PERTURBATIONS OF SPECTRA OF OPERATOR MATRICES

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Abstract. In this article M_C denotes a 2×2 operator matrix of the form $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, which is acting on the product of Banach or Hilbert spaces $X \oplus Y$. We investigate sets $\bigcap\limits_{C \in \mathcal{L}(Y,X)} \sigma_\tau(M_C)$, where $\sigma_\tau(M_C)$ can be equal to the left (right), essential, left (right) Fredholm, Weyl or Browder spectrum of M_C. Thus, generalizations and extensions of various well-known and recent results of H. Du and J. Pan (Proc. Amer. Math. Soc. **121** (1994), 761–766), J. K. Han, H. Y. Lee and W. Y. Lee (Proc. Amer. Math. Soc. **128** (2000), 119–123) and W. Y. Lee (Proc. Amer. Math. Soc. **12**9 (2000), 131–138) are presented.

1. Introduction

Let X, Y and $X \oplus Y$ denote arbitrary infinite dimensional Banach spaces. We use $\mathcal{L}(X,Y)$ to denote the set of all bounded linear operators from X into Y and $\mathcal{L}(X) = \mathcal{L}(X,X)$. We will also consider operators on Hilbert spaces. Thus, H and K are infinite dimensional Hilbert spaces and $H \oplus K$ is their orthogonal sum. We use $\dim H$ to denote the orthogonal dimension of H.

If $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y,X)$, we denote

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{L}(X \oplus Y).$$

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In the case when A, B and C are operators on Hilbert spaces, we will use H instead of X, K instead of Y and the orthogonal sum $H \oplus K$ instead of $X \oplus Y$.

For $T \in \mathcal{L}(X,Y)$ we use $\mathcal{R}(T)$ and $\mathcal{N}(T)$ to denote the range and kernel of T, respectively.

We use $\mathcal{G}_l(X)$ and $\mathcal{G}_r(X)$, respectively, to denote the set of all left and right invertible operators on X. It is well-known that if $T \in \mathcal{L}(X)$, then $T \in \mathcal{G}_l(X)$ if and only if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is a closed and complemented subspace of X. Also, $T \in \mathcal{G}_r(X)$ if and only if $\mathcal{R}(T) = X$ and $\mathcal{N}(T)$ is a complemented subspace of X. The set of all invertible operators on X is denoted by $\mathcal{G}(X)$.

Let $T \in \mathcal{L}(X)$, $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim X/\mathcal{R}(T)$. Sets of left and right Fredholm operators, respectively, are defined as

$$\Phi_l(X) = \{T \in \mathcal{L}(X) : \mathcal{R}(T) \text{ is a closed and complemented subspace of } X \\ \text{and } \alpha(T) < \infty\}$$

and

$$\Phi_r(X) = \{T \in \mathcal{L}(X) : \mathcal{N}(T) \text{ is a complemented subspace of } X \\ \text{and } \beta(T) < \infty\}.$$

The set of Fredholm operators is defined as

$$\Phi(X) = \Phi_l(X) \cap \Phi_r(X) = \{T \in \mathcal{L}(X) : \alpha(T) < \infty \text{ and } \beta(T) < \infty\}.$$

For a left or right Fredholm operator T the index is defined as $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. The set of Weyl operators is defined as

$$\Phi_0(X) = \{T \in \Phi(X) : \mathrm{ind}(T) = 0\}.$$

For $T \in \mathcal{L}(X)$ consider the following inclusions: $\{0\} \subset \mathcal{N}(T) \subset \mathcal{N}(T^2) \subset \cdots$ and $X \supset \mathcal{R}(T) \supset \mathcal{R}(T^2) \supset \cdots$. The ascent of T, denoted by $\mathrm{asc}(T)$, is defined as the least k (if it exists) for which $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$ holds. If such k does not exist, then we say that the ascent of A is equal to infinity. The descent of T, denoted by $\operatorname{des}(T)$, is defined as the least k (if it exists) for which $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$ is satisfied. If such k does not exist, then we say that the descent of A is equal to infinity. If the ascent and the descent of T are finite, then they are equal [2].

The Drazin inverse of $T \in \mathcal{L}(X)$ is the unique operator $T^D \in \mathcal{L}(X)$ satisfying

$$T^{k+1}A^D = T^k, \quad T^D T T^D = T \quad \text{and} \quad TT^D = T^D T$$

for some nonnegative integer k. The least k in the previous definition is known as the Drazin index of T. It is well-known that T^D exists if and only if $p = \operatorname{asc}(T) = \operatorname{des}(T) < \infty$. In this case the Drazin index of T is equal to p [1].

The set of Browder operators on X is defined as

$$\begin{aligned} \mathcal{B}(X) &= \{T \in \Phi(X) : \operatorname{asc}(T) = \operatorname{des}(T) < \infty\} \\ &= \{T \in \Phi(X) : T^D \text{ exists}\} \\ &= \{T \in \Phi(X) : 0 \notin \operatorname{acc}\sigma(T)\}. \end{aligned}$$

Corresponding spectra of an operator $T \in \mathcal{L}(X)$ are defined as:

the left spectrum: $\sigma_l(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}_l(X)\}$,

the right spectrum: $\sigma_r(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}_r(X)\}$,

the spectrum: $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}(X)\}$,

the left Fredholm spectrum: $\sigma_{le}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_l(T)\}$,

the right Fredholm spectrum: $\sigma_{re}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_r(T)\}$,

the essential spectrum: $\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(X)\}$,

the Weyl spectrum $\sigma_w(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_0(X)\}$,

the Browder spectrum $\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}(X)\}$.

Also, the approximate point spectrum of $T \in \mathcal{L}(X)$ is defined as $\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}$. The defect spectrum of $T \in \mathcal{L}(X)$ is defined as $\sigma_d(T) = \{\lambda \in \mathbb{C} : \mathcal{R}(T - \lambda) \neq X\}$.

All of these spectra are compact nonempty subsets of the complex plane.

An operator $T \in \mathcal{L}(X, Y)$ is g-invertible (or relatively regular), provided that there exists an operator $S \in \mathcal{L}(Y, X)$ satisfying $TST = T$ and $STS = S$. S is known as a g-inverse of T. It is well-known that T is g invertible if and only if $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively, are closed and complemented subspaces of Y and X. In this case TS is the projection from Y onto $\mathcal{R}(T)$ parallel to $\mathcal{N}(S)$ and ST is the projection from X onto $\mathcal{R}(S)$ parallel to $\mathcal{N}(T)$.

Operators in sets $\mathcal{G}_l(X)$, $\mathcal{G}_r(X)$, $\Phi_l(X)$ and $\Phi_r(X)$ are g-invertible operators.

In this article we use a notation of the "relatively regular" spectrum: if $T \in \mathcal{L}(X)$, then $\sigma_g(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not } g\text{-invertible}\}$. Also, the point spectrum of T is denoted by $\sigma_p(T)$. Sets $\sigma_g(T)$ and $\sigma_p(T)$ are not necessary closed and may be empty.

For a Banach space X we use X' to denote the dual space of X. If $T \in \mathcal{L}(X, Y)$, then $T' \in \mathcal{L}(Y', X')$ is the dual operator of T. It is well-known that $(X \oplus Y)' = X' \oplus Y'$ holds. Thus, if $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{L}(X \oplus Y)$, then $M_C' = \begin{bmatrix} A' & 0 \\ C' & B' \end{bmatrix} \in \mathcal{L}(X' \oplus Y')$.

If S is a subset of X, then $S^\circ = \{f \in X' : f|_S = 0\}$ is the annihilator of S.

For a subset K of $\mathbb{C}$ we use $\mathrm{acc}(K)$, $\mathrm{int}(K)$ and ∂K, respectively, to denote the set of all points of accumulation of K, the interior of K and the boundary of K.

Recall that $\partial\sigma(T) \subset \sigma_a(T) \cap \sigma_d(T)$ holds for all $T \in \mathcal{L}(X)$.

