REVERSE ORDER LAW FOR THE MOORE-PENROSE INVERSE

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Abstract

In this paper we present new results related to the reverse order law for the Moore-Penrose inverse of operators on Hilbert spaces. Some finite dimensional results are extended to infinite dimensional settings.

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1 Introduction

If S is a semigroup with the unit 1, and if a, b ∈ S are invertible, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. It is well-known that the reverse order law does not hold for various classes of generalized inverses. Hence, a significant number of papers treat the sufficient or equivalent conditions such that the reverse order law holds in some sense. In this paper we specialize the investigations to the Moore-Penrose inverse of closed range linear bounded operators on Hilbert spaces.

Let $X, Y, Z$ be Hilbert spaces, and let $\mathcal{L}(X,Y)$ denote the set of all linear bounded operators from $X$ to $Y$. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X,X)$. For $A \in \mathcal{L}(X,Y)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of $A$. An operator $B \in \mathcal{L}(Y,X)$ is an inner inverse of $A$, if $ABA = A$ holds. In this case $A$ is inner invertible, or relatively regular. It is well-known that $A$ is inner invertible if and only if $\mathcal{R}(A)$ is closed in $Y$. The Moore-Penrose inverse of $A \in \mathcal{L}(X,Y)$ is the operator $X \in \mathcal{L}(Y,X)$ which satisfies the Penrose equations

\begin{align*}
(1) \ AXA &= A, \quad (2) \ XAX = X, \quad (3) \ (AX)^* = AX, \quad (4) \ (XA)^* =XA.
\end{align*}

The Moore-Penrose inverse of $A$ exists if and only if $\mathcal{R}(A)$ is closed in $Y$. If the Moore-Penrose inverse of $A$ exists, then it is unique, and it is denoted by $A^\dagger$.

If $\theta \subset \{1,2,3,4\}$, and $X$ satisfies the equations $(i)$ for all $i \in \theta$, then $X$ is an $\theta$-inverse of $A$. The set of all $\theta$-inverses of $A$ is denoted by $A\{\theta\}$. If

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\(\mathcal{R}(A)\) is closed, then \(A\{1, 2, 3, 4\} = \{A^\dagger\}\). The theory of generalized inverses on infinite dimensional Hilbert spaces can be found in [4, 8, 10].

It is a classical result of Greville [9], that \((AB)^\dagger = B^\dagger A^\dagger\) if and only if \(\mathcal{R}(A^*AB) \subset \mathcal{R}(B)\) and \(\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)\), in the case when \(A\) and \(B\) are complex (possibly rectangular) matrices. This result is extended for linear bounded operators on Hilbert spaces, by Bouldin [2], [3], and Izumino [12]. Among other things, Bouldin and Izumino used gaps between subspaces. In [13] the reverse order law for the Moore-Penrose inverse is proved in rings with involutions. Then, in [6], the reverse order law for the Moore-Penrose inverse is obtained as a consequence of some set equalities. The reader can find some interesting and related results in [1, 5, 8, 11, 15, 16, 17, 18].

In particular, the paper [15] is related to our investigations. In [15] Tian obtained some very interesting results concerning the sets of generalized inverses of complex rectangular matrices. As a corollary, the reverse order law for the Moore-Penrose inverse follows. Notice that the finite dimensional methods are used in [15] (mostly the rank of a complex matrix).

In this paper we extend some results from [15] to infinite dimensional settings. Among other things, we obtain the reverse order law for the Moore-Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary result. In Section 2 we present the results related to the reverse order rule for the Moore-Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite dimensional settings.

**Lemma 1.1.** Let \(A \in \mathcal{L}(X, Y)\) have a closed range. Then \(A\) has the matrix decomposition with respect to the orthogonal decompositions of spaces \(X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)\) and \(Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)\):

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},
\]

where \(A_1\) is invertible. Moreover,

\[
A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.
\]

The proof is straightforward.
Lemma 1.2. Let \( A \in \mathcal{L}(X, Y) \) have a closed range. Let \( X_1 \) and \( X_2 \) be closed and mutually orthogonal subspaces of \( X \), such that \( X = X_1 \oplus X_2 \). Let \( Y_1 \) and \( Y_2 \) be closed and mutually orthogonal subspaces of \( Y \), such that \( Y = Y_1 \oplus Y_2 \). Then the operator \( A \) has the following matrix representations with respect to the orthogonal sums of subspaces \( X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A) \), and \( Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2 \):

(a) \[
A = \begin{bmatrix}
A_1 & A_2 \\
0 & 0
\end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},
\]
where \( D = A_1 A_1^* + A_2 A_2^* \) maps \( \mathcal{R}(A) \) into itself and \( D > 0 \) (meaning \( D \geq 0 \) invertible). Also,

\[
A^\dagger = \begin{bmatrix}
A_1^* D^{-1} & 0 \\
A_2^* D^{-1} & 0
\end{bmatrix}.
\]

(b) \[
A = \begin{bmatrix}
A_1 & 0 \\
A_2 & 0
\end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},
\]
where \( D = A_1^* A_1 + A_2^* A_2 \) maps \( \mathcal{R}(A^*) \) into itself and \( D > 0 \) (meaning \( D \geq 0 \) invertible). Also,

\[
A^\dagger = \begin{bmatrix}
D^{-1} A_1^* & D^{-1} A_2^* \\
0 & 0
\end{bmatrix}.
\]

Here \( A_i \) denotes different operators in any of these two cases.

Proof. Recall that one special case of this result is proved in [7]. We prove only the result of (a), since the proof of (b) is analogous.

