Generalized invertibility of operator matrices

Milica Z. Kolundžija* and Dragan S. Djordjević*

Abstract

In this paper we consider various aspects of generalized invertibility of the operator matrix $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ acting on a Banach space $X \oplus Y$.

1 Introduction

There are many papers dealing with spectral properties of 2×2 operator matrices, acting on a direct (or orthogonal) sum of Banach or Hilbert spaces (see all references). In this paper we consider some properties related to generalized invertibility, left Browder invertibility, and the point spectrum of a given operator.

Let Z be a Banach space, such that $Z = X \oplus Y$ for some closed and complementary subspaces X and Y. This sum will be also denoted by $\begin{bmatrix} X \\ Y \end{bmatrix}$. If Z is a Hilbert space, then we always assume that X and Y are closed and mutually orthogonal subspaces of Z, so $Z = X \oplus Y$ denotes the orthogonal sum.

Let $\mathcal{L}(X, Y)$ denote the set of all linear bounded operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X, X)$. The set of all finite rank operators from X to Y is denoted by $\mathcal{F}(X, Y)$. For $A \in \mathcal{L}(X, Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$

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to denote the range and the null-space of A, respectively. The ascent $\mathrm{asc}(A)$ and the descent $\mathrm{dsc}(A)$ of A are given by $\mathrm{asc}(A) = \inf\{n \geq 0 : \mathcal{N}(A^n) = \mathcal{N}(A^{n+1})\}$ and $\mathrm{dsc}(A) = \inf\{n \geq 0 : \mathcal{R}(A^n) = \mathcal{R}(A^{n+1})\}$.

If W is a finite dimensional subspace of a Banach space, then $\dim W$ denotes the dimension of W. If W is infinite dimensional, then we simply write $\dim W = \infty$. However, if X is a Hilbert space and W is a closed subspace of X, then $\dim W$ is the orthogonal dimension of W.

If $Z = X \oplus Y$, then any $M \in \mathcal{L}(Z)$ satisfying $M(X) \subset X$, can be decomposed as the following operator matrix

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix},$$

for some $A \in \mathcal{L}(X)$, $C \in \mathcal{L}(Y, X)$ and $B \in \mathcal{L}(Y)$. On the other hand, any choice of A, C, B (linear and bounded operators on the corresponding subspaces), produces a linear and bounded operator M on the space Z, such that X is invariant for M.

If A and B are fixed, then we use the notation M_C to show that M depends on C. For given A and B, we are interested to find C, such that M_C has some prescribed properties. There are several papers that investigate invertibility of 2×2 operator matrices (see [1], [2], [4], [5], [6], [8]).

In this paper we extend some results from Hilbert to Banach space settings. Thus, some recent results from, [2], [3] and [9] are generalized.

2 Generalized inverses of M_C

We need some properties of generalized inverses. Let $B \in \mathcal{L}(X, Y)$ be given. B is relatively regular (inner invertible) if there exists some $D \in \mathcal{L}(Y, X)$ such that $BDB = B$ holds. In this case D is an inner inverse of B. It is well-known that B is relatively regular, if and only if $\mathcal{R}(B)$ and $\mathcal{N}(B)$ are closed and complemented in Y and X, respectively. If $DBD = D$ holds and $D \neq 0$, then B is outer invertible, and D is an outer inverse of B. If $B \neq 0$, then it is a corollary of the Hahn-Banach theorem that there exists some non-zero outer inverse D of B. If D is both inner and outer inverse of B, then D is a reflexive inverse of B. Moreover, if D is an inner inverse of B, then DBD is a reflexive inverse of B.

If $D \in \mathcal{L}(Y, X)$ is a reflexive inverse of $B \in \mathcal{L}(X, Y)$, then BD is the projection from Y onto $\mathcal{R}(B)$ parallel to $\mathcal{N}(D)$, and DB is the projection from X onto $\mathcal{R}(D)$ parallel to $\mathcal{N}(B)$. On the other hand, if $X = U \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(B) \oplus V$ for closed subspaces: U of X and V of Y, then B have the matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} U \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ V \end{bmatrix},$$

and B_1 is invertible. It is easy to see that

$$D = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{N}(B) \end{bmatrix}$$

is the reflexive inverse of B satisfying $\mathcal{R}(B) = U$ and $\mathcal{N}(D) = V$.