The set

$$\bigcap_{C \in \mathcal{L}(K,H)} \sigma(M_C),$$

where $H \oplus K$ is the orthogonal sum of Hilbert spaces H and K, is determined in [4]. Some related results concerning the essential and Weyl spectra for operators on Banach spaces X and Y are proved in [7] and [11].

In this article we determine sets

$$\bigcap_{C\in\mathcal{L}(Y,X)} \sigma_\tau(M_C),$$

where we take $\sigma_\tau(M_C)$ to be equal to one of the following parts of the spectrum: $\sigma_e(M_C)$, $\sigma_w(M_C)$, $\sigma_b(M_C)$, $\sigma_{le}(M_C)$, $\sigma_{re}(M_C)$, $\sigma_l(M_C)$ and $\sigma_r(M_C)$.

The paper is organized as follows. In Section 2 auxiliary results and notions are presented. The essential, Weyl and Browder spectra are investigated in Section 3. Some special classes of operators and related results are also considered in Section 3. In this section we obtain extensions of results from [7] and [11]. In Section 4 we investigate the left and right Fredholm spectra. Finally, in Section 5 we consider perturbations of the left and right spectra. Here a generalization of results from [4] is obtained.

2. Auxiliary results

In this section we recall or introduce necessary notions and results. It is well-known that the product of two relatively regular operators need not to be relatively regular. On the other hand, if a product of two relatively regular operators is again relatively regular, then the following "ghost" of the index theorem can be proved:

Lemma 2.1 (Harte [8]). *If $T\in\mathcal{L}(X,Y)$, $S\in\mathcal{L}(Y,Z)$ and $ST\in\mathcal{L}(X,Z)$ are relatively regular, then*

$$\mathcal{N}(T)\times\mathcal{N}(S)\times Z/\mathcal{R}(ST)\cong\mathcal{N}(ST)\times Y/\mathcal{R}(T)\times Z/\mathcal{R}(S).$$

We introduce the notion of an "isomorphism up to a finite dimensional subspace".

Definition 2.1. We say that two Banach spaces U and V are isomorphic up to a finite dimensional subspace, if one of the following statements hold:

(a) there exists a bounded below operator $J_1 : U \to V$, such that $\dim V/J_1(U) < \infty$, or
(b) there exists a bounded below operator $J_2 : V \to U$, such that $\dim U/J_2(V) < \infty$.

This definition is reminiscent to a condition introduced by V. Müller [14]: If M_1 and M_2 are subspaces of X, then we write $M_1 \subset_e M_2$ if there exists a finite dimensional subspace $F \subset X$ such that $M_1 \subset M_2 + F$. This condition is also used in [9], [13] and [10].

Firstly, we prove the following auxiliary result.

Lemma 2.2. *Let X and Y be Banach spaces, and let M and N be finite dimensional spaces. If $M \oplus X \overset{\sim}{=} N \oplus Y$, then X and Y are isomorphic up to a finite dimensional subspace. Particularly, if $\dim M = \dim N$, then $X \overset{\sim}{=} Y$.*

Proof. Let $\dim M = m$, $\dim N = n$ and $J : M \oplus X \to N \oplus Y$ be a Banach spaces isomorphism. Let $x_1, \dots, x_k \in X$ denote the system of all linearly independent vectors in X, such that $Jx_1, \dots, Jx_k$ are linearly independent modulo Y. Obviously, $0 \leq k \leq n$. There exists a system of $n-k$ vectors $z_1, \dots, z_{n-k}$ in $N \oplus Y$, which are linearly independent modulo $span\{Jx_1, \dots, Jx_k\} \oplus Y$. It follows that $0 \leq n-k \leq n$. Denote by $y_i = J^{-1}z_i$ for all $i = 1, \dots, n-k$, in the case when $n-k > 0$. All vectors $y_1, \dots, y_{n-k}$ must be linearly independent modulo X. In general, it follows that $0 \leq n-k \leq m$. There exists a system of exactly $l = m-(n-k)$ vectors $u_1, \dots u_l$, which are linearly independent modulo $span\{y_1, \dots, y_{n-k}\} \oplus X$. There exists a Banach space X_1, such that

$$span\{x_1, \dots, x_k\} \oplus X_1 = X$$

and

$$M \oplus X = span\{y_1, \dots, y_{n-k}\} \oplus span\{u_1, \dots, u_l\} \oplus span\{x_1, \dots, x_k\} \oplus X_1.$$

Let $v_i = Ju_i$, $i = 1, \dots, l$. Vectors $v_1, \dots, v_l$ are linearly independent modulo

$$span\{Jx_1, \dots, Jx_k\} \oplus span\{z_1, \dots, z_{n-k}\}.$$

Let $Y_1 = J(X_1)$. Then Y_1 is closed, $X_1 \cong Y_1$ and

$$N \oplus Y = span\{Jx_1, \dots, Jx_k\} \oplus span\{z_1, \dots, z_{n-k}\} \oplus span\{v_1, \dots, v_l\} \oplus Y_1.$$

Since $span\{Jx_1, \dots, Jx_k\} \oplus span\{z_1, \dots, z_{n-k}\}$ is linearly independent modulo Y, we conclude

$$N \oplus Y = span\{Jx_1, \dots, Jx_k\} \oplus span\{z_1, \dots, z_{n-k}\} \oplus Y.$$

Hence,

$$\begin{aligned} Y &\cong \frac{N \oplus Y}{span\{Jx_1, \dots, Jx_k\} \oplus span\{z_1, \dots, z_{n-k}\}} \\ &\cong span\{v_1, \dots, v_l\} \oplus Y_1. \end{aligned}$$

We have to add an k-dimensional subspace to X_1, to get a space which is isomorphic to X. We have to add an l-dimensional subspace to Y_1 to get a space which is isomorphic to Y. Since $X_1 \cong Y_1$, we conclude that X and Y are isomorphic up to a finite dimensional subspace.

Particularly, if $m = n$ then $k = l$, so X and Y are isomorphic. □

Remark 2.1. If H and K are Hilbert spaces, then H and K are isomorphic up to a finite dimensional subspace if and only if either H and K are both finite dimensional, or $H \cong K$.

Using the idea from [15], we introduce the following class of operators:

$$\mathcal{S}_+(X) = \{T \in \mathcal{L}(X) : \alpha(\lambda - T) \geq \beta(\lambda - T) \text{ if at least one of these quantities is finite }\},$$
$$\mathcal{S}_-(X) = \{T \in \mathcal{L}(X) : \alpha(\lambda - T) \leq \beta(\lambda - T) \text{ if at least one of these quantities is finite }\}.$$

Recall that an operator $T \in \mathcal{L}(H)$ is quasihyponormal, if $\|T^*Tx\| \leq \|T^2x\|$ holds for all $x \in H$. The class of all quasihyponormal operators on a Hilbert space H is contained in $\mathcal{S}_-(H)$ (see [5]).

We need the following result concerning the Drazin inverse.

Lemma 2.3. *If* $\operatorname{asc}(A) = \operatorname{des}(A) = k$ *and* $\operatorname{asc}(B) = \operatorname{des}(B) = l$*, then the Drazin inverse of* M_C *exists for any* $C \in \mathcal{L}(Y, X)$ *and has the form*

$$M_C^D = \begin{bmatrix} A^D & S \\ 0 & B^D, \end{bmatrix}$$

where

$$S = (A^D)^2 \left[\sum_{n=0}^{l-1} (A^D)^n C B^n \right] (I - BB^D) + (I - AA^D) \left[\sum_{n=0}^{k-1} A^n C (B^D)^n \right] (B^D)^2 - A^D C B^D.$$

If M_C *has the Drazin inverse and* $0 \notin \operatorname{acc} \sigma(A) \cup \operatorname{acc} \sigma(B)$*, then* A *and* B *also have Drazin inverses.*

The result from Lemma 2.3 is well-known for matrices (see the paper of Meyer and Rose [12]). A complete proof for Banach space operators, which is based on the expansion of the resolvent, is given in [3].

Notice that a special case of Lemma 2.3 appears when $A = 0$ and $B = 0$. If $C \neq 0$, then $\operatorname{ind}(M_C) = 2$ and $(M_C)^d = 0$.

