REVERSE ORDER LAW FOR THE MOORE-PENROSE INVERSE

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Abstract

In this paper we present new results related to the reverse order law for the Moore-Penrose inverse of operators on Hilbert spaces. Some finite dimensional results are extended to infinite dimensional settings.

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1 Introduction

If S is a semigroup with the unit 1, and if $a, b \in S$ are invertible, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. It is well-known that the reverse order law does not hold for various classes of generalized inverses. Hence, a significant number of papers treat the sufficient or equivalent conditions such that the reverse order law holds in some sense. In this paper we specialize the investigations to the Moore-Penrose inverse of closed range linear bounded operators on Hilbert spaces.

Let X, Y, Z be Hilbert spaces, and let $\mathcal{L}(X, Y)$ denote the set of all linear bounded operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X, X)$. For $A \in \mathcal{L}(X, Y)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A. An operator $B \in \mathcal{L}(Y, X)$ is an inner inverse of A, if $ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in Y. The Moore-Penrose inverse of $A \in \mathcal{L}(X, Y)$ is the operator $X \in \mathcal{L}(Y, X)$ which satisfies the Penrose equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

The Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed in Y. If the Moore-Penrose inverse of A exists, then it is unique, and it is denoted by $A^\dagger$.

If $\theta \subset \{1, 2, 3, 4\}$, and X satisfies the equations (i) for all $i \in \theta$, then X is an θ-inverse of A. The set of all θ-inverses of A is denoted by $A\{\theta\}$. If

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$\mathcal{R}(A)$ is closed, then $A\{1,2,3,4\} = \{A^\dagger\}$. The theory of generalized inverses on infinite dimensional Hilbert spaces can be found in [4, 8, 10].

It is a classical result of Greville [9], that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$, in the case when A and B are complex (possibly rectangular) matrices. This result is extended for linear bounded operators on Hilbert spaces, by Bouldin [2], [3], and Izumino [12]. Among other things, Bouldin and Izumino used gaps between subspaces. In [13] the reverse order law for the Moore-Penrose inverse is proved in rings with involutions. Then, in [6], the reverse order law for the Moore-Penrose inverse is obtained as a consequence of some set equalities. The reader can find some interesting and related results in [1, 5, 8, 11, 14, 15, 16, 17, 18].

In particular, the paper [15] is related to our investigations. In [15] Tian obtained some very interesting results concerning the sets of generalized inverses of complex rectangular matrices. As a corollary, the reverse order law for the Moore-Penrose inverse follows. Notice that the finite dimensional methods are used in [15] (mostly the rank of a complex matrix).

In this paper we extend some results from [15] to infinite dimensional settings. Among other things, we obtain the reverse order law for the Moore-Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary result. In Section 2 we present the results related to the reverse order rule for the Moore-Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite dimensional settings.

Lemma 1.1. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof is straightforward.

Lemma 1.2. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X, such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y, such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^*A_1 + A_2^*A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i denotes different operators in any of these two cases.

Proof. Recall that one special case of this result is proved in [7]. We prove only the result of (a), since the proof of (b) is analogous.

The operator A has the following representation:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

i.e.

$$A_1 = A_{|X_1} : X_1 \to \mathcal{R}(A), \; A_2 = A_{|X_2} : X_2 \to \mathcal{R}(A),$$

$$A_3 = A_{|X_1} : X_1 \to \mathcal{N}(A^*), \; A_4 = A_{|X_2} : X_2 \to \mathcal{N}(A^*).$$

Furhtermore,

$$A^* = \begin{bmatrix} A_1^* & A_3^* \\ A_2^* & A_4^* \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

From $A^*(\mathcal{N}(A^*)) = \{0\}$ we obtain $A_3^* = 0$ and $A_4^* = 0$, so $A_3 = 0$ and $A_4 = 0$. Hence, $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$. Notice that

$$AA^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^* : \mathcal{R}(A) \to \mathcal{R}(A)$. From $\mathcal{N}(AA^*) = \mathcal{N}(A^*)$ it follows that D is one-one. From $\mathcal{R}(AA^*) = \mathcal{R}(A)$ it follows that D is onto. Hence, D is invertible. Finally, we obtain the form for the Moore-Penrose inverse of A using the formula $A^\dagger = A^*(AA^*)^\dagger$. □

The following result is well-known, and it can be found in [4], p.127.