The operator \( A \) has the following representation:

\[
A = \begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},
\]
i.e.

\[
A_1 = A_{1|X_1} : X_1 \to \mathcal{R}(A), \quad A_2 = A_{1|X_2} : X_2 \to \mathcal{R}(A),
\]

\[
A_3 = A_{1|X_1} : X_1 \to \mathcal{N}(A^*), \quad A_4 = A_{1|X_2} : X_2 \to \mathcal{N}(A^*).
\]

Furthermore,

\[
A^* = \begin{bmatrix}
A_1^* & A_3^* \\
A_2^* & A_4^*
\end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.
\]
From $A^*(N(A^*)) = \{0\}$ we obtain $A_3^* = 0$ and $A_4^* = 0$, so $A_3 = 0$ and $A_4 = 0$. Hence, $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$. Notice that

$$AA^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ N(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^* : \mathcal{R}(A) \to \mathcal{R}(A)$. From $N(AA^*) = N(A^*)$ it follows that $D$ is one-one. From $\mathcal{R}(AA^*) = \mathcal{R}(A)$ it follows that $D$ is onto. Hence, $D$ is invertible. Finally, we obtain the form for the Moore-Penrose inverse of $A$ using the formula $A^\dagger = A^*(AA^*)^{-\dagger}$.

The following result is well-known, and it can be found in [4], p.127.

**Lemma 1.3.** Let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ have closed ranges. Then $AB$ has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range.

Finally, the reader should notice the difference between the following notations. If $A, B \in \mathcal{L}(X)$, then $[A, B] = AB - BA$ denotes the commutator of $A$ and $B$. On the other hand, if $U \in \mathcal{L}(X, Z)$ and $V \in \mathcal{L}(Y, Z)$, then $[U \ V] : \begin{bmatrix} X \\ Y \end{bmatrix} \to Z$ denote the matrix form of the corresponding operator.

## 2 Reverse order law

In this section we prove the results concerning the reverse order law for the Moore-Penrose inverse.

**Theorem 2.1.** Let $X, Y, Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be such that $A, B, AB$ have closed ranges. Then the following statements are equivalent:

(a) $ABB^\dagger A^\dagger AB = AB$;
(b) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$;
(c) $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$;
(d) $A^\dagger ABB^\dagger$ is an idempotent;
(e) $BB^\dagger A^\dagger A$ is an idempotent;
(f) $B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$;
(g) \((A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A;\)

Notice that \(A^\dagger ABB^\dagger\) has a closed range, according to Lemma 1.3. Moreover, \(A^*ABB^*\) also has a closed range: \(\mathcal{R}(B^*A^*A) = B^*(\mathcal{R}(A^*A)) = B^*(\mathcal{R}(A))\) is closed, so \(\mathcal{R}(A^*ABB^*) = A^*A(\mathcal{R}(BB^*)) = A^*A(\mathcal{R}(B)) = \mathcal{R}(A^*AB) = \mathcal{R}((B^*A^*)^*)\) is closed.

**Proof.** Using Lemma 1.1 we conclude that the operator \(B\) has the following matrix form:

\[
B = \begin{bmatrix}
B_1 & 0 \\
0 & 0
\end{bmatrix}: \begin{bmatrix}
\mathcal{R}(B^*) \\
\mathcal{N}(B)
\end{bmatrix} \to \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{N}(B^*)
\end{bmatrix},
\]

where \(B_1\) is invertible. Then

\[
B^\dagger = \begin{bmatrix}
B_1^{-1} & 0 \\
0 & 0
\end{bmatrix}: \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{N}(B^*)
\end{bmatrix} \to \begin{bmatrix}
\mathcal{R}(B^*) \\
\mathcal{N}(B)
\end{bmatrix}.
\]

From Lemma 1.2 it follows that the operator \(A\) has the following matrix form:

\[
A = \begin{bmatrix}
A_1 & A_2 \\
0 & 0
\end{bmatrix}: \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{N}(B^*)
\end{bmatrix} \to \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*)
\end{bmatrix},
\]

where \(D = A_1A_1^* + A_2A_2^*\) is invertible and positive in \(\mathcal{L}(\mathcal{R}(A))\). Then

\[
A^\dagger = \begin{bmatrix}
A_1^*D^{-1} & 0 \\
A_2^*D^{-1} & 0
\end{bmatrix}.
\]

Notice the following:

\[
BB^\dagger = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}: \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{N}(B^*)
\end{bmatrix} \to \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{N}(B^*)
\end{bmatrix},
\]

\[
AA^\dagger = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}: \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*)
\end{bmatrix} \to \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*)
\end{bmatrix},
\]

and

\[
A^\dagger A = \begin{bmatrix}
A_1^*D^{-1}A_1 & A_2^*D^{-1}A_2 \\
A_2^*D^{-1}A_1 & A_2^*D^{-1}A_2
\end{bmatrix}: \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{N}(B^*)
\end{bmatrix} \to \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{N}(B^*)
\end{bmatrix}.
\]

From Lemma 1.3 it follows that \((A^\dagger ABB^\dagger)^\dagger\) has a closed range. We obtain

\[
A^\dagger ABB^\dagger = \begin{bmatrix}
A_1^*D^{-1}A_1 & 0 \\
A_2^*D^{-1}A_1 & 0
\end{bmatrix},
\]

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\[BB^\dagger A^\dagger A = \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}.\]

Consider the following chain of equivalences, which is related to the statement of (a):

\[
ABB^\dagger A^\dagger AB = AB 
\iff \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} [B_1 \ 0] = [A_1 B_1 \ 0] 
\iff \begin{bmatrix} A_1 A_1^* D^{-1} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix} = [A_1 B_1 \ 0] 
\iff A_1 A_1^* D^{-1} A_1 = A_1. \quad (1)
\]

Consequently, the statement (a) is equivalent to (1).

Notice that (1) is equivalent to

\[A_1^* D^{-1} A_1 A_1^* = A_1^*. \quad (2)\]

We consider also the statement (b):

\[
B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger
\iff \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = [B_1^{-1} A_1^* D^{-1} 0] 
\iff [B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1} 0] = [B_1^{-1} A_1^* D^{-1} 0] 
\iff B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1} = B_1^{-1} A_1^* D^{-1} \iff (2).\]

Thus, (a) \iff (1) \iff (2) \iff (b).

