $A$. Then

(1) $a_{e,f}^\dagger = (a^{sf,e}a + 1 - a_{e,f}^\dagger a)^{-1}a^{sf,e} = a^{sf,e}(aa^{sf,e} + 1 - aa_{e,f}^\dagger)^{-1}$,

(2) $a^{sf,e}A^{-1} = a_{e,f}^\dagger A^{-1}$ and $A^{-1}a^{sf,e} = A^{-1}a_{e,f}^\dagger$;

(3) $(a^{sf,e})^\circ = (a_{e,f}^\dagger)^\circ$ and $^\circ(a^{sf,e}) = ^\circ(a_{e,f}^\dagger)$.

We recall the definition of EP elements.

Definition 1.1. An element $a \in A^-$ is EP if $aa^\dagger = a^\dagger a$.

Lemma 1.4. [17] An element $a \in A$ is EP, if $a \in A^-$ and $aA = a^*A$ (or, equivalently, if $a \in A^-$ and $a^\circ = (a^*)^\circ$).
Many authors have investigated various characterizations of EP elements in a ring and $C^*$-algebra (see, for example, [15, 17, 18, 20, 21, 24]), many more still for Banach or Hilbert space operators and matrices (see [1, 2, 4, 5, 6, 8, 9, 10, 14, 16, 19, 22]). In [12], Drivaliaris, Karanasios and Pappas and in [11] Djordjević, J.J. Koliha and I. Straškraba have characterized EP Hilbert space operators and EP $C^*$-algebra elements respectively through several different factorizations. Boasso [3] have recently characterized EP Banach space operators and EP Banach algebra elements using factorizations, extending results of [11, 12].

Now, we state the definition of weighted–EP elements and some characterizations of weighted–EP elements.

**Definition 1.2.** [23] An element $a \in \mathcal{A}$ is said to be weighted–EP with respect to two invertible positive elements $e, f \in \mathcal{A}$ (or weighted–EP w.r.t. $(e, f)$) if both $ea$ and $af^{-1}$ are EP, that is $a \in \mathcal{A}^-$, $ea\mathcal{A} = (ea)^*\mathcal{A}$ and $af^{-1}\mathcal{A} = (af^{-1})^*\mathcal{A}$.

**Theorem 1.4.** [23] Let $\mathcal{A}$ be a unital $C^*$–algebra, and let $e, f$ be invertible positive elements in $\mathcal{A}$. For $a \in \mathcal{A}^-$ the following statements are equivalent:

(i) $a$ is weighted–EP w.r.t. $(e, f)$;

(ii) $aa^\dagger_{e,f} = a^\dagger_{e,f}a$;

(iii) $a^\dagger_{e,f} = a(a^\dagger_{e,f})^2 = (a^\dagger_{e,f})^2a$;

(iv) $a \in a^\dagger_{e,f} \mathcal{A}^{-1} \cap \mathcal{A}^{-1} a^\dagger_{e,f}$

(v) $a \in a^\dagger_{e,f} \mathcal{A} \cap \mathcal{A} a^\dagger_{e,f}$;

(vi) $a\mathcal{A} = a^{*f,e}\mathcal{A}$ and $\mathcal{A}a = \mathcal{A}a^{*f,e}$;

(vii) $a^\circ = (a^{*f,e})^\circ$ and $\circ a = \circ(a^{*f,e})$.

We turn our attention for characterizing weighted–EP elements in terms of factorizations, motivated by papers [3, 11, 12], which are related to similar characterizations of EP elements.
2 Factorization \( a = ba^{*}f, e \)

In this section we characterize weighted–EP elements of \( C^{*} \)-algebras through factorizations of the form \( a = ba^{*}f, e \).

**Theorem 2.1.** Let \( \mathcal{A} \) be a unital \( C^{*} \)-algebra, and let \( e, f \) be invertible positive elements in \( \mathcal{A} \). For \( a \in \mathcal{A}^{-} \) the following statements are equivalent:

(i) \( a \) is weighted–EP w.r.t. \( (e, f) \);

(ii) \( a = ba^{*}f, e = a^{*}f, ec \) for some \( b, c \in \mathcal{A} \);

(iii) \( a^{*}f, e = b_{1}a^{*}f, e = ac_{1} \) and \( aa^{*}f, e = a^{*}f, eb_{2} = c_{2}a \) for some \( b_{1}, b_{2}, c_{1}, c_{2} \in \mathcal{A} \);

(iv) \( a^{*}f, e = b_{3}a^{*}f, e = a^{*}f, e = a^{*}f, e = a_{e, f}b_{4} \) and \( a_{e, f} = c_{3}a = ac_{4} \) for some \( b_{3}, b_{4}, c_{3}, c_{4} \in \mathcal{A} \).

**Proof.** (i) \( \Leftrightarrow \) (ii): By Theorem 1.4, \( a \) is weighted–EP w.r.t. \( (e, f) \) if and only if \( a \in a_{e, f}^{*} A \cap Aa^{*}f, e \), which is equivalent to \( a \in a^{*}f, e A \cap Aa^{*}f, e \), by Lemma 1.2. Thus, the equivalence (i) \( \Leftrightarrow \) (ii) holds.