Lemma 1.3. *Let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ have closed ranges. Then AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range.*

Finally, the reader should notice the difference between the following notations. If $A, B \in \mathcal{L}(X)$, then $[A, B] = AB - BA$ denotes the commutator of A and B. On the other hand, if $U \in \mathcal{L}(X, Z)$ and $V \in \mathcal{L}(Y, Z)$, then $[U \;\; V] : \begin{bmatrix} X \\ Y \end{bmatrix} \to Z$ denote the matrix form of the corresponding operator.

2 Reverse order law

In this section we prove the results concerning the reverse order law for the Moore-Penrose inverse.

Theorem 2.1. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(a) $ABB^\dagger A^\dagger AB = AB$;

(b) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$;

(c) $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$;

(d) $A^\dagger ABB^\dagger$ *is an idempotent;*

(e) $BB^\dagger A^\dagger A$ *is an idempotent;*

(f) $B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$;

(g) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

Notice that $A^\dagger ABB^\dagger$ has a closed range, according to Lemma 1.3. Moreover, A^*ABB^* also has a closed range: $\mathcal{R}(B^*A^*A) = B^*(\mathcal{R}(A^*A)) = B^*(\mathcal{R}(A^*)) = \mathcal{R}((AB)^*)$ is closed, so $\mathcal{R}(A^*ABB^*) = A^*A(\mathcal{R}(BB^*)) = A^*A(\mathcal{R}(B)) = \mathcal{R}(A^*AB) = \mathcal{R}((B^*A^*A)^*)$ is closed.

Proof. Using Lemma 1.1 we conclude that the operator B has the following matrix form:

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From Lemma 1.2 it follows that the operator A has the following matrix form:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

Notice the following:

$$BB^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

$$AA^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

and

$$A^\dagger A = \begin{bmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2 \\ A_2^*D^{-1}A_1 & A_2^*D^{-1}A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

From Lemma 1.3 it follows that $A^\dagger ABB^\dagger$ has a closed range. We obtain

$$A^\dagger ABB^\dagger = \begin{bmatrix} A_1^*D^{-1}A_1 & 0 \\ A_2^*D^{-1}A_1 & 0 \end{bmatrix},$$

$$BB^\dagger A^\dagger A = \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}.$$

Consider the following chain of equivalences, which is related to the statement of (a):

$$\begin{aligned} & ABB^\dagger A^\dagger AB = AB \\ \iff & \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix} \\ \iff & \begin{bmatrix} A_1 A_1^* D^{-1} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\iff A_1 A_1^* D^{-1} A_1 = A_1. \tag{1}$$

Consequently, the statement (a) is equivalent to (1).

Notice that (1) is equivalent to

$$A_1^* D^{-1} A_1 A_1^* = A_1^*. \tag{2}$$

We consider also the statement (b):

$$\begin{aligned} & B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger \\ \iff & \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix} \\ & = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix} \\ \iff & \begin{bmatrix} B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1^{-1} A_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ \iff & B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1} = B_1^{-1} A_1^* D^{-1} \iff (2). \end{aligned}$$

Thus, (a) $\iff$ (1) $\iff$ (2) $\iff$ (b).

In the case of the statement (c) we have:

$$A^\dagger ABB^\dagger = BB^\dagger A^\dagger A \iff \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}$$

$$\iff A_1^* D^{-1} A_2 = 0 \iff A_2^* D^{-1} A_1 = 0. \tag{3}$$

Thus, if (c) holds, i.e. $A_2^* D^{-1} A_1 = 0$, then it is obvious that $A_2 A_2^* D^{-1} A_1 = 0$, so (1) also holds because of:

$$(A_1 A_1^* + A_2 A_2^*) D^{-1} = I \implies A_1 A_1^* D^{-1} A_1 + A_2 A_2^* D^{-1} A_1 = A_1$$
$$\iff A_1 A_1^* D^{-1} A_1 = A_1.$$

On the other hand, suppose that (1) holds. Then $A_2 A_2^* D^{-1} A_1 = 0$, and we have the following

$$A_2 A_2^* D^{-1} A_1 = 0 \Rightarrow \mathcal{R}(D^{-1} A_1) \subset \mathcal{N}(A_2 A_2^*) = \mathcal{N}(A_2^*) \Rightarrow A_2^* D^{-1} A_1 = 0,$$

so (3) is satisfied. Consequently, (c) also holds. We have just proved (c) $\iff$ (3) $\iff$ (1) $\iff$ (a).