3. Perturbations of the essential, Weyl and Browder spectra

In this section X and Y are infinite dimensional Banach spaces. We formulate the following statement.

Proposition 3.1. *For given $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y, X)$, the following inclusion holds:*

$$\sigma_e(M_C) \subset \sigma_e(A) \cup \sigma_e(B).$$

Particularly, if $A \in \Phi(X)$ and $B \in \Phi(Y)$, then $M_C \in \Phi(X \oplus Y)$ for all $C \in \mathcal{L}(Y, X)$.

Proof. We set $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Since $\mathcal{N}(M-\lambda) = \mathcal{N}(A-\lambda) \oplus \mathcal{N}(B-\lambda)$ and $\mathcal{R}(M-\lambda) = \mathcal{R}(A-\lambda) \oplus \mathcal{R}(B-\lambda)$, it follows that $\sigma_{re}(M) = \sigma_{re}(A) \cup \sigma_{re}(B)$. Observe that

$$\begin{bmatrix} I & 0 \\ 0 & nI \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{n}I \end{bmatrix} = \begin{bmatrix} A & \frac{1}{n}C \\ 0 & B \end{bmatrix} \to M \text{ as } n \to \infty.$$

It is well-known that similar operators have the same essential spectrum, so $\sigma_{re}(M_C) = \sigma_e\left(\begin{bmatrix} A & \frac{1}{n}C \\ 0 & B \end{bmatrix}\right)$. Now, the proof follows from the upper semicontinuity of the essential spectrum. $\square$

Extending the method of J. K. Han, H. Y. Lee and W. Y. Lee from [7], we formulate the basic theorem considering the essential invertibility of M_C.

Theorem 3.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given and consider the statements:*

(1) *$M_C \in \Phi(X \oplus Y)$ for some $C \in \mathcal{L}(Y, X)$.*

(2) (2a) *$A \in \Phi_l(X)$;*

(2b) *$B \in \Phi_r(Y)$;*

(2c) *$\mathcal{N}(B)$ and $X/\overline{\mathcal{R}(A)}$ are isomorphic up to a finite dimensional subspace.*

Then (1) $\Longleftrightarrow$ (2).

Proof. (1) $\Longrightarrow$ (2) This implication is proved in [7]. For the convenience of the reader, we give a complete proof.

Let $M_C \in \Phi(X \oplus Y)$ for some $C \in \mathcal{L}(Y, X)$ and denote $B_1 = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}$, $C_1 = \begin{bmatrix} I & C \\ 0 & I \end{bmatrix}$, $A_1 = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$. Obviously, C_1 is invertible in $\mathcal{L}(X \oplus Y)$. From

$$M_C = B_1 C_1 A_1 \in \Phi(X \oplus Y)$$

it follows that $B_1, B_1C_1 \in \Phi_r(X \oplus Y)$ and $A_1, C_1A_1 \in \Phi_l(X \oplus Y)$. We conclude $A \in \Phi_l(X)$ and $B \in \Phi_r(Y)$. Hence (2a) and (2b) are proved.

Applying Lemma 2.1 to $M_C = (B_1C_1)A_1$ we get

$$\mathcal{N}(A) \times \mathcal{N}(B_1C_1) \times (X \oplus Y)/\mathcal{R}(M_C) \stackrel{\sim}{=} \mathcal{N}(M_C) \times X/\mathcal{R}(A) \times Y/\mathcal{R}(B).$$

Applying Lemma 2.1 to B_1C_1 we get

$$\mathcal{N}(B) \times Y/\mathcal{R}(B) \stackrel{\sim}{=} \mathcal{N}(B_1C_1) \times Y/\mathcal{R}(B).$$

Since $\beta(B) < \infty$, from Lemma 2.2 we obtain

$$\mathcal{N}(B) \stackrel{\sim}{=} \mathcal{N}(B_1C_1).$$

Finally, the following holds

$$\mathcal{N}(A) \times \mathcal{N}(B) \times (X \oplus Y)/\mathcal{R}(M_C) \stackrel{\sim}{=} \mathcal{N}(M_C) \times X/\mathcal{R}(A) \times Y/\mathcal{R}(B). \tag{3.1}$$

Since $\mathcal{N}(A)$, $(X \oplus Y)/\mathcal{R}(M_C)$, $\mathcal{N}(M_C)$ and $Y/\mathcal{R}(B)$ are finite dimensional spaces, from Lemma 2.2 it follows that $\mathcal{N}(B)$ and $Y/\mathcal{R}(A)$ are isomorphic up to a finite dimensional subspace. Thus, (2c) is proved.

(2) $\Longrightarrow$ (1) Suppose that $A \in \Phi_l(X)$, $B \in \Phi_r(Y)$ and spaces $\mathcal{N}(B)$ and $Y/\mathcal{R}(A)$ are isomorphic up to a finite dimensional subspace. There exist closed subspaces U and V of X and Y, respectively, such that $\mathcal{R}(A) \oplus U = X$ and $\mathcal{N}(B) \oplus V = Y$. We consider two cases.

Case I. Suppose that there exists a bounded below operator $J : \mathcal{N}(B) \to U$, such that $\dim U/J(\mathcal{N}(B)) < \infty$. There exists a finite dimensional subspace W of X, such that $J(\mathcal{N}(B)) \oplus W = U$. We define $C \in \mathcal{L}(Y, X)$ as follows:

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \longrightarrow \begin{bmatrix} J(N(B)) \\ W \\ \mathcal{R}(A) \end{bmatrix}.$$

Obviously, $\mathcal{R}(C) = J(\mathcal{N}(B))$. Now $\mathcal{R}(M_C) = [\mathcal{R}(A) \oplus J(\mathcal{N}(B))] \oplus \mathcal{R}(B)$ and $\dim(X \oplus Y)/\mathcal{R}(M_C) = \dim W + \beta(B) < \infty$. It also follows that $\mathcal{R}(M_C)$ is closed. On the other hand, if $M_C \begin{bmatrix} x \\ y \end{bmatrix} = 0$, then $y \in \mathcal{N}(B)$ and $Ax = -Cy$, implying $x \in \mathcal{N}(A)$ and $y = 0$. We get that $\mathcal{N}(M_C) = \mathcal{N}(A)$, so $M_C \in \Phi(X \oplus Y)$.

Case II. Let there exist a bounded below operator $J : U \to \mathcal{N}(B)$, such that $\dim \mathcal{N}(B)/J(U) < \infty$. There exists a finite dimensional subspace Z of $\mathcal{N}(B)$, such that $\mathcal{N}(B) = J(U) \oplus Z$. Let $J_1 : J(U) \to U$ denote the inverse of the truncation $J : U \to J(U)$ and define $C \in \mathcal{L}(Y, X)$ as follows:

$$C = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} J(U) \\ Z \\ V \end{bmatrix} \longrightarrow \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

Obviously, $\mathcal{R}(C) = U$. We conclude that $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$, so $\dim(X \oplus Y)/\mathcal{R}(M_C) = \beta(B) < \infty$ and $\mathcal{R}(M_C)$ is closed. Also, $\mathcal{N}(M_C) = \mathcal{N}(A) \oplus Z$, so it follows that $M_C \in \Phi(X \oplus Y)$. $\square$

Immediately, we get the following corollary, concerning perturbations of the essential spectrum.

Corollary 3.1. *For given $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ the following holds:*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}(A, B),$$

where

$\mathcal{W}(A, B) = \{\lambda \in \mathbb{C} : \mathcal{N}(B - \lambda)$ *and* $X/\overline{\mathcal{R}(A - \lambda)}$ *are not isomorphic up to a finite dimensional subspace*$\}$.

Now, we know which part of the set $\sigma_e(A)\cup\sigma_e(B)$ may be perturbed out by choosing a suitable operator $C\in\mathcal{L}(Y,X)$.