(i) \( \Leftrightarrow \) (iii): Notice that, by Theorem 1.3, \( \mathcal{A} = aa^{*}f, e A \), \( \mathcal{A} = Aa^{*}f, e A \), \( a^{*}f, e A = a^{*}f, e a_{e, f} A \) and \( \mathcal{A} = a^{*}f, e \mathcal{A} \). Now (iii) is equivalent to \( a \mathcal{A} = a^{*}f, e \mathcal{A} \) and \( \mathcal{A} a = \mathcal{A} a^{*}f, e \). By Theorem 1.4, these equalities hold if and only if \( a \) is weighted–EP w.r.t. \( (e, f) \).

(i) \( \Leftrightarrow \) (iv): Similarly as the previous part. \( \Box \)

3 Factorization \( a^{*}f, e = sa \)

In this section, the weighted–EP elements of the form \( a^{*}f, e = sa \) or \( a_{e, f}^{*} = sa \) will be characterized.

We start with characterizations of weighted–EP elements via factorizations of the form \( a^{*}f, e = sa \).

**Theorem 3.1.** Let \( \mathcal{A} \) be a unital \( C^{*} \)-algebra, and let \( e, f \) be invertible positive elements in \( \mathcal{A} \). For \( a \in \mathcal{A}^{-} \) the following statements are equivalent:

(i) \( a \) is weighted–EP w.r.t. \( (e, f) \);

(ii) \( \exists s, t \in \mathcal{A} : s^{\circ} = t^{\circ} = \{0\} \) and \( a^{*}f, e = sa = at \);
\[\exists s_1, s_2, t_1, t_2 \in A: a_s^{f,e} = s_1a = at_1 \text{ and } a = s_2a_s^{f,e} = a_s^{f,e}t_2;\]

\[\exists u, v \in A: uA = A = Av \text{ and } a_s^{f,e} = au = va;\]

\[\exists x, y \in A^{-1}: a_s^{f,e}a = xaa_s^{f,e} = aa_s^{f,e}y;\]

\[\exists x_1, y_1 \in A: x_1^o = y_1^o = \{0\} \text{ and } a_s^{f,e}a = x_1aa_s^{f,e} = aa_s^{f,e}y_1;\]

\[\exists x_2, y_2 \in A: Ax_2 = y_2A = x_2aa_s^{f,e} = aa_s^{f,e}y_2;\]

\[\exists x_3, x_4, y_3, y_4 \in A: a_s^{f,e}a = x_3aa_s^{f,e} = aa_s^{f,e}y_3 \text{ and } a_s^{f,e} = x_4aa_s^{f,e}a = a_s^{f,e}y_4;\]

\[\exists z_1, z_2 \in A: a_s^{f,e}a = az_1a_s^{f,e} \text{ and } aa_s^{f,e} = a_s^{f,e}z_2a;\]

\[\exists g_1, h_1 \in A^{-1}: a_s^{f,e}a = ah_1h_1^o a_s^{f,e} \text{ and } aa_s^{f,e} = ah_1h_1^o g_1^o g_1a;\]

\[\exists g_2, h_2 \in A: g_2^o = h_2^o = \{0\}, a_s^{f,e}a = ah_2h_2^o a_s^{f,e} \text{ and } aa_s^{f,e} = a_s^{f,e}g_2^o g_2a;\]

\[\exists g_3, h_3 \in A: Ag_3 = h_3A, a_s^{f,e}a = ah_3h_3^o a_s^{f,e} \text{ and } aa_s^{f,e} = a_s^{f,e}g_3^o g_3a.\]

**Proof.** (i) \(\Rightarrow\) (ii): If \(a\) is weighted–EP w.r.t. \((e, f)\), by Theorem 1.4, \(a \in A_{e,f}^{-1} \cap A^{-1}a_{e,f}^{-1}\), i.e. \(a \in A_{e,f}^{-1} \cap A^{-1}a_{e,f}^{-1}\) by Lemma 1.3. So, there exist \(s, t \in A^{-1}\) such that \(a_s^{f,e} = sa = at\) and the statement (ii) holds.

Similarly, we can prove that (i) implies (iii) and (iv).

(ii) \(\Rightarrow\) (i): The condition (ii) implies \(a^o \subseteq (a_s^{f,e})^o\) and \(a^o \subseteq o(a_s^{f,e})\). Let \(x \in (a_s^{f,e})^o\), then \(sax = a_s^{f,e}x = 0\), by \(s^o = \{0\}\), gives \(ax = 0\). Hence, \(a^o = (a_s^{f,e})^o\) and, analogy, \(a^o = o(a_s^{f,e})\). By Theorem 1.4, \(a\) is weighted–EP w.r.t. \((e, f)\).

In the similar way, we can check (iii) \(\Rightarrow\) (i).