In the case of the statement (c) we have:

\[
A^\dagger ABB^\dagger = BB^\dagger A^\dagger A
\iff \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} 
\iff A_1^* D^{-1} A_2 = 0 \iff A_2^* D^{-1} A_1 = 0. \quad (3)
\]
Thus, if (c) holds, i.e. $A_2^*D^{-1}A_1 = 0$, then it is obvious that $A_2A_2^*D^{-1}A_1 = 0$, so (1) also holds because of:

\[
(A_1A_1^* + A_2A_2^*)D^{-1} = I \iff A_1A_1^*D^{-1}A_1 + A_2A_2^*D^{-1}A_1 = A_1
\]

\[
\iff A_1A_1^*D^{-1}A_1 = A_1.
\]

On the other hand, suppose that (1) holds. Then $A_2A_2^*D^{-1}A_1 = 0$, and we have the following:

\[
A_2A_2^*D^{-1}A_1 = 0 \Rightarrow \mathcal{R}(D^{-1}A_1) \subset \mathcal{N}(A_2^*) = \mathcal{N}(A_2^*) \Rightarrow A_2^*D^{-1}A_1 = 0,
\]

so (3) is satisfied. Consequently, (c) also holds. We have just proved (c) $\iff$ (3) $\iff$ (1) $\iff$ (a).

A straightforward computation shows that (d) is equivalent to

\[
\begin{align*}
A_1D^{-1}A_1A_1^*D^{-1}A_1 &= A_1^*D^{-1}A_1 \\
A_2D^{-1}A_1A_1^*D^{-1}A_1 &= A_2^*D^{-1}A_1
\end{align*}
\]

(4)

If the statement (1) holds, then obviously (4) is satisfied. On the other hand, suppose that (4) holds. Then multiply the first equation of (4) by $A_1$ from the left side, and multiply the second equation of (4) by $A_2$ from the left side. The sum of these two new equations leads to the equation (1).

Notice that (e) is also equivalent to (4). Consequently, (d) $\iff$ (4) $\iff$ (2) $\iff$ (e).

In order to establish (f), we proceed as follows. Let $Q = A_1^*ABB^\dagger$. From Lemma 1.3 we know that $Q$ has a closed range. We use the formula $Q^\dagger = Q^*(QQ^*)^{-1} = (Q^*Q)^+Q^*$. Hence,

\[
(A_1^*ABB^\dagger)^\dagger = (BB^\dagger A_1^*ABB^\dagger)^\dagger BB^\dagger A_1^*ABB^\dagger
\]

\[
= \left[ \begin{array}{c} A_1^*D^{-1}A_1 \ 0 \\ 0 \ 0 \end{array} \right]^\dagger \left[ \begin{array}{c} A_1^*D^{-1}A_1 \\ 0 \end{array} \right] = \left[ \begin{array}{c} A_1^*D^{-1}A_1 \\ 0 \end{array} \right]^\dagger \left[ \begin{array}{c} A_1^*D^{-1}A_1 \\ 0 \end{array} \right]
\]

\[
= \left[ \begin{array}{c} A_1^*D^{-1}A_1^\dagger A_1^*D^{-1}A_1 \ 0 \\ 0 \ A_1^*D^{-1}A_2 \end{array} \right]
\]

We get

\[
B^\dagger(A_1^*ABB^\dagger)^\dagger A^\dagger - B^\dagger A^\dagger = 0
\]

\[
\iff \left[ \begin{array}{c} B_1^\dagger(A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1} - B_1^\dagger A_1^*D^{-1} \\ B_1^\dagger A_1^*D^{-1} \end{array} \right] = 0
\]

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\[ A^*_1 D^{-1} A_1 = A^*_1. \] (5)

We need to prove (1) \(\iff\) (5). Let \(P = A^*_1 D^{-1} A_1\). Obviously, \(P^* = P\).

(1) \(\implies\) (5): We have the following:

\[ P^2 = A^*_1 D^{-1} A_1 A^*_1 D^{-1} A_1 = A^*_1 D^{-1} A_1 = P, \]
\[ P = P^* = P^2 = P^\dagger, \]
\[ (A^*_1 D^{-1} A_1)^\dagger A^*_1 = A^*_1 D^{-1} A_1 A^*_1 = A^*_1. \]

(5) \(\implies\) (1): In this case we have

\[ (A^*_1 D^{-1} A_1)^\dagger A^*_1 = A^*_1, \]
\[ PP^\dagger = (PP^\dagger)^* = (P^*)^\dagger P = P^\dagger P = P \]
\[ PP^\dagger = P^\dagger PP^\dagger = P^\dagger P = P \]
\[ A^*_1 = (A^*_1 D^{-1} A_1)^\dagger A^*_1 = A^*_1 D^{-1} A_1 A^*_1. \]

We have just proved (f) \(\iff\) (1) \(\iff\) (a).

To prove (g) \(\iff\) (f), we use the fact which is already proved for (f), i.e. for \((A^\dagger ABB^\dagger)^\dagger\). Thus, we have

\[ (A^\dagger ABB^\dagger)^\dagger - BB^\dagger A^\dagger A = 0 \iff \begin{cases} (A^*_1 D^{-1} A_1)^\dagger A^*_1 D^{-1} A_1 = A^*_1 D^{-1} A_1, \\ (A^*_1 D^{-1} A_1)^\dagger A^*_1 D^{-1} A_2 = A^*_1 D^{-1} A_2. \end{cases} \]

It is easy to conclude that (g) \(\iff\) (f).

\[ \square \]

Now we prove the following result.