Proposition 3.2. *Assume that there exists an operator $C\in\mathcal{L}(Y,X)$, such that the inclusion $\sigma_e(M_C)\subset\sigma_e(A)\cup\sigma_e(B)$ is proper. Then for any $\lambda\in[\sigma_e(A)\cup\sigma_e(B)]\setminus\sigma_e(M_C)$ it follows that $\lambda\in\sigma_e(A)\cap\sigma_e(B)$.*

Proof. Suppose that $\lambda\in[\sigma_e(A)\setminus\sigma_e(B)]\setminus\sigma_e(M_C)$. Then $A-\lambda\notin\Phi(X)$ and $B-\lambda\in\Phi(Y)$. Since $\alpha(B-\lambda)<\infty$, from Corollary 3.1 we conclude $\beta(A-\lambda)<\infty$. It follows that $\lambda\notin\sigma_e(A)$, and it is in contradiction with the choice of λ. Thus,

$$[\sigma_e(A)\setminus\sigma_e(B)]\setminus\sigma_e(M_C)=\emptyset.$$

Analogously, we can prove

$$[\sigma_e(B)\setminus\sigma_e(A)]\setminus\sigma_e(M_C)=\emptyset.$$

Thus, the theorem is proved. □

For operators in classes $\mathcal{S}_+(X)$ and $\mathcal{S}_-(Y)$ Proposition 3.1 and Theorem 3.1 become more precise.

Proposition 3.3. *If $A\in\mathcal{S}_+(X)$ or $B\in\mathcal{S}_-(Y)$, then for all $C\in\mathcal{L}(Y,X)$ we have*

$$\sigma_e(M_C)=\sigma_e(A)\cup\sigma_e(B).$$

Proof. By Proposition 3.1 it is enough to prove the inclusion $\supset$. Suppose that $\lambda\in[\sigma_e(A)\cup\sigma_e(B)]\setminus\sigma_e(M_C)$. Then $A-\lambda\in\Phi_l(X)$, $B-\lambda\in\Phi_r(Y)$ and $\mathcal{N}(B-\lambda)$ and $X/\mathcal{R}(A-\lambda)$ are isomorphic up to a finite dimensional subspace.

If $A\in\mathcal{S}_+(X)$, then $\beta(A-\lambda)\le\alpha(A-\lambda)<\infty$ and $A-\lambda\in\Phi(X)$. Hence, $\mathcal{N}(B-\lambda)$ must be a finite dimensional subspace and $B-\lambda\in\Phi(Y)$.

If $B \in \mathcal{S}_-(Y)$, then $\alpha(B-\lambda) \leq \beta(B-\lambda) < \infty$ and $B-\lambda \in \Phi(Y)$. Then $X/\mathcal{R}(A-\lambda)$ must be a finite dimensional space.

In both cases we obtain $A-\lambda \in \Phi(X)$ and $B-\lambda \in \Phi(Y)$, which is in contradiction with our assumption $\lambda \in \sigma_e(A) \cup \sigma_e(B)$. $\square$

Now, we consider the Weyl spectrum of M_C.

Theorem 3.2. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given and consider the statements:*

(1) $M_C \in \Phi_0(X \oplus Y)$ *some* $C \in \mathcal{L}(Y, X)$.
(2) (2a) $A \in \Phi_l(X)$;
(2b) $B \in \Phi_r(Y)$;
(2c) $\mathcal{N}(A) \oplus \mathcal{N}(B) \cong X/\overline{\mathcal{R}(A)} \oplus Y/\overline{\mathcal{R}(B)}$.

Then (1) $\Longleftrightarrow$ (2).

Proof. (1) $\Longrightarrow$ (2) W. Y. Lee proved this implication in [11] (actually, it follows from (3.1) and Lemma 2.2).

(2) $\Longrightarrow$ (1) Let $A \in \Phi_l(X)$, $B \in \Phi_r(Y)$ and

$$\mathcal{N}(A) \oplus \mathcal{N}(B) \cong X/\mathcal{R}(A) \oplus Y/\mathcal{R}(B). \tag{3.2}$$

There exist closed subspaces Z and V, such that $X = \mathcal{R}(A) \oplus Z$ and $Y = \mathcal{N}(B) \oplus V$. We consider three cases.

Case I. Let $\alpha(A) = \beta(B) (< \infty)$. From (3.2) and Lemma 2.2 it follows $\mathcal{N}(B) \cong X/\mathcal{R}(A)$ and let $J : \mathcal{N}(B) \to Z$ denote an arbitrary isomorphism. Define $C \in \mathcal{L}(Y, X)$ as follows:

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \longrightarrow \begin{bmatrix} Z \\ \mathcal{R}(A) \end{bmatrix}.$$

We get that $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$, $\mathcal{N}(M_C) = \mathcal{N}(A)$ and M_C is Weyl.

Case II. Let $\alpha(A) < \beta(B) (< \infty)$. From (3.2) and Lemma 2.2 it follows that there exists a bounded below operator $J : Z \to \mathcal{N}(B)$, such that

$\dim \mathcal{N}(B)/J(Z) = \beta(B) - \alpha(A)$. The truncation $J : Z \to J(Z)$ is invertible, so let $J_1 : J(Z) \to Z$ denote the inverse of this truncation. There exists a finite dimensional subspace Z_1, such that $J(Z) \oplus Z_1 = \mathcal{N}(B)$ and $\dim Z_1 = \beta(B) - \alpha(A)$. Define $C \in \mathcal{L}(Y, X)$ as

$$C = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} J(Z) \\ Z_1 \\ V \end{bmatrix} \longrightarrow \begin{bmatrix} Z \\ \mathcal{R}(A) \end{bmatrix}.$$

We get that $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$ and $\mathcal{N}(M_C) = \mathcal{N}(A) \oplus Z_1$, so we conclude that M_C is Weyl.

Case III. Let $\beta(B) < \alpha(A)(< \infty)$. From (3.2) and Lemma 2.2 it follows that there exists a bounded below operator $J : \mathcal{N}(B) \to Z$, such that $\dim Z/J(\mathcal{N}(B)) = \alpha(A) - \beta(B)$. There exists a finite dimensional subspace Z_2 such that $J(\mathcal{N}(B)) \oplus Z_2 = Z$ and $\dim Z_2 = \alpha(A) - \beta(B)$. We define $C \in \mathcal{L}(Y, X)$ as

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \longrightarrow \begin{bmatrix} J(\mathcal{N}(B)) \\ Z_2 \\ \mathcal{R}(A) \end{bmatrix}.$$

It follows that $\mathcal{R}(M_C) = [\mathcal{R}(A) \oplus J(\mathcal{N}(B))] \oplus \mathcal{R}(B)$, $\mathcal{N}(M_C) = \mathcal{N}(A)$, and we conclude that M_C is Weyl. $\square$

As a corollary, we get the following result.

Corollary 3.2. *For given $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ the following holds:*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_w(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}_0(A, B),$$

where

$$\mathcal{W}_0(A, B) = \{\lambda \in \mathbb{C} : \mathcal{N}(A - \lambda) \oplus \mathcal{N}(B - \lambda) \textit{ is not isomorphic to } X/\overline{\mathcal{R}(A - \lambda)} \oplus Y/\overline{\mathcal{R}(B - \lambda)}\}.$$

Now we formulate the following result for the Browder spectrum.

Corollary 3.3. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given. Consider the following statements:*

(1) (1a) $A \in \Phi_l(X)$;

(1b) $B \in \Phi_r(Y)$;

(1c) $\mathcal{N}(B)$ *and* $X/\overline{\mathcal{R}(A)}$ *are isomorphic up to a finite dimensional subspace;*

(1d) A *has the Drazin inverse;*

(1e) B *has the Drazin inverse.*

(2) $M_C \in \mathcal{B}(X \oplus Y)$ *for some* $C \in \mathcal{L}(Y, X)$.

Then (1) $\Longrightarrow$ (2).

Moreover, if $0 \notin \mathrm{acc}(\sigma(A) \cup \sigma(B))$, *then* (1) $\Longleftrightarrow$ (2).

Proof. The proof follows from Theorem 3.1 and Lemma 2.3. $\square$

More details will be obtained for perturbations of the Browder spectrum.