(iv) \(\Rightarrow\) (i): From the assumption (iv), we deduce that \(a_s^{f,e}A = auA = aA\) and \(Aa_s^{f,e} = AuA = Aa\) which gives that the condition (i) is satisfied, by Theorem 1.4.

(i) \(\Rightarrow\) (v): Let \(x = (a_{e,f}^{-1})^o a_{e,f}^{-1} + 1 - aa_{e,f}^{-1} = (aa_{e,f}^{-1} + 1 - aa_{e,f}^{-1})^{-1}\) and \(y = a_{e,f}^{-1} a_{e,f}^{-1} + 1 - a_{e,f}^{-1} a = (a_{e,f}^{-1} a + 1 - a_{e,f}^{-1} a)^{-1}\). Then \(x, y \in A^{-1}\), \(a_s^{f,e} = y^{-1} a_{e,f}^{-1}\) and \(a_s^{f,e} = a_{e,f}^{-1}\), by Lemma 1.3. Now, we can verify that \(aa_s^{f,e}x = xaa_s^{f,e} = a_{e,f}^{-1}\) and \(a_s^{f,e}y = yaa_s^{f,e} = a_{e,f}^{-1}\). Further,

\[a_s^{f,e}a = y^{-1}(a_{e,f}^{-1} a) = y^{-1} aa_{e,f}^{-1} = y^{-1}(aa_s^{f,e}x) = y^{-1} xaa_s^{f,e}\]
and

\[ a\f_e = (aa_e^*)^{-1} = a_e^* ax^{-1} = (ya\f_e a)x^{-1} = a\f_e ax^{-1}, \]

i.e. \( a\f_e = ta\f_e = a\f_e z^{-1} \), for \( t = y^{-1}x \) and \( z = yx^{-1} \). Therefore, the condition \((v)\) holds.

It is clear that the condition \((v)\) implies \((vi)-(viii)\).

\((vi) \Rightarrow (i):\) Using \((vi)\), we obtain \((a\f_e a)^o = (a\f_e a)^o\) and \(o(a\f_e a) = o(a\f_e a)\). Observe that, by Theorem 1.3, \((a\f_e a)^o = e\), \((a\f_e a)^o = (a\f_e a)^o\), \(o(a\f_e a) = o(a\f_e a)\) and \(o(a\f_e a) = o(a\f_e a)\). Hence, \(o = (a\f_e a)^o\) and \(o = o(a\f_e a)\) and, by Theorem 1.4, \(a\) is weighted–EP w.r.t. \((e,f)\).

Analog, we check that \((vii) \vee (ix) \Rightarrow (i)\).

\((vii) \Rightarrow (i):\) Applying the hypothesis \((vii)\) and the equalities \(a\mathcal{A} = a\f_e a\mathcal{A}\), \(A_1 = A\f_e a\mathcal{A}\), \(a\f_e A = a\f_e a\mathcal{A}\), \(a\f_e = a\f_e a\mathcal{A}\), we get \(a\mathcal{A} = a\f_e A\) and \(A_1 = A\f_e a\mathcal{A}\). Thus, by Theorem 1.4, \(a\) is weighted–EP w.r.t. \((e,f)\).

\((i) \Rightarrow (ix):\) It is well-known that \((i)\) gives that \(a = a\f_e x_1 = x_2 a\f_e\) and \(a\f_e = ax_3 = x_4 a\), for some \(x_1, x_2, x_3, x_4 \in \mathcal{A}\). Now, we conclude that \(a\f_e a = a(x_3 x_2)\f_e a\) and \(a\f_e = a\f_e(x_1 x_4) a\). So, \((ix)\) holds.

\((i) \Rightarrow (x):\) The condition \((i)\) implies that there exist \(g_1, g_2 \in \mathcal{A}^{-1}\) such that \(a\f_e = a h_1 = g_1 a\) which gives \(a = h_1 \f_e a\f_e = a\f_e g_1\f_e a\). Therefore, \((x)\) is satisfied.

Obviously, \((x) \Rightarrow (xi) \wedge (xii)\).

\((xi) \Rightarrow (i):\) Since \(o(a\f_e a) = o(ah_3 \f_e a\f_e a), \) then \(o(a\f_e a) = o(ah_3 h_2 a), \) and, by \(o(h_2) = \{0\}, o(a\f_e a) = o(a). \) Similarly, from \((aa\f_e a)^o = (g_2 a \f_e a\f_e g_2 a)^o\) and \(g_2^o = \{0\}\), we have \((a\f_e a)^o = o(a). \) Hence, \(a\) is weighted–EP w.r.t. \((e,f)\), by Theorem 1.4.

\((xii) \Rightarrow (i):\) The assumption \((xii)\) gives \(a\f_e a\mathcal{A} = ah_3 \f_e A = ah_3 (ah_2)^e a\mathcal{A}. \) Therefore, \(a\f_e A = ah_3 a\mathcal{A}, \) \(a\mathcal{A} = a\mathcal{A}. \) In the same way, \(Aa\f_e = Aa\f_e g_3 a\) and \(Ag_3 = A\mathcal{A} \) imply \(Aa\f_e = Aa. \) Therefore, \(a\) is weighted–EP w.r.t. \((e,f)\), by Theorem 1.4.