If H, K are Hilbert spaces, and $A \in \mathcal{L}(H, K)$, then the Moore-Penrose inverse of A is the unique operator $A^\dagger \in \mathcal{L}(K, H)$ (in the case when it exists) which satisfies:

$$AA^\dagger A = A, \ A^\dagger AA^\dagger = A^\dagger, \ (AA^\dagger)^* = AA^\dagger, \ (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse of $A \in \mathcal{L}(H, K)$ exists if and only if $\mathcal{R}(A)$ is closed. If $A \in \mathcal{L}(H, K)$ is left (right) invertible, then $A^\dagger$ is a left (right) inverse of A.

In this section we investigate relative regularity of M_C, and the corresponding relatively regular spectrum σ_g. Notice that for $A \in \mathcal{L}(X)$ the relatively regular spectrum of A is defined as

$$\sigma_g(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not relatively regular}\}.$$

Definition 2.1. [1] *If X and Y are Banach spaces, then X can be embedded in Y, if there exists a left invertible operator $W \in \mathcal{L}(X, Y)$. The notation is $X \preceq Y$.*

If X and Y are Hilbert spaces, then $X \preceq Y$ if and only if $\dim X \leq \dim Y$.

Theorem 2.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be relatively regular. If $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$, then there exists some $C \in \mathcal{L}(Y, X)$ such that M_C is relative regular.*

Proof. Let $A_1 \in \mathcal{L}(X)$ and $B_1 \in \mathcal{L}(Y)$ denote reflexive inverses of A and B, respectively. Then $Y = \mathcal{R}(B_1) \oplus \mathcal{N}(B)$ and $X = \mathcal{N}(A_1) \oplus \mathcal{R}(A)$. Let $J : \mathcal{N}(B) \to \mathcal{N}(A_1)$ be a left invertible mapping and let $J_1 : \mathcal{N}(A_1) \to \mathcal{N}(B)$ be a left inverse of J. Define $C \in \mathcal{L}(Y, X)$ and $C_1 \in \mathcal{L}(X, Y)$ in the following way:

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix},$$

$$C_1 = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix}.$$

Consider the operator $N = \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} \in \mathcal{L}(X \oplus Y)$. Then we find

$$NM_C = \begin{bmatrix} A_1A & A_1C \\ C_1A & C_1C + B_1B \end{bmatrix}.$$

Since $\mathcal{R}(C) \subset \mathcal{N}(A_1)$ and $\mathcal{R}(A) \subset \mathcal{N}(C_1)$, we have $A_1C = 0$ and $C_1A = 0$, respectively. Also, B_1B is the projection from Y onto $\mathcal{R}(B_1)$ parallel to $\mathcal{N}(B)$, and C_1C is the projection from Y onto $\mathcal{N}(B)$ parallel to $\mathcal{R}(B_1)$. Hence $C_1C + B_1B = I$, and

$$NM_C = \begin{bmatrix} A_1A & 0 \\ 0 & I \end{bmatrix}.$$

Since $AA_1A = A$ and $A_1AA_1 = A_1$, we have

$$M_CNM_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} A_1A & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} AA_1A & C \\ 0 & B \end{bmatrix} = M_C,$$

and M_C is relatively regular.

$\square$

As a corollary, we get the following results.

Corollary 2.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given operators. Then the following inclusion holds:*

$$\bigcap_{C\in\mathcal{L}(Y,X)} \sigma_g(M_C) \subseteq \sigma_g(A) \cup \sigma_g(B) \cup \{\lambda \in \mathbb{C} : \mathcal{N}(B - \lambda I) \preceq X/\mathcal{R}(A - \lambda I) \text{ does not hold}\}.$$

We state the following result concerning the Moore-Penrose inverse of M_C.