Theorem 3.3. *If $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$, then*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_b(M_C) \subset \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}(A,B) \cup \mathcal{W}_1(A,B), \tag{3.3}$$

where $\mathcal{W}(A,B)$ is defined in Corollary 3.1 and

$$\mathcal{W}_1(A,B) = \{\lambda \in \mathbb{C} : \ (A-\lambda)^D \text{ does not exist, or } (B-\lambda)^D \text{ does not exist}\}.$$

If $\mathrm{acc}\,\sigma(A) \cup \mathrm{acc}\,\sigma(B) = \emptyset$, *then equality holds in* (3.3).

If $\sigma_a(A) = \sigma(A)$ *and* $\sigma_d(B) = \sigma(B)$, *then equality holds in* (3.3).

If $\sigma(A) \cup \sigma(B)$ *does not have interior points, then equality holds in* (3.3).

Proof. The result (3.3) follows immediately from Theorem 3.3. If a trivial assumption $\mathrm{acc}\,\sigma(A) \cup \mathrm{acc}\,\sigma(B) = \emptyset$ is satisfied, from Theorem 3.3 it follows that equality holds in (3.3).

Suppose that $\sigma_a(A) = \sigma(A)$ and $\sigma_d(B) = \sigma(B)$ hold and

$$\lambda \notin \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_b(M_C).$$

There exists some $C \in \mathcal{L}(Y,X)$ such that $M_C - \lambda \in \mathcal{B}(X \oplus Y)$. From Theorem 3.1 it follows that $A-\lambda \in \Phi_l(X)$, $B-\lambda \in \Phi_r(X)$ and $X/\mathcal{R}(A-\lambda)$ isomorphic to $\mathcal{N}(B-\lambda)$ up to a finite dimensional subspace. Let $\mathrm{asc}(M_C-\lambda) = \mathrm{des}(M_C-\lambda) = p < \infty$. Also, $\lambda \notin \mathrm{acc}\,\sigma(M_C)$. Hence, there exists an $\epsilon > 0$, such that if $0 < |z-\lambda| < \epsilon$, then $z \notin \sigma(M_C - \lambda)$. For such z the operator $M_C - \lambda - z$ is invertible and it is easy to prove that $A - \lambda - z$ is left invertible and $B - \lambda - z$ is right invertible. It follows that $\lambda \notin \mathrm{acc}\,\sigma_a(A) \cup \mathrm{acc}\,\sigma_d(B) = \mathrm{acc}\,\sigma(A) \cup \mathrm{acc}\,\sigma(B)$. From Lemma 2.3 it follows that $A-\lambda$ and $B-\lambda$ have Drazin inverses.

Let $\mathrm{int}(\sigma(A) \cup \sigma(B)) = \emptyset$. If $\lambda \notin \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_b(M_C)$, in the same way as above we can prove that $\lambda \notin \mathrm{acc}\,\sigma_a(A) \cup \mathrm{acc}\,\sigma_d(B)$. We will prove that $\lambda \notin \mathrm{acc}\,\sigma(A) \cup \mathrm{acc}\,\sigma(B)$. Since λ can not be an interior point of $\sigma(A) \cup \sigma(B)$, it follows that λ must be a boundary point of $\sigma(A) \cup \sigma(B)$. If $\lambda \in \mathrm{acc}\,\sigma(A)$, then there exists a sequence $(x_n)_n$, $x_n \in \partial\sigma(A) \subset \sigma_a(A)$, such that $\lim x_n = \lambda$. It follows that $\lambda \in \mathrm{acc}\,\sigma_a(A)$ and this is in contradiction with our previous statement $\lambda \notin \mathrm{acc}\,\sigma_a(A) \cup \mathrm{acc}\,\sigma_d(B)$. We conclude that $\lambda \notin \mathrm{acc}\,\sigma(A)$. Similarly, since $\partial\sigma(B) \subset \sigma_d(B)$, we get $\lambda \notin \mathrm{acc}\,\sigma(B)$.

Now, from Lemma 2.3 it follows that $A-\lambda$ and $B-\lambda$ have Drazin inverses. $\square$

4. Perturbations of the left and right Fredholm spectra

We formulate an analogous statement as Proposition 3.1.

Proposition 4.1. *For given $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y,X)$, the following inclusion holds:*

$$\sigma_{re}(M_C) \subset \sigma_{re}(A) \cup \sigma_{re}(B).$$

Particularly, if $A \in \Phi_r(X)$ and $B \in \Phi_r(Y)$, then $M_C \in \Phi_r(X \oplus Y)$ for all $C \in \mathcal{L}(Y,X)$.

The notion of the embedded spaces is introduced.

Definition 4.1. Let X and Y be Banach spaces. We say that X can be embedded in Y and write $X \preceq Y$ if and only if there exists a left invertible operator $J : X \to Y$. We say that X can essentially be embedded in Y and write $X \prec Y$, if and only $X \preceq Y$ and $Y/T(X)$ is an infinite dimensional linear space for all $T \in \mathcal{L}(X, Y)$.

Remark 4.1. Obviously, $X \preceq Y$ if and only if there exists a right invertible operator $J_1 : Y \to X$.

If H and K are Hilbert spaces, then $H \preceq K$ if and only if $\dim H \leq \dim K$. Also, $H \prec K$ if and only if $\dim H < \dim K$ and K is infinite dimensional. Here $\dim H$ denotes the orthogonal dimension of H.

The main result of this section follows.

Theorem 4.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given operators and consider the following statements:*

(1) (1a) $B \in \Phi_r(Y)$;
 (1b) $(A \in \Phi_r(X))$, *or* ($\mathcal{R}(A)$ *is closed and complemented in* X *and* $X/\mathcal{R}(A) \preceq \mathcal{N}(B)$).

(2) $M_C \in \Phi_r(X \oplus Y)$ *for some* $C \in \mathcal{L}(Y, X)$.

(3) (3a) $B \in \Phi_r(Y)$;
 (3b) $A \in \Phi_r(X)$, *or* $\mathcal{R}(A)$ *is not closed, or* $\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$ *does not hold.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) Let $B \in \Phi_r(Y)$. If $A \in \Phi_r(X)$, then from Proposition 4.1 we get that $M_C \in \Phi_r(X \oplus Y)$ for all $C \in \mathcal{L}(Y, X)$.

Hence, assume $B \in \Phi_r(Y)$, $A \notin \Phi_r(X)$, $\mathcal{R}(A)$ is closed and complemented in X and $X/\mathcal{R}(A) \preceq \mathcal{N}(B)$. There exists a closed subspace U of X such that $\mathcal{R}(A) \oplus U = X$. Let $J : U \to \mathcal{N}(B)$ be a left invertible operator and $J_1 : \mathcal{N}(B) \to U$ its left inverse. There exists a closed subspace V of Y such that $\mathcal{N}(B) \oplus V = Y$. Define an operator $C \in \mathcal{L}(Y, X)$ in the following way:

$$C = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

Then $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$ and $\beta(M_C) = \beta(B) < \infty$. Hence, $M_C \in \Phi_r(X \oplus Y)$.

(2) $\Longrightarrow$ (3) Let $M_C \in \Phi_r(X \oplus Y)$ for some $C \in \mathcal{L}(Y, X)$. Then $\mathcal{R}(M_C) \subset [\mathcal{R}(A) + \mathcal{R}(C)] \oplus \mathcal{R}(B)$. If $x_1, \dots, x_n \in X$ are linearly independent modulo $\mathcal{R}(A) + \mathcal{R}(C)$ and $y_1, \dots, y_m \in Y$ are linearly independent modulo $\mathcal{R}(B)$, then $n + m \le \beta(M_C) < \infty$. Hence, $\beta(B) < \infty$ and $B \in \Phi_r(Y)$. Thus, we have proved the statement (3a).