We continue with characterizations of weighted–EP elements via factorizations of the form \(a_e = sa.\)

**Theorem 3.2.** Let \(\mathcal{A}\) be a unital C*-–algebra, and let \(e, f\) be invertible positive elements in \(\mathcal{A}\). For \(a \in \mathcal{A}^{-}\) the following statements are equivalent:

\[(i)\] \(a\) is weighted–EP w.r.t. \((e,f)\);

\[(ii)\] \(\exists s, t \in \mathcal{A}: s^o = t = \{0\} \) and \(a_e = sa = at;\)

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(iii) \( \exists s_1, s_2, t_1, t_2 \in \mathcal{A} : a_{e,f}^\dagger s_1 = at_1 \) and \( a = s_2 a_{e,f}^\dagger t_2 \);

(iv) \( \exists u, v \in \mathcal{A} : uA = A = Av \) and \( a_{e,f}^\dagger = au = va \);

(v) \( \exists x, y \in \mathcal{A}^{-1} : a_{e,f}^\dagger a = xaa_{e,f}^\dagger = aa_{e,f}^\dagger y \);

(vi) \( \exists x_1, y_1 \in \mathcal{A} : x_1^\circ = \circ y_1 = \{0\} \) and \( a_{e,f}^\dagger a = x_1 a_{e,f}^\dagger = a_{e,f}^\dagger y_1 \);

(vii) \( \exists x_2, y_2 \in \mathcal{A} : Ax_2 = A = y_2 A \) and \( a_{e,f}^\dagger a = x_2 a_{e,f}^\dagger = a_{e,f}^\dagger y_2 \);

(viii) \( \exists x_3, x_4, y_3, y_4 \in \mathcal{A} : a_{e,f}^\dagger a = x_3 a_{e,f}^\dagger = a_{e,f}^\dagger y_3 \) and \( a_{e,f}^\dagger = x_4 a_{e,f}^\dagger a = a_{e,f}^\dagger y_4 \);

(ix) \( \exists z_1, z_2 \in \mathcal{A} : a_{e,f}^\dagger a = az_1 a_{e,f}^\dagger \) and \( a_{e,f}^\dagger = a_{e,f}^\dagger z_2 a \).

Proof. Similarly as the proof of Theorem 3.1, using Lemma 1.2 and Lemma 1.3. \( \square \)

4 Factorization \( a = e^{-1}ucvf \)

In this section, we give characterizations of weighted–EP elements through factorizations of the form \( a = e^{-1}ucvf \).

Theorem 4.1. Let \( e, f \) be invertible positive elements in \( \mathcal{A} \). If \( a \in \mathcal{A}^{-} \), then the following statements are equivalent:

(i) \( a \) is weighted–EP w.r.t. \( (e,f) \);

(ii) \( \exists c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf = e^{-1}fv^*d^*u^*e^{-1}f, vA = A = Au, cA = dA \) and \( Ac = Ad \);

(iii) \( \exists c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf = e^{-1}fv^*d^*u^*e^{-1}f, u^0 = \{0\} = \circ v, c^0 = d^0 \) and \( \circ c = \circ d \);

(iv) \( \exists c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf, a_{e,f}^\dagger = e^{-1}udvf, vA = A = Au, cA = dA \) and \( Ac = Ad \);

(v) \( \exists c, d, u, v \in \mathcal{A} : a = e^{-1}ucvf, a_{e,f}^\dagger = e^{-1}udvf, u^0 = \{0\} = \circ v, c^0 = d^0 \) and \( \circ c = \circ d \).
(vi) \( \exists c, d, u, v \in \mathcal{A} : a^{*f,e} a = uc, aa^{*f,e} = udv, vA = A = Au, cA = dA \) and \( \mathcal{A}c = \mathcal{A}d \).

(vii) \( \exists c, d, u, v \in \mathcal{A} : a^{*f,e} a = uc, aa^{*f,e} = udv, u^o = \{0\} = e v, c^o = d^o \) and \( c^o = d^o \).

Proof. (ii) \( \Rightarrow \) (i): If \( a = e^{-1} uc = e^{-1} f v^* d^* u^* e^{-1} f \), for some \( c, d, u, v \in \mathcal{A} \) satisfying \( vA = A = Au, cA = dA \) and \( \mathcal{A}c = \mathcal{A}d \), then \( a^* = f e^{-1} udvf \). Further

\[
\begin{align*}
aA &= e^{-1} uc v A = e^{-1} uc A = e^{-1} ud A \\
     &= e^{-1} udv A = e^{-1} udvf A = f^{-1} a^* e A = a^{*f,e} A
\end{align*}
\]

and

\[
\begin{align*}
Aa &= Ae^{-1} uc v f = Auc v f = Ac v f = Ad v f \\
     &= Adu v f = Ae^{-1} udvf = A f^{-1} a^* e = Aa^{*f,e}.
\end{align*}
\]

By Theorem 1.4, we deduce that \( a \) is weighted–EP w.r.t. \((e,f)\).