A straightforward computation shows that (d) is equivalent to

$$\begin{cases} A_1^* D^{-1} A_1 A_1^* D^{-1} A_1 = A_1^* D^{-1} A_1 \\ A_2^* D^{-1} A_1 A_1^* D^{-1} A_1 = A_2^* D^{-1} A_1 \end{cases} \tag{4}$$

If the statement (1) holds, then obviously (4) is satisfied. On the other hand, suppose that (4) holds. Then multiply the first equation of (4) by A_1 from the left side, and multiply the second equation of (4) by A_2 from the left side. The sum of these two new equations leads to the equation (1).

Notice that (e) is also equivalent to (4). Consequently, (d) $\iff$ (4) $\iff$ (2) $\iff$ (e).

In order to establish (f), we proceed as follows. Let $Q = A^\dagger A B B^\dagger$. From Lemma 1.3 we know that Q has a closed range. We use the formula $Q^\dagger = Q^*(QQ^*)^\dagger = (Q^*Q)^\dagger Q^*$. Hence,

$$\begin{aligned} (A^\dagger A B B^\dagger)^\dagger &= (BB^\dagger A^\dagger A A^\dagger A B B^\dagger)^\dagger B B^\dagger A^\dagger A = (BB^\dagger A^\dagger A B B^\dagger)^\dagger B B^\dagger A^\dagger A \\ &= \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (A_1^* D^{-1} A_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_1 & (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

We get

$$\begin{aligned} & B^\dagger (A^\dagger A B B^\dagger)^\dagger A^\dagger - B^\dagger A^\dagger = 0 \\ \iff & \begin{bmatrix} B_1^{-1} (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} - B_1^{-1} A_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

$$\iff (A_1^* D^{-1} A_1)^\dagger A_1^* = A_1^*. \tag{5}$$

We need to prove (1) $\iff$ (5). Let $P = A_1^* D^{-1} A_1$. Obviously, $P^* = P$.
(1)$\Rightarrow$ (5): We have the following:

$$\begin{gathered} P^2 = A_1^* D^{-1} A_1 A_1^* D^{-1} A_1 = A_1^* D^{-1} A_1 = P, \\ P = P^* = P^2 = P^\dagger, \\ (A_1^* D^{-1} A_1)^\dagger A_1^* = A_1^* D^{-1} A_1 A_1^* = A_1^*. \end{gathered}$$

(5)$\Rightarrow$(1): In this case we have

$$\begin{gathered} (A_1^* D^{-1} A_1)^\dagger A_1^* = A_1^*, \\ P^\dagger P = P, \\ PP^\dagger = (PP^\dagger)^* = (P^*)^\dagger P^* = P^\dagger P \\ P^\dagger = P^\dagger P P^\dagger = PP^\dagger = P^\dagger P = P \\ A_1^* = (A_1^* D^{-1} A_1)^\dagger A_1^* = A_1^* D^{-1} A_1 A_1^*. \end{gathered}$$

We have just proved (f) $\iff$ (1) $\iff$ (a).

To prove (g) $\iff$ (f), we use the fact which is already proved for (f), i.e. for $(A^\dagger A B B^\dagger)^\dagger$. Thus, we have

$$(A^\dagger A B B^\dagger)^\dagger - B B^\dagger A^\dagger A = 0 \iff \begin{cases} (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_1 = A_1^* D^{-1} A_1, \\ (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_2 = A_1^* D^{-1} A_2. \end{cases}$$

It is easy to conclude that (g) $\iff$ (f).

□

Now we prove the following result.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;

(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;

(c) *The following statements are equivalent:*

  (1) $(AB)^\dagger = B^\dagger A^\dagger$*;*

  (2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$*;*

(3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;

(4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. The following products will be useful:

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, (AB)^\dagger = \begin{bmatrix} (A_1B_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we find the equivalent expressions for our statements in terms of A_1, A_2 and B_1.