Moreover, assume that the statement (3b) does not hold. Then $A \notin \Phi_r(X)$, $\mathcal{R}(A)$ is closed and $\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$. It follows that $X/\mathcal{R}(A)$ is an infinite dimensional space and hence $\dfrac{X}{\mathcal{R}(A) + C(\mathcal{N}(B))}$ is an infinite dimensional linear space. Let $z_1, \dots, z_n \in X$ be linearly independent modulo $\mathcal{R}(A) + C(\mathcal{N}(B))$. We will prove that $z_1, \dots, z_n$ are linearly independent modulo $(X \oplus Y)/\mathcal{R}(M_C)$. Suppose that there exist complex numbers $\alpha_1, \dots \alpha_n$, such that $\alpha_1 z_1 + \cdots \alpha_n z_n = z \in \mathcal{R}(M_C)$. Then there exists a vector $x \in X \oplus Y$, such that $M_C x = z$. We can find $u \in X$ and $v \in Y$ such that $x = u + v$. Since $z = (Au + Cv) \oplus Bv \in X$ and $Bv \in Y$, we get $Bv = 0$ and $\alpha_1 z_1 \cdots + \alpha_n z_n = z \in \mathcal{R}(A) + C(\mathcal{N}(B))$. This is in contradiction with the choice of $z_1, \dots, z_n$, so $z_1, \dots, z_n \in X$ must be linearly independent modulo $\mathcal{R}(M_C)$. It follows that $(X \oplus Y)/\mathcal{R}(M_C)$ is an infinite dimensional linear space, so $M_C \notin \Phi_r(X \oplus Y)$. This is in contradiction with our previous assumption $M_C \in \Phi_r(X \oplus Y)$. Thus, we have proved that the statement (3b) holds. $\square$

As a corollary we get the following result.

Corollary 4.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given. Then*

$$\begin{aligned}
&\sigma_{re}(B) \cup \{\lambda \in \sigma_{re}(A) : \mathcal{R}(A-\lambda) \text{ is closed and } \mathcal{N}(B-\lambda) \prec X/\mathcal{R}(A-\lambda)\} \\
&\subset \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{re}(M_C) \\
&\subset \sigma_{re}(B) \cup \{\lambda \in \sigma_{re}(A) : \mathcal{R}(A-\lambda) \text{ is not closed and complemented}\} \cup \\
&\cup \{\lambda \in \sigma_{re}(A) : X/\overline{\mathcal{R}(A-\lambda)} \preceq \mathcal{N}(B-\lambda) \text{ does not hold}\}.
\end{aligned}$$

Analogously, we can prove similar results for the left Fredholm spectrum.

Theorem 4.2. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given operators and consider the following statements:*

(1) (1a) *$A \in \Phi_l(X)$;*
 (1b) *($B \in \Phi_l(Y)$), or ($\mathcal{R}(B)$ and $\mathcal{N}(B)$ are closed and complemented subspaces of Y and $\mathcal{N}(B) \preceq X/\overline{\mathcal{R}(A)}$).*

(2) *$M_C \in \Phi_l(X \oplus Y)$ for some $C \in \mathcal{L}(Y,X)$.*

(3) (3a) *$A \in \Phi_l(X)$;*
 (3b) *$B \in \Phi_l(Y)$, or $\mathcal{R}(B)$ is not closed, or $\mathcal{R}(A)^\circ \prec \mathcal{N}(B)'$ does not hold.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) If $A \in \Phi_l(X)$ and $B \in \Phi_l(Y)$, then from Proposition 4.1 it follows that $M_C \in \Phi_l(X \oplus Y)$. Otherwise, let (1) hold and $B \notin \Phi_l(Y)$. There exist closed subspaces U of X and V, W of Y, such that $\mathcal{R}(A) \oplus U = X$ and $\mathcal{N}(B) \oplus V = \mathcal{R}(B) \oplus W = Y$. Let $J : \mathcal{N}(B) \to U$ be an arbitrary left invertible operator. There exists a closed subspace Z such that $\mathcal{R}(J) \oplus Z = U$. Define $C \in \mathcal{L}(Y,X)$ as

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

Then $\mathcal{R}(M_C) = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus \mathcal{R}(B)$. From the decomposition

$$X \oplus Y = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus Z \oplus \mathcal{R}(B) \oplus W$$

it follows that $\mathcal{R}(M_C)$ is closed. Also, it is easy to verify $\mathcal{N}(M_C) = \mathcal{N}(A)$. Hence, $M_C \in \Phi_l(X \oplus Y)$.

(2) $\Longrightarrow$ (3) Suppose that $M_C \in \Phi_l(X \oplus Y)$ for some $C \in \mathcal{L}(Y, X)$. If $f \in X'$, we may take that $f|_Y = 0$. Hence, $(X \oplus Y)' = X' \oplus Y'$. Notice that

$$M_C' = \begin{bmatrix} A' & 0 \\ C' & B' \end{bmatrix} \in \Phi_r(X' \oplus Y').$$

In the same way as in Theorem 4.1 we can prove $A' \in \Phi_r(X')$, so $A \in \Phi_l(X)$. Thus, (3a) is proved.

Suppose that (3b) does not hold. Then $B \notin \Phi_l(Y)$, $\mathcal{R}(B)$ is closed and $\mathcal{R}(A)^\circ \prec \mathcal{N}(B)'$ holds. Notice that $\mathcal{R}(A)^\circ = \mathcal{N}(A')$ and

$$\mathcal{N}(B)' \cong Y'/\mathcal{N}(B)^\circ = Y'/\mathcal{R}(B').$$

Since $B' \notin \Phi_r(Y')$, we know that $Y'/\mathcal{R}(B')$ is an infinite dimensional space. In the same way as in Theorem 4.1 we can prove that $(X' \oplus Y')/\mathcal{R}(M_C')$ is an infinite dimensional linear space. Hence $M_C' \notin \Phi_r(X \oplus Y)$ and $M_C \notin \Phi_l(X \oplus Y)$. Thus, (3b) is proved. $\square$

The following result concerning the perturbation of the left Fredholm spectrum holds.

Corollary 4.2. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given operators. Then*

$$\begin{aligned}
&\sigma_{le}(A) \cup \{\lambda \in \sigma_{le}(B) : \mathcal{R}(B-\lambda) \textit{ is closed and } \mathcal{R}(A-\lambda)^\circ \prec \mathcal{N}(B-\lambda)'\} \\
&\subset \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{le}(M_C) \\
&\subset \sigma_{le}(A) \cup \{\lambda \in \sigma_{le}(B) : \mathcal{R}(B-\lambda) \textit{ and } \mathcal{N}(B-\lambda) \\
&\qquad\qquad \textit{are not closed and complemented}\} \cup \\
&\cup \{\lambda \in \sigma_{le}(B) : \mathcal{N}(B-\lambda) \preceq X/\overline{\mathcal{R}(A-\lambda)} \textit{ does not hold}\}.
\end{aligned}$$

Remark 4.2. Notice the difference between statements in Theorem 4.1 (1b) and Theorem 4.2 (1b). The reason is that the spaces $\mathcal{L}(Y', X')$ and $\mathcal{L}(X, Y)$

are not isomorphic. Precisely, the mapping $T \mapsto T'$ from $\mathcal{L}(X,Y)$ into $\mathcal{L}(Y',X')$ is injective, but not necessarily surjective.

Finally, we get the result for perturbations of the Fredholm spectrum for Hilbert space operators. This result may be also obtained from Corollary 3.1.

Corollary 4.3. *Let $H \oplus K$ be the orthogonal sum of infinite dimensional Hilbert spaces. Then*

$$\bigcap_{C \in \mathcal{L}(K,H)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}_2(A,B),$$

where

$$W_2(A,B) = \{\lambda \in \mathbb{C} : \ \dim \mathcal{N}(B-\lambda) \neq \dim \mathcal{R}(A-\lambda)^{\perp} \textit{ and at least one of these spaces is infinite dimensional}\}.$$

5. Perturbations of the left and right spectra

We begin with the following statement, which can be proved in the same way as Proposition 3.1.

Proposition 5.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given. Then the inclusion*

$$\sigma_l(M_C) \subset \sigma_l(A) \cup \sigma_l(B)$$

holds for any $C \in \mathcal{L}(Y,X)$. Particularly, if A and B are left invertible, then M_C is left invertible for all $C \in \mathcal{L}(Y,X)$.

For a left invertibility of an operator matrix we can prove the following result.