(i) \( \Rightarrow \) (ii): Since \( a \) is weighted–EP w.r.t. \((e,f)\), we have \( aA = a^{*f,e}A \) and \( Aa = Aa^{*f,e} \). Let \( u = v = 1, c = ea f^{-1} \) and \( d = ea^{*f,e} f^{-1} \). Now, we obtain

\[
e^{-1} cA = af^{-1} A = aA = a^{*f,e} A = a^{*f,e} f^{-1} A = e^{-1} dA,
\]

and

\[
Ac f = Aea = Aa = Aa^{*f,e} = Aa^{*f,e} = Ad f,
\]

implying \( cA = dA \) and \( Ac = Ad \). The rest is obviously.

(iii) \( \Rightarrow \) (i): Assume that there exist \( c, d, u, v \in \mathcal{A} \) satisfying \( a = e^{-1} uc v f = e^{-1} f v^* d^* u^* e^{-1} f \), \( u^o = \{0\} = e v, c^o = d^o \) and \( c^o = d^o \). To prove that \( a^o = (a^{*f,e})^o \), let \( x \in a^o \), i.e. \( e^{-1} uc v f x = 0 \). Now, \( uc v f x = 0 \) and, by \( u^o = \{0\}, \), \( cv f x = 0 \). So, \( v f x \in c^o = d^o \); that is, \( d v f x = 0 \) which gives \( a^{*f,e} x = e^{-1} udvf x = 0 \). Hence, \( a^o \subseteq (a^{*f,e})^o \). The reverse inclusion follows similarly. The conditions \( \{0\} = e v \) and \( c^o = d^o \) imply \( c^o = (a^{*f,e})^o \), analogy. Thus, \( a \) is weighted–EP w.r.t. \((e,f)\), by Theorem 1.4.

(i) \( \Rightarrow \) (iii): Because \( a \) is weighted–EP w.r.t. \((e,f)\), then \( a^o = (a^{*f,e})^o \) and \( a^o = (a^{*f,e})^o \). We can show that \( (af^{-1})^o = (a^{*f,e} f^{-1})^o \) and \( (ea)^o = (ea^{*f,e})^o \). For \( u = v = 1, c = ea f^{-1} \) and \( d = ea^{*f,e} f^{-1} \), we obtain

\[
c^o = (e^{-1} c)^o = (af^{-1})^o = (a^{*f,e} f^{-1})^o = (e^{-1} d)^o = d^o
\]

and

\[
o c = (o (e a)) = o (ea^{*f,e}) = o (df) = o d.
\]
(iv) ⇒ (i): We can verify that (iv) gives \( aA = a_{e,f}^\dagger A \) and \( Aa = Aa_{e,f}^\dagger \) in the same way as in the part (ii) ⇒ (i). By the equality (1), we conclude that \( aA = a_{e,f}^\dagger A \) and \( Aa = Aa_{e,f}^\dagger \) and, by Theorem 1.4, (i) holds.

(i) ⇒ (iv): The statements (i) implies \( aA = a_{e,f}^\dagger A = a_{e,f}^\dagger A \) and \( Aa = Aa_{e,f}^\dagger = Aa_{e,f}^\dagger \). The condition (iv) follows on choosing \( u = v = 1 \), \( c = eaf^{-1} \) and \( d = eaf_{e,f}^{-1} \).

(v) ⇒ (i): As in the part (iii) ⇒ (i), we get \( a^o = (ae_{e,f}^\dagger)^o \) and \( o_a = o(a_{e,f}^\dagger) \) which yields (i), by (1) and Theorem 1.4.

(i) ⇒ (v): By the choose \( u = v = 1 \), \( c = eaf^{-1} \) and \( d = eaf_{e,f}^{-1} \).

(vi) ⇒ (i): From the hypothesis (vi), we can check that \( a^*f_c a_A = a^*f_c a_A = a^*f_c a_A = a^*f_c e_A \). This equalities give \( a^*f_c A = aA \) and \( aA = a^*f_c e_A \), i.e. (i) is satisfied.

(vi) ⇒ (i) ∧ (vii): It follows for \( u = v = 1 \), \( c = a^*f_c a \) and \( d = aa^*f_c e \).

(vii) ⇒ (i): Using (vii), we have \( (a^*f_c a)^o = (aa^*f_c e)^o \) and \( o(a^*f_c a) = o(aa^*f_c e) \) which yields \( a^o = (a_{e,f}^\dagger)^o \) and \( o(a_{e,f}^\dagger) = o_a \). So, (i) holds.

