ON INTEGRAL REPRESENTATION
OF THE GENERALIZED INVERSE $A^{(2)}_{T,S}$

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Abstract

We present a general integral representation for the generalized inverse $A^{(2)}_{T,S}$, which extends earlier result on the Moore-Penrose inverse, weighted Moore-Penrose inverse and Drazin inverse.

Keywords: Generalized $A^{(2)}_{T,S}$ inverse, integral representation.

1 Introduction

Goretsch [3] presented an integral representation of the Moore-Penrose inverse $T^\dagger$ of a bounded linear operator $T \in \mathcal{L}(H_1, H_2)$ with closed range $\mathcal{R}(T)$ in Hilbert space

$$T^\dagger = \int_0^\infty e^{T^*Tt}T^*dt,$$  \hspace{1cm} (1)

where $H_1, H_2$ are Hilbert spaces.

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Wei and Wu [7] extended the result of Groetsch to the weighted Moore-Penrose inverse of matrix $A \in \mathbb{C}^{m \times n}$,

$$A_{M,N}^\dagger = \int_0^\infty \exp(-A^\# At)A^\# dt,$$

(2)

where $A^\# = N^{-1}A^* M$, $M$ and $N$ are Hermitian positive definite matrices of order $m$ and $n$, respectively.

Gonzalez, Koliha and Wei [2] gave a simple integral representation of the Drazin inverse $a^D$ in Banach algebras: let $a \in \mathcal{A}$ be a Drazin invertible element of a finite Drazin index $k \geq 1$ such that the nonzero spectrum of $a^{m+1}$ lies in the open right half of the complex plane for some $m \geq k$. Then

$$a^D = \int_0^\infty \exp(-a^{m+1} t) a^m dt.$$

(3)

The above-mentioned results motivate us to investigate the outer inverse $A^{(2)}_{T,S}$ of a matrix $A \in \mathbb{C}^{m \times n}$, since we have observed that the traditional generalized inverses (see [1]), such as the Moore-Penrose inverse $A^\dagger$, the weighted Moore-Penrose inverse $A_{M,N}^\dagger$, the Drazin inverse $A^D$, the group inverse $A^g$, etc., are outer inverses with prescribed range and kernel. The generalized inverse $A^{(2)}_{T,S}$ of $A \in \mathbb{C}^{m \times n}$ is the matrix $X \in \mathbb{C}^{n \times m}$ satisfying

$$XAX = X, \ R(X) = T, \ N(X) = S.$$

Recently, Wei [6] established the integral representation for the generalized inverse $A^{(2)}_{T,S}$ of a matrix $A \in \mathbb{C}^{m \times n}$, since we have observed that the traditional generalized inverses (see [1]), such as the Moore-Penrose inverse $A^\dagger$, the weighted Moore-Penrose inverse $A_{M,N}^\dagger$, the Drazin inverse $A^D$, the group inverse $A^g$, etc., are outer inverses with prescribed range and kernel. The generalized inverse $A^{(2)}_{T,S}$ of $A \in \mathbb{C}^{m \times n}$ is the matrix $X \in \mathbb{C}^{n \times m}$ satisfying

$$XAX = X, \ R(X) = T, \ N(X) = S.$$

In this paper we will give a general integral representation for the generalized inverse $A^{(2)}_{T,S}$ which drops the restriction on the spectrum of $GA$ and extends the earlier result on Drazin inverse [2].

Fundamental lemmas are needed in what follows.

**Lemma 1.1** Let $A \in \mathbb{C}^{m \times n}$ be of rank $r$, let $T$ be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let $S$ be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose $G \in \mathbb{C}^{n \times n}$ such that $R(G) = T$ and $N(G) = S$. If $A$ has an outer inverse $A^{(2)}_{T,S}$, then $\text{ind}(GA) = \text{ind}(AG) = 1$. Further, we have

$$A^{(2)}_{T,S} = (GA)^g G = G(AG)^g.$$

(5)
Lemma 1.2 Let $A \in \mathbb{C}^{n \times n}$ be a nonsingular matrix with $\Re \sigma(A) > 0$. Then

$$A^{-1} = \int_0^\infty \text{ext}(-At)dt.$$  \hspace{0.5cm} (6)

In this paper for any matrix $A \in \mathbb{C}^{n \times n}$ we denote its spectrum by $\sigma(A)$. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represents the range and the null space of $A$, respectively. We define the index of $A$, written $\text{ind}(A)$, to be the least nonnegative $k$ for which $\mathcal{N}(A^k) = \mathcal{N}(A^{k+1})$ holds.

2 Main results

In this section we will present a general integral representation of the generalized inverse $A_{(2)}^{(T, S)}$. Throughout this section, we let $A$, $T$ and $S$ to be the same as in Lemma 1.1. In addition, let $G \in \mathbb{C}^{n \times m}$ be such that

$$\mathcal{R}(G) = T \quad \text{and} \quad \mathcal{N}(G) = S.$$ \hspace{0.5cm} (7)

First we develop the algebraic structures of $A$ and $G$.

**Theorem 2.1** Let $A$, $T$ and $S$ be the same as in Lemma 1.1 and $G \in \mathbb{C}^{n \times m}$ satisfies (7). Then we have

$$A = Q^{-1} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} P, \quad G = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q, \quad A_{(2)}^{(T, S)} = P^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q,$$ \hspace{0.5cm} (8)

where $P$, $Q$, $A_{11}$ and $G_{11}$ are nonsingular matrices.