(a)
1. $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$. Here $A_1B_1(A_1B_1)^\dagger$ is Hermitian, so $[A_1A_1^*, D^{-1}] = 0$.
2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so 2. $\Longleftrightarrow$ 3.
4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: 1. $\Longleftrightarrow$ 2., 4. $\Longrightarrow$ 2. and 1. $\Longrightarrow$ 4.

We prove 1. $\Longleftrightarrow$ 2. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \iff (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by A_1B_1 from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned} &(A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} \\ &\iff (A_1B_1)^\dagger(A_1A_1^* + A_2A_2^*) = (A_1B_1)^\dagger A_1A_1^* \\ &\iff (A_1B_1)^\dagger A_2A_2^* = 0 \iff \mathcal{R}(A_2A_2^*) \subset \mathcal{N}((A_1B_1)^\dagger) \\ &\iff \mathcal{R}(A_2) \subset \mathcal{N}((A_1B_1)^*) \iff B_1^*A_1^*A_2 = 0 \iff A_1^*A_2 = 0, \end{aligned}$$

Therefore, we have just proved that 1. $\Leftrightarrow$ 2.

Now we prove 1. $\Longrightarrow$ 4. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by A_1B_1 from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4. holds.

Finally, we prove 4. $\Longrightarrow$ 2. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2. holds.

Notice that the equivalence 3. $\Longleftrightarrow$ 4. is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.

3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, 2. $\Longleftrightarrow$ 3.

4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove 1. $\Rightarrow$ 4. $\Rightarrow$ 2. $\Rightarrow$ 1.

Suppose that 1. holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by A_1B_1 from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, 1. $\Rightarrow$ 4.

Suppose that 4. holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2. holds. The second equality of 2. also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of the Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, 4. $\Rightarrow$ 2.

In order to prove that 2. $\Longrightarrow$ 1., we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, 2. $\Rightarrow$ 1.

Notice that 3. $\Longleftrightarrow$ 4. is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b). $\square$

We also prove the following result.

Theorem 2.3. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;

(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;

(c) *The following three statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;

(3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of A_1, A_2 and B_1, for our assumptions.

(a)

1. $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger A_1 = A_1$ and $A_1B_1(A_1B_1)^\dagger A_2 = 0$. The first equality on the right side of the equivalence always holds, so: $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger A_2 = 0$.
2. $A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow A_1^*A_2 = 0$.
3. $\mathcal{R}(A^*AB) \subset \mathcal{R}(B) \Leftrightarrow BB^\dagger A^*AB = A^*AB \Leftrightarrow A_2^*A_1 = 0$ (see the proof of Theorem 2.2, the part (a) 2. and 3.).
4. $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$ (see Theorem 2.2 (a) 4.).

To prove that 1.⇔2., we see that $A_1B_1(A_1B_1)^\dagger A_2 = 0 \Leftrightarrow \mathcal{R}(A_2) \subset \mathcal{N}((A_1B_1)(A_1B_1)^\dagger) = \mathcal{N}((A_1B_1)^\dagger) = \mathcal{N}((A_1B_1)^*) = \mathcal{N}(B_1^*A_1^*) = \mathcal{N}(A_1^*) \Leftrightarrow A_1^*A_2 = 0$.

Now, we prove that 2.⇔4. If $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*D^{-1}A_1 = A_1 \Leftrightarrow A_1A_1^*A_1 = DA_1 \Leftrightarrow A_2A_2^*A_1 = 0 \Leftrightarrow A_1^*A_2A_2^* = 0 \Leftrightarrow \mathcal{R}(A_2A_2^*) \subset \mathcal{N}(A_1^*) \Leftrightarrow \mathcal{R}(A_2) \subset \mathcal{N}(A_1^*) \Leftrightarrow A_1^*A_2 = 0$. On the other hand, if $A_1^*A_2 = 0$, then $A_1A_1^*D = A_1A_1^*A_1A_1^*$ is Hermitian, so $A_1A_1^*$ commutes with D. This implies $[A_1A_1^*, D^{-1}] = 0$ and $A_1A_1^*D^{-1}A_1 = A_1$.