**Theorem 2.2.** Let \(X, Y, Z\) be Hilbert spaces, and let \(A \in \mathcal{L}(Y, Z)\), \(B \in \mathcal{L}(X, Y)\) be such that \(A, B, AB\) have closed ranges. Then the following statements hold:

(a) \(AB(AB)^\dagger = ABB^\dagger A^\dagger \iff A^* AB = BB^\dagger A^* AB \iff \mathcal{R}(A^* AB) \subseteq \mathcal{R}(B) \iff B^\dagger A^\dagger \in (AB)^\{1, 2, 3\};\)

(b) \((AB)^\dagger AB = B^\dagger A^\dagger AB \iff ABB^* = ABB^* A^\dagger A \iff \mathcal{R}(BB^* A^*) \subseteq \mathcal{R}(A^*) \iff B^\dagger A^\dagger \in (AB)^\{1, 2, 4\};\)

(c) The following statements are equivalent:

\[ \begin{align*}
(1) \ & (AB)^\dagger = B^\dagger A^\dagger; \\
(2) \ & AB(AB)^\dagger = ABB^\dagger A^\dagger \text{ and } (AB)^\dagger AB = B^\dagger A^\dagger AB; \end{align*} \]
(3) \( A^*AB = BB^\dagger A^*AB \) and \( ABB^* = ABB^*A^\dagger A \);
(4) \( \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \) and \( \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \).

**Proof.** The operators \( A \) and \( B \) have the same matrix representations as in the previous theorem. The following products will be useful:

\[
AB = \begin{bmatrix}
    A_1B_1 & 0 \\
    0 & 0
\end{bmatrix},
(AB)^\dagger = \begin{bmatrix}
    (A_1B_1)^\dagger & 0 \\
    0 & 0
\end{bmatrix},
B^\dagger A^* = \begin{bmatrix}
    B_1^{-1}A_1^*D^{-1} & 0 \\
    0 & 0
\end{bmatrix}.
\]

First, we find the equivalent expressions for our statements in terms of \( A_1 \), \( A_2 \) and \( B_1 \).

(a) \( 1. \) \( AB(AB)^\dagger = ABB^\dagger A^\dagger \iff A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \). Here \( A_1B_1(A_1B_1)^\dagger \) is Hermitian, so \( [A_1A_1^*D^{-1}] = 0 \).

2. \( A^*AB = BB^\dagger A^*AB \iff A_2^*A_1 = 0 \).

3. Notice that \( \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \) if and only if \( BB^\dagger A^*AB = A^*AB \), so 2. \( \iff \) 3.

4. If we check properly the Penrose equations, then we see that:
   \( B^\dagger A^\dagger \in (AB)\{1, 2, 3\} \iff A_1A_1^*D^{-1}A_1 = A_1 \) and \( [A_1A_1^*, D^{-1}] = 0 \).

Now, we prove the following: 1. \( \iff \) 2., 4. \( \iff \) 2. and 1. \( \iff \) 4.

We prove 1. \( \iff \) 2. Notice that

\[
A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \iff (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.
\]

The last statement is obtained by multiplying the first expression by \( (A_1B_1)^\dagger \) from the left side, or multiplying the second expression by \( A_1B_1 \) from the left side, and using \( A_1A_1^* = A_1B_1B_1^{-1}A_1^* \). Now, there is a chain of the equivalences:

\[
(A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}
\]

\[
\iff (A_1B_1)^\dagger (A_1A_1^* + A_2A_2^*) = (A_1B_1)^\dagger A_1A_1^*
\]

\[
\iff (A_1B_1)^\dagger A_2A_2^* = 0 \iff \mathcal{R}(A_2A_2^*) \subseteq \mathcal{N}((A_1B_1)^\dagger)
\]

\[
\iff \mathcal{R}(A_2) \subseteq \mathcal{N}((A_1B_1)^*) \iff B_1^*A_1^*A_2 = 0 \iff A_1^*A_2 = 0,
\]

Therefore, we have just proved that 1. \( \iff \) 2.

Now we prove 1. \( \iff \) 4. If we multiply \( A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \) by \( A_1B_1 \) from the right side, we get \( A_1A_1^*D^{-1}A_1 = A_1 \). Thus, 4. holds.
Finally, we prove 4. \(\implies\) 2. If \(A_1 A_1^* D^{-1} A_1 = A_1\) and \([A_1 A_1^*, D^{-1}] = 0\), then \(A_1 A_1^* A_1 = D A_1 = A_1 A_1^* A_1 + A_2 A_2^* A_1\), implying that \(A_2 A_2^* A_1 = 0\). Hence, \(\mathcal{R}(A_1) \subseteq \mathcal{N}(A_2 A_2^*) = \mathcal{N}(A_2^*),\) so \(A_2^* A_1 = 0\). Thus, 2. holds.

Notice that the equivalence 3. \(\iff\) 4. is proved in [8], also.

(b) 1. \((AB)^\dagger AB = B^\dagger A^\dagger AB \iff (A_1 B_1)^\dagger A_1 B_1 = B_1^{-1} A_1^* D^{-1} A_1 B_1\). Moreover, \((A_1 B_1)^\dagger A_1 B_1\) is Hermitian, so \([B_1 B_1^*, A_1^* D^{-1} A_1] = 0\).

2. \(A B B^* A^\dagger A \iff A_1 B_1 B_1^* A_1^* D^{-1} A_1 = A_1 B_1 B_1^*\) and \(A_1 B_1 B_1^* A_1^* D^{-1} A_2 = 0\).

3. Notice that \(\mathcal{R}(B B^* A^*) \subseteq \mathcal{R}(A^*)\) if and only if \(A_1^* A B B^* A^* = B B^* A^*,\) which is equivalent to \(A B B^* A^* A = A B B^*\.\) Hence, 2. \(\iff\) 3.