Theorem 5.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given operators and consider statements:*

(1) (1a) $A \in \mathcal{G}_l(X)$;
 (1b) $\mathcal{N}(B) \preceq X/\overline{\mathcal{R}(A)}$;
 (1c) *B is g-invertible.*

(2) $M_C \in \mathcal{G}_l(X \oplus Y)$ *for some* $C \in \mathcal{L}(Y, X)$.

(3) (3a) $A \in \mathcal{G}_l(X)$;
 (3b) $X/\overline{\mathcal{R}(A)} \prec \mathcal{N}(B)$ *does not hold.*

Then (1) $\Longrightarrow$ (2).

If $H \oplus K$ is the orthogonal sum of infinite dimensional Hilbert spaces H and K, then (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) Assume that $A \in \mathcal{G}_l(X)$, $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$ and B is g-invertible. Let $B_1 \in \mathcal{L}(Y)$ denote a g-inverse of B. Then $Y = \mathcal{R}(B_1) \oplus \mathcal{N}(B)$. Let $A_1 \in \mathcal{L}(X)$ be a left inverse of A. Then $X = \mathcal{N}(A_1) \oplus \mathcal{R}(A)$. Let $J : \mathcal{N}(B) \to \mathcal{N}(A_1)$ be a left invertible mapping and $J_1 : \mathcal{N}(A_1) \to \mathcal{N}(B)$ denote a left inverse of J. Hence, $\mathcal{N}(A_1) = \mathcal{R}(J) \oplus \mathcal{N}(J_1)$. Define $C \in \mathcal{L}(X \oplus Y)$ in the following way:

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix}.$$

Notice that from the decomposition

$$X \oplus Y = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus \mathcal{N}(J_1) \oplus \mathcal{R}(B) \oplus \mathcal{N}(B_1)$$

it follows that $\mathcal{R}(M_C) = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus \mathcal{R}(B)$ is closed.

Define $C_1 \in \mathcal{L}(X, Y)$ in the following way:

$$C_1 = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix}.$$

Consider the operator $N = \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} \in \mathcal{L}(X \oplus Y)$. Then we find

$$NM_C = \begin{bmatrix} A_1A & A_1C \\ C_1A & C_1C + B_1B \end{bmatrix}.$$

Notice that: $A_1A = I$, $A_1C = 0$ since $\mathcal{R}(C) \subset \mathcal{N}(A_1)$, $C_1A = 0$ since $\mathcal{R}(A) \subset \mathcal{N}(C_1)$. Also, B_1B is the projection from Y onto $\mathcal{R}(B_1)$ parallel to $\mathcal{N}(B)$ and C_1C is the projection form Y onto $\mathcal{N}(B)$ parallel to $\mathcal{R}(B_1)$. Hence $C_1C + B_1B = I$ and N is the left inverse of M_C. Thus, (2) is proved.

(2) $\Longrightarrow$ (3) Let H and K be Hilbert spaces and let M_C be left invertible. It immediately follows that A is left invertible, hence (3a) is proved.

Suppose that $H/\mathcal{R}(A) \prec \mathcal{N}(B)$ holds, i.e. $\dim \mathcal{R}(A)^\perp < \dim \mathcal{N}(B)$, where $\dim H$ denotes the orthogonal dimension of a Hilbert space H (see Remark 4.2).

Assume $\mathcal{N}(C) \cap \mathcal{N}(B) \neq \{0\}$. Then for all non-zero vectors $z \in \mathcal{N}(C) \cap \mathcal{N}(B)$ we have $M_C z = 0$. We conclude that M_C is not one-to-one and $M_C \notin \mathcal{G}_l(H \oplus K)$.

We conclude that $\mathcal{N}(C) \cap \mathcal{N}(B) = \{0\}$ holds. Hence, $C|_{\mathcal{N}(B)}$ is one-to-one. By [6, Problem 42] it follows that $\dim \overline{C(\mathcal{N}(B))} = \dim \mathcal{N}(B)$. Hence,

$$\dim \overline{C(\mathcal{N}(B))} = \alpha(B) > \beta(A).$$

Since $\mathcal{R}(A)$ is closed, we get $\mathcal{R}(A) \cap \overline{C(\mathcal{N}(B))} \neq \{0\}$. We take a non-zero vector $y_1 \in \mathcal{R}(A) \cap \overline{C(\mathcal{N}(B))}$. There exist: some $y_2 \in H$ and a sequence (z_n) in $\mathcal{N}(B)$, such that $Ay_2 = y_1 = \lim Cz_n$. Obviously $\lim z_n \neq 0$, so we may assume that there exists an $\epsilon > 0$, such that for all n: $\|z_n\| \geq \epsilon$. Notice that $\|y_2 - z_n\| \geq \sqrt{\|y_2\|^2 + \epsilon^2}$. Now,

$$\lim \left\| M_C \frac{y_2 - z_n}{\|y_2 - z_n\|} \right\| \leq \frac{1}{\sqrt{\|y_2\|^2 + \epsilon^2}} \lim \|Ay_2 - Cz_n - Bz_n\| = 0.$$

It follows that $M_C \notin \mathcal{G}_l(H \oplus K)$. Thus, (3b) is proved. $\square$

As a corollary, we get the following result.

Corollary 5.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given. Then the following inclusion holds:*

$$\bigcap_{C\in\mathcal{L}(Y,X)} \sigma_l(M_C) \subset \sigma_l(A) \cup \sigma_g(B) \cup$$
$$\cup \{\lambda \in \mathbb{C} : \mathcal{N}(B-\lambda) \preceq X/\overline{\mathcal{R}(A-\lambda)} \text{ does not hold}\}.$$

If $H \oplus K$ is the orthogonal sum of infinite dimensional Hilbert spaces H and K, then

$$\sigma_l(A) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{R}(A-\lambda)^\perp < \dim \mathcal{N}(B-\lambda)\} \subset \bigcap_{C\in\mathcal{L}(K,H)} \sigma_l(M_C).$$

Analogously, we can prove a similar result concerning the right spectrum and right invertibility of M_C

Theorem 5.2. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given and consider the statements*

(1) (1a) $B \in \mathcal{G}_r(Y)$;
 (1b) $X/\overline{\mathcal{R}(A)} \preceq \mathcal{N}(B)$.
 (1c) *A is g-invertible.*
(2) *$M_C \in \mathcal{G}_r(X \oplus Y)$ for some $C \in \mathcal{L}(Y,X)$.*
(3) (3a) $B \in \mathcal{G}_r(Y)$;
 (3b) *$\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$ does not hold.*

Then (1) $\Longrightarrow$ (2).

If $H \oplus K$ is the orthogonal sum of infinite dimensional Hilbert spaces, then (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) Let B_1 be a right inverse of B and A_1 be a g-inverse of A. Then $X = \mathcal{R}(A) \oplus \mathcal{N}(A_1)$ and $Y = \mathcal{N}(B) \oplus \mathcal{R}(B_1) = \mathcal{R}(B) \oplus \mathcal{N}(B_1)$. There exists a left invertible operator $J : \mathcal{N}(A_1) \to \mathcal{N}(B)$ and denote by

$J_1 : \mathcal{N}(B) \to \mathcal{N}(A_1)$ its left inverse. Define an operator $C \in \mathcal{L}(Y, X)$ in the following way:

$$C = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix}.$$

Then $\mathcal{R}(M_C) = X \oplus Y$. Since $\mathcal{N}(M_C) = \mathcal{N}(A)$, from the decomposition

$$X \oplus Y = \mathcal{N}(A) \oplus \mathcal{R}(A_1) \oplus Y$$

we conclude that $\mathcal{N}(M_C)$ is a complemented subspace of $X \oplus Y$. Hence, M_C is right invertible and (2) is proved.

(2) $\Longrightarrow$ (3). Let M_C be right invertible. It immediately follows that $B \in \mathcal{G}_r(K)$ and (3a) is proved.