From Theorem 2.2 we know that 3.⇔4.

(b)

1. $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow B_1(A_1B_1)^\dagger A_1 = A_1^*D^{-1}A_1$ and $A_2^*D^{-1}A_1 = 0$.
2. $A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow [B_1B_1^*, A_1^*D^{-1}A_1] = 0$ and $A_1^*D^{-1}A_2 = 0$.
3. $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \iff A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$ and $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ (Theorem 2.2 (b) parts 2. and 3.).
4. $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ (Theorem 2.2 (b) part 4.).

1. $\Longrightarrow$ 4. We multiply the expression $B_1(A_1B_1)^\dagger A_1 = A_1^*D^{-1}A_1$ by A_1 from the left side, and by B_1 from the right side, and thus obtain $A_1A_1^*D^{-1}A_1 = A_1$. Also, we obtain that $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ is Hermitian. Hence, $A_1^*D^{-1}A_1B_1B_1^*$ is Hermitian, and we get $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

4. $\Longrightarrow$ 1. If 4. holds, then it is easy to see that $B_1^{-1}A_1^*D^{-1}A_1B_1(A_1B_1)^\dagger$ is the Moore-Penrose inverse of A_1B_1 (check the Penrose equations). This implies $B_1(A_1B_1)^\dagger A_1 = A_1^*D^{-1}A_1$. Now, we obtain that $A_1 = A_1A_1^*D^{-1}A_1$. From $(A_1A_1^* + A_2A_2^*)D^{-1}A_1 = A_1$ it follows that $A_2A_2^*D^{-1}A_1 = 0$, so $\mathcal{R}(D^{-1}A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, and $A_2^*D^{-1}A_1 = 0$.

2. $\Longrightarrow$ 3. If 2. holds, then $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$ is trivially satisfied. Moreover, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ is equivalent to $A_1A_1^*D^{-1}A_1 = A_1$, which follows from $A_1^*D^{-1}A_2 = 0$.

3. $\Longrightarrow$ 2. From the proof of Theorem 2.2, part (b) 4., it follows that $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Now, in the usual manner, we get that $A_2A_2^*D^{-1}A_1 = 0$, so $A_1^*D^{-1}A_2 = 0$.

2. $\Longleftrightarrow$ 4. Obvious.

The part (c) follows from the parts (a) and (b). □

We also prove the following result.

Theorem 2.4. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. The following statements hold.*

(a) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger \Leftrightarrow B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$.

(b) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A \Leftrightarrow (A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

(c) *The following three statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(3) $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger$.

Notice that $ABB^\dagger$ and $A^\dagger AB$ have closed ranges. This is explained in the further proof.

Proof. The operators A and B have the same matrix representations as in the previous theorem.

(a) Notice that $\mathcal{R}(ABB^\dagger) = \mathcal{R}(AB)$ is closed, so there exists $(ABB^\dagger)^\dagger$.

1. $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger \Leftrightarrow A_1^\dagger = A_1^* D^{-1}$ (the existence of $A_1^\dagger$ follows from the assumptions).
2. $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger \Leftrightarrow A_1^\dagger = A_1^* D^{-1}$, so 1.⇔2.
3. $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$ (see Theorem 2.2, (a) parts 3. and 4.).

1. $\Longrightarrow$ 3. If $A_1^\dagger = A_1^*D^{-1}$, then $A_1^\dagger D = A_1^*$ and $A_1A_1^\dagger = A_1A_1^*D^{-1}$ is Hermitian, so $[A_1A_1^*, D^{-1}] = 0$. Moreover, $A_1A_1^\dagger A_2A_2^* = 0$. We conclude $\mathcal{R}(A_2A_2^*) \subset \mathcal{N}(A_1A_1^\dagger) = \mathcal{N}(A_1^*)$, so $A_1^*A_2A_2^* = 0$ and $A_2^*A_1 = 0$. Now, $(A_1A_1^* + A_2A_2^*)A_1 = A_1A_1^*A_1$, so $A_1 = D^{-1}A_1A_1^*A_1 = A_1A_1^*D^{-1}A_1$.