4. The Penrose equations imply that: \(B^\dagger A^\dagger \in (AB)\{1, 2, 4\} \iff A_1 A_1^* D^{-1} A_1 = A_1\) and \([B_1 B_1^*, A_1^* D^{-1} A_1] = 0\).

We prove 1. \(\Rightarrow\) 4. \(\Rightarrow\) 2. \(\Rightarrow\) 1.

Suppose that 1. holds. If we multiply \((A_1 B_1)^\dagger A_1 B_1 = B_1^{-1} A_1^* D^{-1} A_1 B_1\) by \(A_1 B_1\) from the left side, we obtain \(A_1 = A_1 A_1^* D^{-1} A_1\). Furthermore, \([B_1 B_1^*, A_1^* D^{-1} A_1] = 0\) holds. Therefore, 1. \(\Rightarrow\) 4.

Suppose that 4. holds. Obviously, \(A_1 B_1 B_1^* A_1^* D^{-1} A_1 = A_1 A_1^* D^{-1} A_1 B_1 B_1^* = A_1 B_1 B_1^*\). Thus, the first equality of 2. holds. The second equality of 2. also holds, since \(A_1^* D^{-1} A_2 = 0 \iff A_1 A_1^* D^{-1} A_1 = A_1\), which is shown in the proof of the Theorem 2.1. Here we use again \([B_1 B_1^*, A_1^* D^{-1} A_1] = 0\). Consequently, 4. \(\Rightarrow\) 2.

In order to prove that 2. \(\implies\) 1., we multiply \(A_1 B_1 B_1^* A_1^* D^{-1} A_1 = A_1 B_1 B_1^*\) by \((A_1 B_1)^\dagger\) from the left side. It follows that \(B_1^* A_1^* D^{-1} A_1 = (A_1 B_1)^\dagger A_1 B_1 B_1^*\), so \((A_1 B_1)^\dagger A_1 B_1 = B_1^* A_1^* D^{-1} A_1 (B_1^*)^{-1}\) which is equivalent to \((A_1 B_1)^\dagger A_1 B_1 = B_1^{-1} A_1^* D^{-1} A_1 B_1\). Hence, 2. \(\Rightarrow\) 1.

Notice that 3. \(\iff\) 4. is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b).

We also prove the following result.

**Theorem 2.3.** Let \(X, Y, Z\) be Hilbert spaces, and let \(A \in \mathcal{L}(Y, Z), B \in \mathcal{L}(X, Y)\) be such that \(A, B, AB\) have closed ranges. Then we have:

(a) \(AB(AB)^\dagger A = ABB^\dagger \iff A^* ABB^\dagger = BB^\dagger A^* A \iff \mathcal{R}(A^* AB) \subseteq \mathcal{R}(B) \iff B^\dagger A^\dagger \in (AB)\{1, 2, 3\};\)

(b) \(B(AB)^\dagger AB = A^\dagger AB \iff A^\dagger ABB^* = BB^* A^\dagger A \iff \mathcal{R}(BB^* A^*) \subseteq \mathcal{R}(A^* \iff B^\dagger A^\dagger \in (AB)\{1, 2, 4\};\)
The following three statements are equivalent:

1. \((AB)^\dagger = B^\dagger A^\dagger\);
2. \(AB(AB)^\dagger A = ABB^\dagger\) and \(B(AB)^\dagger AB = A^\dagger AB\);
3. \(A^*ABB^\dagger = BB^\dagger A^* A\) and \(A^\dagger ABB^* = BB^* A^\dagger A\).

Proof. The operators \(A\) and \(B\) have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of \(A_1, A_2\) and \(B_1\), for our assumptions.

(a) 
1. \(AB(AB)^\dagger A = ABB^\dagger \iff A_1 B_1 (A_1 B_1)^\dagger A_1 = A_1\) and \(A_1 B_1 (A_1 B_1)^\dagger A_2 = 0\). The first equality on the right side of the equivalence always holds, so: \(AB(AB)^\dagger A = ABB^\dagger \iff A_1 B_1 (A_1 B_1)^\dagger A_1 = A_1\) and \(A_1 B_1 (A_1 B_1)^\dagger A_2 = 0\).
2. \(A^*ABB^\dagger = BB^\dagger A^* A \iff A_1^* A_2 = 0\).
3. \(\mathcal{R}(A^* AB) \subset \mathcal{R}(B) \iff BB^\dagger A^* AB = A^* AB \iff A_2^* A_1 = 0\) (see the proof of Theorem 2.2, the part (a) 2. and 3.).
4. \(B^\dagger A^\dagger \in (AB)\{1, 2, 3\} \iff A_1 A_1^* D^{-1} A_1 = A_1\) and \([A_1 A_1^*, D^{-1}] = 0\) (see Theorem 2.2 (a) 4.).

To prove that 1.\(\iff\)2., we see that \(A_1 B_1 (A_1 B_1)^\dagger A_2 = 0 \iff \mathcal{R}(A_2) \subset \mathcal{N}((A_1 B_1)(A_1 B_1)^\dagger) = \mathcal{N}((A_1 B_1)^\dagger) = \mathcal{N}(A_1^* A_1) = \mathcal{N}(A_1^* A_1) \iff A_1^* A_2 = 0\).

Now, we prove that 2.\(\iff\)4. If \([A_1 A_1^*, D^{-1}] = 0\), then \(A_1 A_1^* D^{-1} A_1 = A_1 \iff A_1 A_1^* A_1 = DA_1 \iff A_2 A_2^* A_1 = 0 \iff A_1 A_2 A_2^* = 0 \iff \mathcal{R}(A_2 A_2^*) \subset \mathcal{N}(A_1^*) \iff \mathcal{R}(A_2) \subset \mathcal{N}(A_1^*) \iff A_1^* A_2 = 0\). On the other hand, if \(A_1^* A_2 = 0\), then \(A_1 A_1^* D = A_1 A_1^* A_1 A_1^* A_1 = D\) is Hermitian, so \(A_1 A_1^*\) commutes with \(D\). This implies \([A_1 A_1^*, D^{-1}] = 0\) and \(A_1 A_1^* D^{-1} A_1 = A_1\).