**Proof.** It follows from Lemma 1.1 that

$$\text{ind} \,(AG) = \text{ind} \,(GA) = 1.$$ 

There is a Jordan canonical form of $AG$ and $GA$ as follows:

$$GA = P^{-1} \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} P, \quad AG = Q^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q,$$

where $C$ and $D$ are invertible matrices of the same order. Partition $A$ and $G$ as

$$A = Q^{-1} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} P, \quad G = P^{-1} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} Q.$$ 

It is easy to check that

$$(GA)^pG = P^{-1} \begin{bmatrix} C^{-p} & 0 \\ 0 & 0 \end{bmatrix} pP^{-1} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} Q$$

$$= P^{-1} \begin{bmatrix} C^{-1}G_{11} & C^{-1}G_{12} \\ 0 & 0 \end{bmatrix} Q.$$
Similarly, we have
\[ G(AG)^g = P^{-1} \begin{bmatrix} G_{11}D^{-1} & 0 \\ G_{21}D^{-1} & 0 \end{bmatrix} Q. \]

Since \( A_{T,S}^{(2)} = (GA)^g G = G(AG)^g \), we have
\[ C^{-1}G_{12} = 0 \quad \text{and} \quad G_{21}D^{-1} = 0, \]
i.e.
\[ G_{12} = 0 \quad \text{and} \quad G_{21} = 0. \]

Applying a little algebra, we obtain
\[ GA = P^{-1} \begin{bmatrix} G_{11}A_{11} & G_{11}A_{12} \\ G_{22}A_{21} & G_{22}A_{22} \end{bmatrix} P = P^{-1} \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} P, \]
and
\[ AG = Q^{-1} \begin{bmatrix} A_{11}G_{11} & A_{12}G_{22} \\ A_{21}G_{11} & A_{22}G_{22} \end{bmatrix} Q = Q^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q. \]

We deduce that
\[ G_{11}A_{11} = C \text{ (nonsingular)}, \quad A_{11}G_{11} = D \text{ (nonsingular)}, \]
that is to say both \( A_{11} \) and \( G_{11} \) are invertible. From \( G_{11}A_{12} = 0 \) and \( A_{21}G_{11} = 0 \), we obtain
\[ A_{12} = 0 \quad \text{and} \quad A_{21} = 0. \]

Finally, from the facts \( G = GA_{T,S}^{(2)} = A_{T,S}^{(2)}AG \) and \( AA_{T,S}^{(2)} = AG(AG)^g \), with \( A_{T,S}^{(2)} A = (GA)^g GA \), we have
\[ G = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} Q = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix} QQ^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q, \]
i.e. \( G_{22} = 0. \)

Thus, we get
\[ A = Q^{-1} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} P, \quad G = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q \]
and
\[ A_{T,S}^{(2)} = (GA)^g G = P^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q. \]

We have just finished the proof.

Now we are in a position to derive the general integral representation for the generalized inverse \( A_{T,S}^{(2)}. \)
Theorem 2.2 Suppose that $A$, $T$ and $S$ be the same as in Lemma 1.1 and $G \in \mathbb{C}^{n \times m}$ satisfying (7). Then we have

$$A^{(2)}_{T,S} = \int_0^\infty \exp \left[ -G(GA)^*GAt \right] G(GA)^*Gdt. \quad (9)$$

Proof. It follows from Theorem 2.1 that

$$A = Q^{-1} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} P, \quad G = P^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q \text{ and } A^{(2)}_{T,S} = P^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

We denote

$$QQ^* = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad \text{and} \quad (P^{-1})^*P^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

It is obviously that $Q_{11}$ and $P_{11}$ are Hermitian positive definite matrices, their square roots $Q_{11}^{1/2}$ and $P_{11}^{1/2}$ are also Hermitian positive definite matrices. By a direct computation we have

$$G(GA)^*Q = P^{-1} \begin{bmatrix} G_{11}Q_{11}(G_{11}A_{11}G_{11})^*P_{11}G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q$$

and

$$G(GA)^*GA = P^{-1} \begin{bmatrix} G_{11}Q_{11}(G_{11}A_{11}G_{11})^*P_{11}G_{11}A_{11} & 0 \\ 0 & 0 \end{bmatrix} Q.$$

We notice that

$$\sigma[G_{11}Q_{11}(G_{11}A_{11}G_{11})^*P_{11}G_{11}A_{11}] =$$

$$= \sigma[P_{11}^{1/2}Q_{11}^{1/2}(G_{11}A_{11}G_{11})^*P_{11}^{1/2}P_{11}^{1/2}(G_{11}A_{11}G_{11})]$$

$$= \sigma([P_{11}^{1/2}G_{11}A_{11}G_{11}Q_{11}^{1/2}]^*(P_{11}^{1/2}G_{11}A_{11}G_{11}Q_{11}^{1/2})) > 0.$$

It follows from Lemma 1.2 that

$$\int_0^\infty \exp \left[ -G(GA)^*GAt \right] G(GA)^*Gdt$$

$$= P^{-1} \left[ \int_0^\infty [-G_{11}Q_{11}(G_{11}A_{11}G_{11})^*P_{11}G_{11}A_{11}]dt \right] 0 \right] P \times$$

$$\times P^{-1} \begin{bmatrix} G_{11}Q_{11}(G_{11}A_{11}G_{11})^*P_{11}G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q$$

$$= P^{-1} \begin{bmatrix} G_{11}Q_{11}(G_{11}A_{11}G_{11})^*P_{11}G_{11}A_{11}^{-1}G_{11}Q_{11}(G_{11}A_{11}G_{11})^*P_{11}G_{11} & 0 \\ 0 & 0 \end{bmatrix} Q$$

$$= P^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q = A^{(2)}_{T,S}.$$

The proof is complete.
3 Concluding remarks

In this paper we have developed the integral representation for the generalized inverse $A^{(2)}_{T,S}$ of a complex matrix $A$. In our opinion, it is worth establishing the same result in Hilbert spaces or $C^*$-algebras.

References