Theorem 2.2. *Let H, K be mutually orthogonal Hilbert spaces and $Z = H \oplus K$. If $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ both have closed ranges, and if* $\mathrm{nul}(B) = \mathrm{def}(A)$*, then there exists some $C \in \mathcal{L}(K, H)$ such that M_C has a closed range, and*

$$M_C^\dagger = \begin{bmatrix} A^\dagger & 0 \\ C^\dagger & B^\dagger \end{bmatrix}.$$

Proof. Recall the notations from the proof of Theorem 2.1, with one assumption: J is invertible. We have the following:

$$NM_CN = \begin{bmatrix} A_1A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} = \begin{bmatrix} A_1AA_1 & 0 \\ C_1 & B_1 \end{bmatrix} = N,$$

and

$$M_CN = \begin{bmatrix} AA_1 + CC_1 & CB_1 \\ BC_1 & BB_1 \end{bmatrix}.$$

Since $\mathcal{R}(B_1) = \mathcal{N}(C)$ and $\mathcal{R}(C_1) = \mathcal{N}(B)$, it follows that $CB_1 = 0$ and $C_1B = 0$. Also, AA_1 is the projection on $\mathcal{R}(A)$ parallel to $\mathcal{N}(A_1)$. Since J is invertible, we have that CC_1 is the projection on $\mathcal{N}(A_1)$ parallel to $\mathcal{R}(A)$. Hence, $AA_1 + CC_1 = I$. Thus, N is a reflexive inverse of M_C.

Now, we take $A_1 = A^\dagger$ and $B_1 = B^\dagger$. Then all previous results holds, with one more nice property: we have orthogonal decompositions. Precisely, $X = \mathcal{N}(A_1) \oplus \mathcal{R}(A) = \mathcal{N}(A^*) \oplus \mathcal{R}(A)$ and $Y = \mathcal{N}(B) \oplus \mathcal{R}(B_1) = \mathcal{N}(B) \oplus \mathcal{R}(B^*)$. Since J is invertible, we have $J_1 = J^{-1}$ and consequently $C_1 = C^\dagger$. The operator N_C is still a reflexive inverse of M_C. Furthermore, we have

$$NM_C = \begin{bmatrix} A^\dagger A & 0 \\ 0 & I \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix},$$

and

$$M_CN = \begin{bmatrix} I & 0 \\ 0 & BB^\dagger \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Projections NM_C and M_CN are obviously selfadjoint, so $N = M_C^\dagger$. $\square$

3 Left Browder invertibility of M_C

An operator $A \in \mathcal{L}(X,Y)$ is right Fredholm, if $\mathcal{N}(A)$ is a complemented subspace of X, and $\operatorname{def}(A) = \dim Y/\mathcal{R}(A) < \infty$. The set of all right Fredholm operators from X to Y is denoted by $\Phi_r(X,Y)$. An operator $A \in \mathcal{L}(X,Y)$ is left Fredholm, if $\operatorname{nul}(A) = \dim \mathcal{N}(A) < \infty$ and $\mathcal{R}(A)$ is a closed and complemented subspace of Y. The set of all left Fredholm operators from X to Y is denoted by $\Phi_l(X,Y)$. The set of Fredholm operators from X to Y is defined as $\Phi(X,Y) = \Phi_l(X,Y) \cap \Phi_r(X,Y)$. The abbreviations $\Phi_l(X)$, $\Phi_r(X)$ and $\Phi(X)$ are clear.

An operator $T \in B(X)$ is left Browder, if it is left Fredholm with finite ascent. Analogously, T is right Browder, if it is right Fredholm with finite descent. These classes of operators are denoted, respectively, by $\mathcal{B}_l(X)$ and $\mathcal{B}_r(X)$. The set of all Browder operators on X is defined as $\mathcal{B}(X) = \mathcal{B}_l(X) \cap \mathcal{B}_r(X)$.

Among left Browder operators, we distinguish one new class of operators as follows:

$$\mathcal{B}_{lc}(X) = \{T \in B_l(X) : \overline{\mathcal{R}(T) + \mathcal{N}(T^{\operatorname{asc}(T)})} \text{ is complemented in } X\}.$$

Analogously, among right Browder operators we distinguish the following class of operators:

$$\mathcal{B}_{rc}(X) = \{T \in B_r(X) : \overline{\mathcal{R}(T^{\operatorname{dsc}(T)}) + \mathcal{N}(T)} \text{ is complemented in } X\}.$$

Now, we prove the following result concerning the left Browder invertibility of M_C. See also [2] for a Hilbert space case.