5 Factorization \( a = bc \)

For an invertible positive element \( f \in A \), we consider a factorization of \( a \in A \) of the form

\[
(4) \quad a = bc, \quad f^{-1}b^*A = A = cA.
\]

**Lemma 5.1.** Let \( e, f, h \) be invertible positive elements in \( A \). If \( a \in A \) has a factorization (4), then \( a \) is regular and \( a_{e,h}^1 = c_{f,h}^1 b_{e,f}^1 \).

**Proof.** Since \( f^{-1}b^*A = A = cA \), by Lemma 1.1, \( bf^{-1}, c^* \in A^\sim \) and \( (bf^{-1})^o = \{0\} = (c^*)^o \). Thus, the elements \( b \) and \( c \) are regular. Also, by the hypothesis \( f^{-1}b^*A = A = cA \), there exist \( x, y \in A \) such that \( f^{-1}b^*y = 1 = cx \). Then,

\[
(5) \quad b_{e,f}^1 = f^{-1}(fb_{e,f}^1) = f^{-1}b^*(b_{e,f}^1)^*f f^{-1}b^*y = f^{-1}(bb_{e,f}^1) = y = f^{-1}b^* = 1
\]

and

\[
(6) \quad cc_{f,h}^1 = cc_{f,h}^1 1 = cc_{f,h}^1 c = cc_{f,h} = c = 1.
\]

Now, we can easy check that \( (bc)_{e,h}^1 = c_{f,h}^1 b_{e,f}^1 \). \( \square \)
**Lemma 5.2.** Let $e, f, h$ be invertible positive elements in $A$. If $a \in A$ has a factorization (4), then

(i) $bA = aA$;

(ii) $c^*A = a^*A$;

(iii) $c^o = a^o$;

(iv) $(b^o)^o = (a^o)^o$;

(v) $[(eb)^*]^o = [(ea)^*]^o$;

(vi) $(ch^{-1})^o = (ah^{-1})^o$;

(vii) $(b_{e,f}^* b_{e,f}^*)^o e,f \in \mathcal{A}^{-1}$ and $(b_{e,f}^* b_{e,f}^*)^o e,f, f^{-1} b^o e$;

(viii) $(c_{f,h}^* c_{f,h}^*)^o f,h \in \mathcal{A}^{-1}$ and $(c_{f,h}^* c_{f,h}^*)^o f,h, f^{-1} c^o e$;

(ix) $b^* e b \in \mathcal{A}^{-1}$ and $b_{e,f}^* = (b^* e b)^{-1} b^o e$;

(x) $c^* f h \in \mathcal{A}^{-1}$ and $c_{f,h}^* = h^{-1} c^o (ch^{-1} c^o)^{-1}$.

**Proof.**

(i) The condition $cA = A$ implies $bA = bcA = aA$.

(ii) From the equality $f^o b^* A = A$, we get $c^* A = c^* f A = c^* f f^{-1} b^* A = (bc)^* A = a^* A$.

(iii) Notice that, $c^o \subseteq a^o$. If $x \in a^o$, then $bf^{-1} f x = 0$. By Lemma 1.1, we observe that $(bf)^o = \{0\}$ which gives $f x = 0$. Now, we deduce that $c x = 0$ and $a^o \subseteq c^o$. Hence, $c^o = a^o$.

(iv) Because $(c^o)^o = \{0\}$, by Lemma 1.1, then $x \in (a^o)^o \iff a^o x = 0 \iff c^o b^o x = 0 \iff b^o x = 0 \iff x \in (b^o)^o$.

By (5) and (6), it follows (vii)-(viii).

(ix) Since $bc = a = a a^\dagger e,h a a^\dagger e,h a = a a^\dagger e,h e^{-1} (a^\dagger e,h)^* a^o e a = bca_{e,h} e^{-1} (a^\dagger e,h)^* a^o e b c$, then $c^\dagger e,h e^{-1} (a^\dagger e,h)^* c^o b^* e b = b_{e,f}^* (bca_{e,h} e^{-1} (a^\dagger e,h)^* a^o e b c) c_{f,h}^* = b_{e,f}^* b c e f_{e,h}^* = 1$ implies $b^* e b \in \mathcal{A}^{-1}$. We can easily check that $b_{e,f}^* = (b^* e b)^{-1} b^o e$.

Considering $a^*$ we verify (x) similarly as in the proof of part (ix).
In the following result, we characterize weighted–EP elements through their factorizations of the form $a = bc$.