From Theorem 2.2 we know that 3.\(\iff\)4.

(b) 
1. \(B(AB)^\dagger AB = A^\dagger AB \iff B_1 (A_1 B_1)^\dagger A_1 = A_1^* D^{-1} A_1\) and \(A_2^* D^{-1} A_1 = 0\).
2. \(A^\dagger ABB^* = BB^* A^\dagger A \iff [B_1 B_1^*, A_1^* D^{-1} A_1] = 0\) and \(A_1^* D^{-1} A_2 = 0\).
3. \(\mathcal{R}(BB^* A^*) \subseteq \mathcal{R}(A^*) \iff A_1 B_1 B_1^* A_1^* D^{-1} A_2 = 0\) and \(A_1 B_1 B_1^* A_1^* D^{-1} A_1 = A_1 B_1 B_1^*\) (Theorem 2.2 (b) parts 2. and 3.).
4. \(B^\dagger A^\dagger \in (AB)\{1, 2, 4\} \iff A_1 A_1^* D^{-1} A_1 = A_1\) and \([B_1 B_1^*, A_1^* D^{-1} A_1] = 0\) (Theorem 2.2 (b) part 4.).
Let $A_1$ from the left side, and by $B_1$ from the right side, and thus obtain $A_1A_1^*D^{-1}A_1 = A_1$. Also, we obtain that $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ is Hermitian. Hence, $A_1^*D^{-1}A_1B_1B_1^*$ is Hermitian, and we get $[B_1B_1^* , A_1^*D^{-1}A_1] = 0$. 

4. $\implies$ 1. If 4. holds, then it is easy to see that $B_1^{-1}A_1^*D^{-1}A_1B_1(A_1B_1)^\dagger$ is the Moore-Penrose inverse of $A_1B_1$ (check the Penrose equations). This implies $B_1(A_1B_1)^\dagger A_1 = A_1^*D^{-1}A_1$. Now, we obtain that $A_1 = A_1A_1^*D^{-1}A_1$.

From $(A_1A_1^* + A_2A_2^*)D^{-1}A_1 = A_1$ it follows that $A_2A_2^*D^{-1}A_1 = 0$, so $\mathcal{R}(D^{-1}A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, and $A_2^*D^{-1}A_1 = 0$.

2. $\implies$ 3. If 2. holds, then $A_1B_1B_1^*D^{-1}A_1 = A_1B_1B_1^*$ is equivalent to $A_1A_1^*D^{-1}A_1 = A_1$, which follows from $A_1^*D^{-1}A_2 = 0$.

3. $\implies$ 2. From the proof of Theorem 2.2, part (b) 4., it follows that $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Now, in the usual manner, we get that $A_2A_2^*D^{-1}A_1 = 0$, so $A_1^*D^{-1}A_2 = 0$.

2. $\iff$ 4. Obvious.

The part (c) follows from the parts (a) and (b). 

We also prove the following result.

**Theorem 2.4.** Let $X, Y, Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that $A, B, AB$ have closed ranges. The following statements hold.

(a) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger \iff B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger \iff \mathcal{R}(A^*AB) \subset \mathcal{R}(B)$.

(b) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A \iff (A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger \iff \mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$.

(c) The following three statements are equivalent:

1. $(AB)^\dagger = B^\dagger A^\dagger$;
2. $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ and $(A^\dagger AB)^\dagger = B^\dagger A^\dagger$;
3. $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger$ and $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger$.

Notice that $ABB^\dagger$ and $A^\dagger AB$ have closed ranges. This is explained in the further proof.

**Proof.** The operators $A$ and $B$ have the same matrix representations as in the previous theorem.

(a) Notice that $\mathcal{R}(ABB^\dagger) = \mathcal{R}(AB)$ is closed, so there exists $(ABB^\dagger)^\dagger$. 

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1. \((ABB^\dagger)^\dagger = BB^\dagger A^\dagger \Leftrightarrow A_1^\dagger = A_1^* D^{-1}\) (the existence of \(A_1^\dagger\) follows from the assumptions).

2. \(B^\dagger (ABB^\dagger)^\dagger = B^\dagger A^\dagger \Leftrightarrow A_1^\dagger = A_1^* D^{-1}\), so 1. \(\Rightarrow\) 2.

3. \(\mathcal{R}(A^* AB) \subseteq \mathcal{R}(B) \Leftrightarrow A_1 A_1^* D^{-1} A_1 = A_1\) and \([A_1 A_1^*, D^{-1}] = 0\) (see Theorem 2.2, (a) parts 3. and 4.).

1. \(\Rightarrow\) 3. If \(A_1^\dagger = A_1^* D^{-1}\), then \(A_1^\dagger D = A_1^*\) and \(A_1 A_1^\dagger = A_1 A_1^* D^{-1}\) is Hermitian, so \([A_1 A_1^*, D^{-1}] = 0\). Moreover, \(A_1 A_1^\dagger A_2 A_2^* = 0\). We conclude \(\mathcal{R}(A_2 A_2^*) \subseteq \mathcal{N}(A_1 A_1^\dagger) = \mathcal{N}(A_1^*),\) so \(A_1^* A_2 A_2^* = 0\) and \(A_2 A_1 = 0\). Now, \((A_1 A_1^\dagger + A_2 A_2^*) A_1 = A_1 A_1^\dagger A_1,\) so \(A_1 = D^{-1} A_1 A_1^\dagger A_1 = A_1 A_1^* D^{-1} A_1\).