Theorem 3.1. *Suppose that the following hold: $A \in \mathcal{B}_{lc}(X)$, B is relatively regular, and $\mathcal{N}(B)$ is isomorphic to $X/\overline{(\mathcal{R}(A) + \mathcal{N}(A^{\operatorname{asc}(A)}))}$. Then there exists some $C \in \mathcal{L}(Y,X)$ such that $M_C \in B_l(Z)$.*

Proof. Let $A \in \mathcal{B}_{lc}(X)$, $\operatorname{asc}(A) = p$, and let W be a closed subspace of X such that $X = \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \oplus W$. Since $\mathcal{N}(B)$ is complemented, then $Y = \mathcal{N}(B) \oplus V$ for a closed subspace V. Since there exists a linear bounded and invertible operator $T : \mathcal{N}(B) \to W$, we can define operator $C : Y \to X$

by

$$C = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} \frac{W}{\mathcal{R}(A)+\mathcal{N}(A^p)} \end{bmatrix}.$$

We prove that M_C is left Fredholm. Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(M_C)$, so it is $Ax+Cy=0$ and $By=0$. We have $Ax=-Cy=-Ty \in \mathcal{R}(A) \cap W \subseteq \overline{\mathcal{R}(A)+\mathcal{N}(A^p)} \cap W = \{0\}$. Since $y \in \mathcal{N}(B)$ we have $Cy=Ty$, so $x \in \mathcal{N}(A)$ and $Ty=0$. Since T is invertible, we have $y=0$. It means that $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(A) \oplus \{0\}$, so $\mathcal{N}(M_C) \subseteq \mathcal{N}(A) \oplus \{0\}$. It follows that $\mathrm{nul}(M_C) \leq \mathrm{nul}(A) < \infty$.

Notice that we have obviously $\mathcal{N}(A) \subset \mathcal{N}(M_C)$, so actually we have $\mathrm{nul}(M_C) = \mathrm{nul}(A)$.

Let S be a reflexive inverse of A, let K be a reflexive inverse of B, and let $L = \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. We prove that $N = \begin{bmatrix} S & 0 \\ L & K \end{bmatrix}$ is an inner inverse of M_C. We have

$$M_C N M_C = \begin{bmatrix} ASA+CLA & ASC+CLC+CKB \\ BLA & BLC+BKB \end{bmatrix}.$$

Since $\mathcal{R}(A) \subseteq \overline{\mathcal{R}(A)+\mathcal{N}(A^p)} = \mathcal{N}(L)$, we have $LA=0$ which induces $BLA=0$ and $CLA=0$. From the fact that S is a reflexive inverse of A, we have $ASA=A$, and AS is a projection from X on $\mathcal{R}(A)$. Since $\mathcal{R}(C)=W$, $W \cap \mathcal{R}(A) = \{0\}$ and AS is a projection on $\mathcal{R}(A)$, it follows that $ASC=0$. Analogously, from the fact that K is a reflexive inverse of B, we have $BKB = B$ and KB is a projection from Y on V. Since $V=\mathcal{N}(C)$ and $\mathcal{R}(KB)=V$, it holds $CKB=0$. We have that $LC = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix}$, so $\mathcal{R}(LC) \subseteq \mathcal{N}(B)$ and then $BLC=0$. Obviously, $CLC=C$ holds.

It follows that

$$\begin{bmatrix} ASA+CLA & ASC+CLC+CKB \\ BLA & BLC+BKB \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = M_C.$$

Thus M_C is relatively regular. This induces $M_C \in \Phi_l$.

Now, we prove that $\mathrm{asc}(M_C) < \infty$. It is enough to prove that $\mathcal{N}(M_C^{p+1}) \subseteq$

$\mathcal{N}(M_C^p)$. Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(M_C^{p+1})$, then

$$\begin{cases} A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y + CB^py = 0, \\ B^{p+1}y = 0. \end{cases}$$

Since $B^py \in \mathcal{N}(B)$, it follows that $A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = -CB^py \in \mathcal{R}(A) \cap W \subseteq \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \cap W = \{0\}$. Thus

$$\begin{cases} A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = 0, \\ CB^py = 0. \end{cases}$$

From the definition of C and from $B^p \in \mathcal{N}(B)$, we know that $CB^py = TB^py = 0$. Since T is invertible, we conclude that $B^py = 0$.