**Theorem 5.1.** Let $e$, $f$, $h$ be invertible positive elements in $\mathcal{A}$. If $a \in \mathcal{A}$ has a factorization (4), then $a \in \mathcal{A}^-$ and the following conditions are equivalent

(i) $a$ is weighted–EP w.r.t. $(e, h)$;

(ii) $bb^\dagger_{e,f} = c^\dagger_{f,h} c$;

(iii) $c^\circ = [(eb)^\circ] = (b^\circ)^\circ = (ch^{-1})^\circ$;

(iv) $c^\circ = (eb)$ and $c^\circ = (h^{-1} c)^\circ$;

(v) $c^\circ A = eb A$ and $b A = h^{-1} c^\circ A$;

(vi) $Ac = Ab^* e$ and $Ab^* = A ch^{-1}$;

(vii) $\exists u \in \mathcal{A}^{-1}: c = ub^\dagger_{e,f}$ and $b = c^\dagger_{f,h} u$;

(viii) $\exists x, y \in \mathcal{A}^{-1}: c = xb^* e$ and $b^* = ych^{-1}$;

(ix) $A^{-1} c = A^{-1} b^* e$ and $A^{-1} b^* = A^{-1} ch^{-1}$;

(x) $c^\circ A^{-1} = eb A^{-1}$ and $b A^{-1} = h^{-1} c^\circ A^{-1}$;

(xi) $\exists x, y \in \mathcal{A}: x^\circ = y^\circ = \{0\}$, $c = xb^* e$ and $b^* = ych^{-1}$;

(xii) $\exists x, x_1, y_1 \in \mathcal{A}: c = xb^* e$, $b^* = x_1 c$, $b^* = ych^{-1}$ and $ch^{-1} = y_1 b^*$;

(xiii) $\exists x, y \in \mathcal{A}$: $xA = yA = \mathcal{A}$, $c^\circ = eb x$ and $b = h^{-1} c^\circ y$;

(xiv) $a \in h^{-1} c^\circ A \cap Ab^* e$ (or $a \in c^\dagger_{f,h} A \cap Ab^\dagger_{e,f}$);

(xv) $a^\dagger_{e,h} \in b A \cap Ac$;

(xvi) $b(b^* eb)^{-1} b^* e = h^{-1} c^\circ (ch^{-1} c^\circ)^{-1} c$.

(xvii) $b = c^\dagger_{f,h} cb$, $c = cbb^\dagger_{e,f}$, $b^\dagger_{e,f} = b^\dagger_{e,f} c^\dagger_{f,h} c$ and $c^\dagger_{f,h} = bb^\dagger_{e,f} c^\dagger_{f,h}$;

(xviii) $A^{-1} c = A^{-1} b^\dagger_{e,f}$ and $b A^{-1} = c^\dagger_{f,h} A^{-1}$;

(xix) $\exists u \in \mathcal{A}: u^\circ = u^\circ = \{0\}$, $c = ub^\dagger_{e,f}$ and $b = c^\dagger_{f,h} u$;

(xx) $\exists u \in \mathcal{A}$: $Au = u A = \mathcal{A}$, $c = ub^\dagger_{e,f}$ and $b = c^\dagger_{f,h} u$.
∃ v ∈ A : v₀ = ∘v = {0}, \( b_{e,f}^\dagger = vc \) and \( c_{f,h}^\dagger = bv \);

∃ v ∈ A : \( Av = v.A = A \), \( b_{e,f}^\dagger = vc \) and \( c_{f,h}^\dagger = bv \);

∃ u, u₁, v₁ ∈ A : \( c = ub_{e,f}^\dagger, b_{e,f}^\dagger = vc \), \( b = c_{f,h}^\dagger u₁ \) and \( c_{f,h}^\dagger = bv₁ \).

**Proof.** (i) ⇔ (ii): By Theorem 1.4, a is weighted–EP w.r.t. \( (e,h) \) if and only if \( aa_{e,h}^\dagger = a_{e,h}^\dagger a \) which is equivalent to \( bb_{e,f}^\dagger = c_{f,h}^\dagger c \), by Lemma 5.1, (5) and (6).

(i) ⇔ (iii): The element \( a \) is weighted–EP w.r.t. \( (e,h) \) if and only if \( ea \) and \( af⁻¹ \) are EP, that is, \( (ea)^o = [(ea)^*]^o \) and \( (ah⁻¹)^o = [(ah⁻¹)^*]^o \).

Notice that, by Lemma 1.2 and Lemma 5.2, these equalities are equivalent to \( c^o = [(eb)^*]^o \) and \( (ch⁻¹)^o = (b^*)^o \).

(iv) ⇔ (iii): This part can be check using involution.

(i) ⇔ (v): It is well-known that \( a \) is weighted–EP w.r.t. \( (e,h) \) if and only if \( ea.A = (ea)^*A \) and \( ah⁻¹.A = (ah⁻¹)^*A \), i.e. \( ea.A = a^*A \) and \( a.A = h⁻¹a^*A \).

Observe that, from Lemma 5.2, \( eb.A = ea.A \) and \( h⁻¹c^*A = h⁻¹a^*A \).

Now, we conclude that \( ea.A = a^*A \) and \( a.A = (ah⁻¹)^*A \) is equivalent to \( eb.A = c^*A \) and \( b.A = h⁻¹c^*A \).

(vi) ⇔ (v): Applying the involution we verify this equivalence.

(ii) ⇒ (vii): Suppose that \( bb_{e,f}^\dagger = c_{f,h}^\dagger c \). Let \( u = eb \) and let \( v = b_{e,f}^\dagger c_{f,h}^\dagger \).