3. \(\Rightarrow\) 1. If 3. holds, then it is easy to see that \(A_1^* D^{-1}\) is the Moore-Penrose inverse of \(A_1\) (check the Penrose equations).

(b) We see that \(\mathcal{R}((A^\dagger AB)^*) = \mathcal{R}(B^* A^\dagger A) = \mathcal{R}(B^* A^*) = \mathcal{R}((AB)^*)\) is closed, so \((A^\dagger AB)^\dagger\) exists. Notice that

\[
B^\dagger A^\dagger A = \begin{bmatrix}
B_1^{-1} A_1^* D^{-1} A_1 & B_1^{-1} A_1^* D^{-1} A_2 \\
0 & 0
\end{bmatrix}
\]

and \(A^\dagger AB = \begin{bmatrix}
A_1^* D^{-1} A_1 B_1 & 0 \\
A_2^* D^{-1} A_1 B_1 & 0
\end{bmatrix}\). Using the formula \(T^\dagger = (T^*)^\dagger T^*\), we obtain

\[
(A^\dagger AB)^\dagger = \begin{bmatrix}
(B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 & (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_2 \\
0 & 0
\end{bmatrix}.
\]

1. \((A^\dagger AB)^\dagger = B^\dagger A^\dagger A \Leftrightarrow (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 = B_1^{-1} A_1^* D^{-1} A_1\) and \((B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_2 = B_1^{-1} A_1^* D^{-1} A_2\).

2. \((A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger B \Leftrightarrow B_1 (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* = A_1^*.

3. \(\mathcal{R}(BB^* A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow A_1 A_1^* D^{-1} A_1 = A_1\) and \([B_1 B_1^*, A_1^* D^{-1} A_1] = 0\).

1. \(\Rightarrow\) 2. We multiply the first equality of 1. by \(A_1^*\) from the right side, and we multiply the second equality of 1. by \(A_2^*\) from the right side. By summing the obtained equalities we obtain 2.

2. \(\Rightarrow\) 1. This is obvious.

2. \(\Rightarrow\) 3. If we multiply \(B_1 (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* = A_1^*\) by \(B_1^* A_1^* D^{-1} A_1\) from left, and by \(D^{-1} A_1 B_1\) from right side, we
get $A_1^*D^{-1}A_1 = A_1^*D^{-1}A_1A_1^*D^{-1}A_1$. Now, $A_1^*D^{-1}A_1$ is the orthogonal projection onto a subspace of $\mathcal{R}(A_1^*)$, so it follows that $A_1A_1^*D^{-1}A_1 = A_1$.

Since $(B_1^*A_1^*D^{-1}A_1B_1)\dagger B_1^*A_1^*D^{-1}A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ is Hermitian, we obtain $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

3. $\implies$ 2. Using the formula $T^\dagger = (T^*T)^\dagger T^*$, we have:

\[(B_1^*A_1^*D^{-1}A_1B_1)\dagger B_1^*A_1^*D^{-1/2} = (D^{-1/2}A_1B_1)^\dagger ,\]

which means that

\[B_1(B_1^*A_1^*D^{-1/2}D^{-1/2}A_1B_1)\dagger B_1^*A_1^* = B_1(D^{-1/2}A_1B_1)^\dagger D^{1/2}.\]

We wish to show that 3. implies $B_1(D^{-1/2}A_1B_1)^\dagger D^{1/2} = A_1^*$.

This means that we will show $(D^{-1/2}A_1B_1)\dagger = B_1^{-1}A_1^*D^{-1/2}$, by proving that the last expression satisfies all four Penrose equations provided that the conditions from 3. are valid. Hence,

\[D^{-1/2}A_1B_1B_1^{-1}A_1^*D^{-1/2}D^{-1/2}A_1B_1 = D^{-1/2}A_1A_1^*D^{-1}A_1B_1 = D^{-1/2}A_1B_1,\]

\[B_1^{-1}A_1^*D^{-1/2}D^{-1/2}A_1B_1B_1^{-1}A_1^*D^{-1/2} = B_1^{-1}A_1^*D^{-1}A_1A_1^*D^{-1/2} = B_1^{-1}A_1^*D^{-1/2},\]

\[D^{-1/2}A_1B_1B_1^{-1}A_1^*D^{-1/2} = D^{-1/2}A_1A_1^*D^{-1/2} \text{ is Hermitian},\]

\[B_1^{-1}A_1^*D^{-1/2}D^{-1/2}A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1 \text{ is Hermitian, since } [B_1B_1^*, A_1^*D^{-1}A_1] = 0.\]

(c) Follows from (a) and (b).

\[\square\]

**Theorem 2.5.** Let $X, Y, Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that $A, B, AB$ have closed ranges. Then we have:

(a) $B^\dagger = (AB)^\dagger A \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(A^*AB)$.

(b) $A^\dagger = B(AB)^\dagger \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^*A^*)$.

**Proof.** (a) We keep the matrix forms of $A$ and $B$ as in previous theorems.
1. It is easy to obtain that $B^\dagger = (AB)^\dagger A$ if and only if $I = (A_1 B_1)^\dagger A_1 B_1$ and $(A_1 B_1)^\dagger A_2 = 0$. Hence, 1. is equivalent to the following two conditions: $A_1$ is one-one with closed range, and $(A_1^* A_1 B_1)^\dagger A_2 = 0$. Hence, 1. is equivalent to the following two conditions: $A_1$ is one-one with closed range, and $(A_1^* A_1 B_1)^\dagger A_2 = 0$.

2. $\mathcal{R}(B) = \mathcal{R}(A^* AB)$ if and only if $\mathcal{R}(A_1^* A_1 B_1) = \mathcal{R}(B)$ and $A_2^* A_1 B_1 = 0$. Hence, 2. is equivalent to the following two conditions: $A_1$ is one-one with closed range and $A_1^* A_2 = 0$.