From the fact that $A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = 0$, we have that $x_1 = A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y + CB^{p-1}y \in \mathcal{N}(A)$. Then

$$\begin{cases} A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y - x_1 + CB^{p-1}y = 0, \\ B^py = 0. \end{cases}$$

Thus $B^{p-1}y \in \mathcal{N}(B)$. It induces that $A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y - x_1 = -CB^{p-1}y \in (\mathcal{R}(A) + \mathcal{N}(A)) \cap W \subseteq \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \cap W = \{0\}$, then $B^{p-1}y = 0$ and $A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y = x_1$. Since $x_1 \in \mathcal{N}(A)$, it follows that $A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + CB^{p-2}y \in \mathcal{N}(A^2)$. Let $x_2 = A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + CB^{p-2}y$. Then

$$\begin{cases} A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + ACB^{p-3}y - x_2 + CB^{p-2}y = 0 \\ B^{p-1}y = 0. \end{cases}$$

If we continue this process, we gets

$$\begin{cases} A^2x + ACy - x_{p-1} + CBy = 0 \\ B^2y = 0, \end{cases}$$

where $x_{p-1} \in \mathcal{N}(A^{p-1})$. Then there exists $x_p \in \mathcal{N}(A^p)$ such that

$$\begin{cases} Ax + Cy - x_p = 0 \\ By = 0. \end{cases}$$

Thus $Ax - x_p = -Cy \in \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \cap W = \{0\}$. It follows that $x \in \mathcal{N}(A^{p+1}) = \mathcal{N}(A^p)$ and $y = 0$, so $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(M_C^p)$. Since $\mathcal{N}(M_C^{p+1}) \subseteq \mathcal{N}(M_C^p)$, we get $\mathrm{asc}(M_C) \leq p$. $\square$

4 Point spectrum of M_C

In this section we investigate the one-one property of M_C.

Theorem 4.1. *Suppose that $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ satisfy the following: A is left invertible, $\mathcal{N}(B)$ is complemented, and $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$. Then there exists some $C \in \mathcal{L}(Y, X)$ such that M_C is one-one.*

Proof. There exist closed subspaces V of Y and W of X, such that $Y = \mathcal{N}(B) \oplus V$ and $X = W \oplus \mathcal{R}(A)$. Since $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$, there exists a left invertible operator $C_0 \in \mathcal{L}(\mathcal{N}(B), W)$. Define $C \in \mathcal{L}(Y, X)$ as follows:

$$C = \begin{bmatrix} C_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} W \\ \mathcal{R}(A) \end{bmatrix}.$$

We prove that M_C is injective. Let $z = \begin{bmatrix} x \\ y \end{bmatrix} \in (X \oplus Y)$. From $M_C z = 0$, we have

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then, $Ax + Cy = 0$ and $By = 0$. From the first equation we have $Ax = -Cy \in \mathcal{R}(A) \cap \mathcal{R}(C) \subseteq \mathcal{R}(A) \cap W = \{0\}$. Now, we have $Ax = Cy = 0$. Since A is injective, we get $x = 0$. From $By = 0$, it follows that $y \in \mathcal{N}(B)$. Now, we have $Cy = C_0 y = 0$. Since C_0 is left invertible, it is also injective. From $C_0 y = 0$ we conclude that $y = 0$. Thus, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and this proves that M_C is injective. The proof is completed.

□

As a corollary, we obtain the following result. Notice that $\sigma_l(A)$ denotes the left spectrum of A.

Corollary 4.1. *For the given operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$, we have*

$$\begin{aligned} \bigcap_{C \in \mathcal{L}(X,Y)} \sigma_p(M_C) \subseteq \; & \sigma_l(A) \cup \\ & \cup \{\lambda \in \mathbb{C} : \mathcal{N}(B - \lambda I) \preceq X/\mathcal{R}(A - \lambda I) \text{ does not hold}\} \\ & \cup \{\lambda \in \mathbb{C} : \mathcal{N}(B - \lambda I) \text{ is not complemented in } Y\} \end{aligned}$$

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Milica Z. Kolundžija

Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, P.O. Box 224, 18000 Niš, Serbia

E-mail: `milica.kolundzija@gmail.com`

Dragan S. Djordjević
Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, P.O. Box 224, 18000 Niš, Serbia
E-mail: `dragan@pmf.ni.ac.rs`