Then

\[ c = cc_{f,h}^\dagger c = cbb_{e,f}^\dagger = ub_{e,f}^\dagger, \quad b = bb_{e,f}^\dagger b = c_{f,h}^\dagger cb = c_{f,h}^\dagger u, \]

and

\[ uv = ub_{e,f}^\dagger c_{f,h}^\dagger = cc_{f,h}^\dagger = 1 = b_{e,f}^\dagger b = b_{e,f}^\dagger c_{f,h}^\dagger u = vu. \]

Hence, \( u ∈ A⁻¹ \) and the condition (vii) holds.

(vii) ⇒ (vii): If there exists \( u ∈ A⁻¹ \) such that \( c = ub_{e,f}^\dagger \) and \( b = c_{f,h}^\dagger u \), then

\[ c = x'b_{f,h}^\dagger e \quad \text{and} \quad b = c_{h,f}^\dagger y' \quad \text{for} \quad x' = ub_{e,f}^\dagger (b_{e,f}^\dagger)^*f \quad \text{and} \quad y' = (c_{f,h}^\dagger f)_{f,h}^\dagger. \]

For \( x = x'f⁻¹ \) and \( y = (y')^*f \), we see that \( c = xb^*e \), \( b^* = ych⁻¹ \) and, by Lemma 5.2, \( x, y ∈ A⁻¹ \).

The following implications can be proved easily:

(viii) ⇒ (vi);

(viii) ⇒ (ix) ⇔ (x);

(viii) ⇒ (xii) ⇒ (iii);

(viii) ⇒ (xiii) ⇒ (v).
(i) ⇔ (xiv): Notice that, by Theorem 1.4, \( a \) is weighted–EP w.r.t. \((e,h)\) if and only if \( a \in a^\dagger_{e,h} A \cap A a^\dagger_{e,h} \), which is equivalent to \( a \in h^{-1} a^* A \cap A a^* e = h^{-1} c^* A \cap A b^* e \).

(i) ⇒ (xv): By Theorem 1.4, \( a \) is weighted–EP w.r.t. \((e,h)\) \(\Leftrightarrow a \in a^\dagger_{e,f} A^{-1} \cap A^{-1} a^\dagger_{e,f} \). Consequently, \( a^\dagger_{e,f} \in a A \cap A a = b A \cap A c \).

(xv) ⇒ (i): Since \( a^\dagger_{e,h} \in b A \cap A c \), then \( a^\dagger_{e,f} \in a A \cap A a \). Therefore, for some \( x, y \in a A \), \( a^\dagger_{e,f} = ax = ya \), which gives
\[
a^\dagger_{e,f} - aa^\dagger_{e,f} a^\dagger_{e,f} = (a - aa^\dagger_{e,f} a)x = 0
\]

and
\[
a^\dagger_{e,f} - a^\dagger_{e,f} a^\dagger_{e,f} a = y(a - aa^\dagger_{e,f} a) = 0.
\]

By Theorem 1.4, \( a^\dagger_{e,f} = aa^\dagger_{e,f} a^\dagger_{e,f} = a^\dagger_{e,f} a^\dagger_{e,f} a \) implies that \( a \) is weighted–EP w.r.t. \((e,h)\).

(xvi) ⇔ (ii): Obviously, by statements (ix) and (x) of Lemma 5.2.

(ii) ⇒ (xvii): By elementary computations.

(xvii) ⇒ (i): The assumption (xvii) can be written as \((1 - c^\dagger_{f,h} c)b = 0, c(1 - bb^\dagger_{e,f}) = 0, b^\dagger_{e,f} (1 - c^\dagger_{f,h} c) = 0 \) and \((1 - bb^\dagger_{e,f}) c^\dagger_{f,h} = 0 \) implying
\[
a A = b A \subseteq (1 - c^\dagger_{f,h} c)^o = h^{-1} c^* A = h^{-1} a^* A,
\]
\[
(a^* e)^o = (b^* e)^o = (1 - bb^\dagger_{e,f}) A \subseteq c^o = a^o,
\]
\[
a^o = c^o = (1 - c^\dagger_{f,h} c) A \subseteq (b^\dagger_{e,f})^o = (b^* e)^o = (a^* e)^o,
\]

and
\[
h^{-1} a^* A = h^{-1} c^* A = c^\dagger_{f,h} A \subseteq (1 - bb^\dagger_{e,f})^o = b A = a A.
\]

Thus, \( ah^{-1} A = a A = h^{-1} a^* A \) and \((ea)^o = a^o = (a^* e)^o \) which gives that \( ah^{-1} \) and \( ea \) are EP elements, that is, \( a \) is weighted–EP w.r.t. \((e,h)\).

Note that for \( u = cb \) and \( v = b^\dagger_{e,f} c^\dagger_{f,h} \), we can show:

(vii) ⇔ (xviii);

(vii) ⇒ (xix) \lor (xx) \lor (xxiii) ⇒ (iii);

(vii) ⇒ (xx) \lor (xxii) ⇒ (vi).
References


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