To prove the equivalence 1. $\iff$ 2., we have the following:

$(A_1 B_1)^\dagger A_2 = 0 \iff \mathcal{R}(A_2) \subset \mathcal{N}((A_1 B_1)^\dagger) = \mathcal{N}((A_1 B_1)^*)$

$\iff (A_1^* A_1 B_1)^\dagger A_2 = 0 \iff A_1^* A_2 = 0$.

(b) From the part (a) it follows that $(B^*)^\dagger = A^*(B^* A^*)^\dagger$ if and only if $\mathcal{R}(B) = \mathcal{R}(A^* AB)$. Now, we change $A^*$ by $B'$ and $B^*$ by $A'$, to obtain that (b) holds.

We need the following auxiliary result.

**Lemma 2.1.** Let $X, Y$ be Hilbert spaces, let $C \in \mathcal{L}(X, Y)$ have a closed range, and let $D \in \mathcal{L}(Y)$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $[D, C C^\dagger] = 0$.

**Proof.** $\implies$: We consider the orthogonal decompositions $X = \mathcal{R}(C^*) \oplus \mathcal{N}(C)$ and $Y = \mathcal{R}(C) \oplus \mathcal{N}(C^*)$. Then the operators $C$ and $D$ have the corresponding matrix forms as follows:

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where $C_1$ is invertible, and

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where $D_3 = D_2^*$. It follows that

$$DC = \begin{bmatrix} D_1 C_1 & 0 \\ D_3 C_1 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix}.$$

Hence, $\mathcal{R}(DC) = \mathcal{R}(C)$ implies $D_3 = 0$ and $D_2 = 0$, so $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$. Since $D$ is Hermitian and invertible, we obtain that $D_1$ and $D_4$ are also
Hermitian and invertible. Since $C^\dagger = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, we obtain that $DCC^\dagger = CC^\dagger D$ holds.

$\iff$ : If $D$ is invertible and $DCC^\dagger = CC^\dagger D$, then

$$\mathcal{R}(DC) = \mathcal{R}(DCC^\dagger) = \mathcal{R}(CC^\dagger D) = \mathcal{R}(CC^\dagger) = \mathcal{R}(C).$$

Finally, we prove the following results.

**Theorem 2.6.** Let $X, Y, Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that $A, B, AB$ have closed ranges. Then we have:

(a) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \iff \mathcal{R}(AA^* AB) = \mathcal{R}(AB)$;

(b) $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger \iff \mathcal{R}(B^* B(AB)^*) = \mathcal{R}((AB)^*)$.

Notice that $A^\dagger AB$ and $ABB^\dagger$ have closed ranges.

**Proof.** (a) Notice that

$$\mathcal{R}((A^\dagger AB)^*) = \mathcal{R}(B^* A^\dagger A) = B^* \mathcal{R}(A^\dagger A) = B^* \mathcal{R}(A^*) = \mathcal{R}((AB)^*)$$

is closed, so $\mathcal{R}(A^\dagger AB)$ is closed. First, let we see how our conditions looks like in the terms of their components.

1. Let us denote $T = A^\dagger AB$. We find $T^\dagger$ as follows

$$T^\dagger = (T^* T)^\dagger = \begin{bmatrix} (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 & (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}.$$ 

Now, it is easy to see that $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ is equivalent with

$$(A_1 B_1)^\dagger = (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} = (D^{-1/2} A_1 B_1)^\dagger D^{-1/2}.$$ 

2. It is obvious that $AA^* AB = \begin{bmatrix} D A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix}$, so 2. holds if and only if $\mathcal{R}(DA_1 B_1) = \mathcal{R}(A_1 B_1)$.  

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1. ⇒ 2. From the third Penrose equation for \((A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}\), we see that \(A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}\) is Hermitian. So, we have the following equivalences:

\[
\begin{align*}
A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2} & \text{ is Hermitian} \\
\iff & \quad D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1} \text{ is Hermitian} \\
\iff & \quad [D, D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger] = 0 \\
\iff & \quad D^{1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger = D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger D \\
\iff & \quad DA_1B_1(D^{-1/2}A_1B_1)^\dagger = A_1B_1(D^{-1/2}A_1B_1)^\dagger D.
\end{align*}
\]

Now,

\[
\mathcal{R}(DA_1B_1) = \mathcal{R}(DA_1B_1(A_1B_1)^\dagger) = \mathcal{R}(A_1B_1(A_1B_1)^\dagger D) = \mathcal{R}(A_1B_1).
\]

2. ⇒ 1. If \(\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1)\), then we apply Lemma 2.1 to obtain \([D, A_1B_1(A_1B_1)^\dagger] = 0\). Now, from the previous implication it follows that \(A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}\) is Hermitian. Notice that \(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1B_1\) is the orthogonal projection onto

\[
\mathcal{R}((A_1B_1)^*D^{-1/2}) \subset \mathcal{R}((A_1B_1)^*),
\]

so \(A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1B_1 = A_1B_1\). Finally, it is not difficult to verify that \((A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}\) holds.

(b) According to (a), we have the following equivalences:

\[
\begin{align*}
(AB)^\dagger & = (A^\dagger AB)^\dagger A^\dagger \iff \mathcal{R}(AA^*AB) = \mathcal{R}(AB) \\
(B^*A^*)^\dagger & = (A^*)^\dagger (B^*A^\dagger A)^\dagger \iff \mathcal{R}(AA^*AB) = \mathcal{R}(A) \\
& \text{(now take } A' = B^* \text{ and } B' = A^*) \\
(A'B')^\dagger & = B^\dagger (ABB^\dagger)^\dagger \iff \mathcal{R}(BB^*B'^*A'^*) = \mathcal{R}(B'^*A'^*).
\end{align*}
\]

\[\square\]

Finally, it is interesting to see how much of this extends to \(C^*\)-algebras. This will be a subject of further investigations.

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References


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