Assume that (3b) is not satisfied. Then $\dim \mathcal{N}(B) < \dim \mathcal{R}(A)^\perp$. Consider the conjugate operator of M_C:

$$M_C^* = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix} \in \mathcal{L}(H \oplus K).$$

If $\mathcal{N}(C^*) \cap \mathcal{N}(A^*) \neq \emptyset$, then there exists some $z \in \mathcal{N}(C^*) \cap \mathcal{N}(A^*)$ and $z \neq 0$. It follows $M_C^* z = 0$, M_C^* is not left invertible and hence M_C is not right invertible.

We conclude that $\mathcal{N}(C^*) \cap \mathcal{N}(A^*) = \{0\}$ holds. Then

$$\dim \overline{C(\mathcal{N}(A^*))} = \dim \mathcal{N}(A^*) > \dim \mathcal{N}(B) = \dim \mathcal{R}(B^*)^\perp.$$

Since $\mathcal{R}(B)$ is closed, we obtain

$$\overline{C(N(A^*))} \cap \mathcal{R}(B) \neq \{0\}.$$

We can prove that $M_C^* \notin \mathcal{G}_l(H \oplus K)$ holds similarly as in the proof of Theorem 5.1. Thus, (3b) is proved. $\square$

As a corollary, we get the following result.

Corollary 5.2. *For given $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ the inclusion*

$$\bigcap_{C\in\mathcal{L}(Y,X)} \sigma_r(M_C) \subset \sigma_r(B) \cup \sigma_g(A) \cup$$
$$\cup \{\lambda \in \mathbb{C} : X/\overline{\mathcal{R}(A-\lambda)} \preceq \mathcal{N}(B-\lambda) \text{ does not hold}\}$$

holds.

Moreover, if $H \oplus K$ is the orthogonal sum of Hilbert spaces, then

$$\sigma_r(B) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(B-\lambda) < \dim \mathcal{R}(A-\lambda)^{\perp}\} \subset \bigcap_{C\in\mathcal{L}(K,H)} \sigma_r(M_C)$$

holds, where $\dim H$ *denotes the orthogonal dimension of a Hilbert space H.*

The main result of Du and Pan follows.

Corollary 5.3. *Let $H \oplus K$ be the orthogonal sum of infinite dimensional Hilbert spaces. For given $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ the following equality holds:*

$$\bigcap_{C\in\mathcal{L}(K,H)} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(B-\lambda) \neq \dim \mathcal{R}(A-\lambda)^{\perp}\}.$$

Proof. The proof follows from Corollary 5.1, Corollary 5.2 and the following facts: $\sigma_g(A) \subset \sigma(A)$, $\sigma_g(B) \subset \sigma(B)$ and $\sigma(T) = \sigma_l(T) \cup \sigma_r(T)$ for all $T \in \mathcal{L}(H)$. $\square$

In the rest of this section we will consider special classes of operators and related result.

Theorem 5.3. *Let H and K be Hilbert spaces, $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$.*

If $A \in \mathcal{S}_+(H)$ and $B \in \mathcal{S}_-(K)$, then for all $C \in \mathcal{L}(K,H)$ we have $\sigma_l(M_C) = \sigma_l(A) \cup \sigma_l(B)$.

If $A \in \mathcal{S}_+(H)$ or $B \in \mathcal{S}_-(K)$, then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.

Proof. Since $\sigma_l(A) \subset \sigma_l(M_C)$, by Proposition 5.1 it is enough to prove that $\sigma_l(B) \subset \sigma_l(M_C)$. Suppose that $\lambda \in \sigma_l(B) \setminus \sigma_l(M_C)$. From Corollary 5.1 we get that $A-\lambda$ is left invertible and $\dim \mathcal{N}(B-\lambda) \leq \dim \mathcal{R}(A-\lambda)^{\perp}$. Since $A \in \mathcal{S}_+$, we conclude $\beta(A-\lambda) \leq \alpha(A-\lambda) = 0$. Now $\alpha(B-\lambda) = 0$ and $\beta(B-\lambda) = 0$. Hence, $A-\lambda$ and $B-\lambda$ are invertible and $M_C - \lambda$ must be invertible. Thus, the equality $\sigma_l(M_C) = \sigma_l(A) \cup \sigma_l(B)$ is proved.

To prove the second equality, notice that $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$ [4] (or repeat the proof of Proposition 3.1 for the spectrum instead of the essential spectrum). Let $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C)$. From Corollary 5.3 we get $A-\lambda$ is left invertible, $B-\lambda$ is right invertible and $\alpha(B-\lambda) = \beta(A-\lambda)$.

If $A \in \mathcal{S}_-(H)$, then we get

$$\alpha(B-\lambda) = \beta(A-\lambda) \leq \alpha(A-\lambda) = 0.$$

Hence, $A-\lambda$ and $B-\lambda$ are invertible, which is in contradiction with the assumption $\lambda \in \sigma(A) \cup \sigma(B)$.

If $B \in \mathcal{S}_-(K)$, then

$$\beta(A-\lambda) = \alpha(B-\lambda) \leq \beta(B-\lambda) = 0.$$

We also get that $A-\lambda$ and $B-\lambda$ are invertible.

Thus, $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for any $C \in \mathcal{L}(K,H)$. $\square$

Finally, we consider four block operator matrices. For given $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$ and $C \in \mathcal{L}(K,H)$, let us take $T \in \mathcal{L}(H,K)$ and

$$G_T = \begin{bmatrix} A & C \\ T & B \end{bmatrix}.$$

We prove the following result.

Theorem 5.4. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$ and $C \in \mathcal{L}(K,H)$ be given operators and $\lambda \in \mathbb{C} \setminus \sigma_l(A)$.*

(1) *If $\mathcal{N}(C) \cap \mathcal{N}(B-\lambda) \neq \{0\}$, then $\lambda \in \sigma_p(G_T)$ for all $T \in \mathcal{L}(H,K)$.*
(2) *If $\mathcal{R}(A-\lambda) \cap \mathcal{R}(C) \neq \{0\}$, then there exists a rank-one operator $T \in \mathcal{L}(H,K)$, such that $\lambda \in \sigma_p(G_T)$.*

(3) *If neither* (a) *nor* (b) *is satisfied, then* $\lambda \notin \sigma_p(G_T)$ *for all* $T \in B(H,K)$.

Proof. To prove (1), suppose that $\mathcal{N}(C) \cap \mathcal{N}(B-\lambda) \neq \{0\}$. There exists a non-zero vector $v \in \mathcal{N}(C) \cap \mathcal{N}(B-\lambda)$, so $(G_T - \lambda)v = 0$ for all $T \in \mathcal{L}(H,K)$, and $\lambda \in \sigma_p(G_T)$.

To prove (2), suppose that $\mathcal{R}(A-\lambda) \cap \mathcal{R}(C) \neq \{0\}$. Let us take an arbitrary non-zero vector $z \in \mathcal{R}(A-\lambda) \cap \mathcal{R}(C)$. There exists an operator $A_1 : \mathcal{R}(A-\lambda) \to H$, such that $A_1(A-\lambda) = I_H$ and $(A-\lambda)A_1 = I_{\mathcal{R}(A-\lambda)}$. There exist vectors: $x_1 = A_1 z \in H$, and $x_2 \in K$, such that $Cx_2 = z$. We define a rank-one operator $T \in \mathcal{L}(H,K)$, such that for any $x \in H$:

$$T(x) = \frac{1}{\|x_1\|^2}(x, x_1)(B-\lambda)x_2.$$

Taking $x = -x_1 + x_2$, we get $(G_T - \lambda)x = 0$, so $\lambda \in \sigma_p(G_T)$.

To prove (3), suppose that neither (1) nor (2) is satisfied. Let $0 \neq x \in \mathcal{N}(G_T - \lambda)$ for some $T \in \mathcal{L}(H,K)$. Then $x = u + v$, $u \in H$, $v \in K$ and

$$(A-\lambda)u + Cv = 0 = Tu + (B-\lambda)v.$$

Since $\mathcal{R}(A-\lambda) \cap \mathcal{R}(C) = \{0\}$, we get $(A-\lambda)u = Cv = 0$. Also, $u = 0$, $v \in \mathcal{N}(C) \cap \mathcal{N}(B-\lambda)$ and $v = 0$. The obtained contradiction completes the proof. □

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