3. $\Longrightarrow$ 1. If 3. holds, then it is easy to see that $A_1^*D^{-1}$ is the Moore-Penrose inverse of A_1 (check the Penrose equations).

(b) We see that $\mathcal{R}((A^\dagger AB)^*) = \mathcal{R}(B^*A^\dagger A) = \mathcal{R}(B^*A^*) = \mathcal{R}((AB)^*)$ is closed, so $(A^\dagger AB)^\dagger$ exists. Notice that

$$B^\dagger A^\dagger A = \begin{bmatrix} B_1^{-1}A_1^*D^{-1}A_1 & B_1^{-1}A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}$$

and $A^\dagger AB = \begin{bmatrix} A_1^*D^{-1}A_1B_1 & 0 \\ A_2^*D^{-1}A_1B_1 & 0 \end{bmatrix}$. Using the formula $T^\dagger = (T^*T)^\dagger T^*$, we obtain

$$(A^\dagger AB)^\dagger = \begin{bmatrix} (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_1 & (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}.$$

1. $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A \Leftrightarrow (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_1 = B_1^{-1}A_1^*D^{-1}A_1$ and $(B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_2 = B_1^{-1}A_1^*D^{-1}A_2$.
2. $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger \Leftrightarrow B_1(B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^* = A_1^*$.
3. $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*) \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

   1. $\Longrightarrow$ 2. We multiply the first equality of 1. by A_1^* from the right side, and we multiply the second equality of 1. by A_2^* from the right side. By summing the obtained equalities we obtain 2.

   2. $\Longrightarrow$ 1. This is obvious.

   2. $\Longrightarrow$ 3. If we multiply $B_1(B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^* = A_1^*$ by $B_1^*A_1^*D^{-1}A_1$ from left, and by $D^{-1}A_1B_1$ from right side, we

get $A_1^* D^{-1} A_1 = A_1^* D^{-1} A_1 A_1^* D^{-1} A_1$. Now, $A_1^* D^{-1} A_1$ is the orthogonal projection onto a subspace of $\mathcal{R}(A_1^*)$, so it follows that $A_1 A_1^* D^{-1} A_1 = A_1$.

Since $(B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 B_1 = B_1^{-1} A_1^* D^{-1} A_1 B_1$ is Hermitian, we obtain $[B_1 B_1^*, A_1^* D^{-1} A_1] = 0$.

3. $\Longrightarrow$ 2. Using the formula $T^\dagger = (T^*T)^\dagger T^*$, we have:

$$(B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1/2} = (D^{-1/2} A_1 B_1)^\dagger,$$

which means that

$$B_1 (B_1^* A_1^* D^{-1/2} D^{-1/2} A_1 B_1)^\dagger B_1^* A_1^* = B_1 (D^{-1/2} A_1 B_1)^\dagger D^{1/2}.$$

We wish to show that 3. implies $B_1 (D^{-1/2} A_1 B_1)^\dagger D^{1/2} = A_1^*$. This means that we will show $(D^{-1/2} A_1 B_1)^\dagger = B_1^{-1} A_1^* D^{-1/2}$, by proving that the last expression satisfies all four Penrose equations provided that the conditions from 3. are valid. Hence,

$$\begin{aligned} D^{-1/2} A_1 B_1 B_1^{-1} A_1^* D^{-1/2} D^{-1/2} A_1 B_1 &= D^{-1/2} A_1 A_1^* D^{-1} A_1 B_1 \\ &= D^{-1/2} A_1 B_1, \end{aligned}$$

$$\begin{aligned} B_1^{-1} A_1^* D^{-1/2} D^{-1/2} A_1 B_1 B_1^{-1} A_1^* D^{-1/2} &= B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1/2} \\ &= B_1^{-1} A_1^* D^{-1/2}, \end{aligned}$$

$$D^{-1/2} A_1 B_1 B_1^{-1} A_1^* D^{-1/2} = D^{-1/2} A_1 A_1^* D^{-1/2} \text{ is Hermitian,}$$

$$\begin{aligned} B_1^{-1} A_1^* D^{-1/2} D^{-1/2} A_1 B_1 = B_1^{-1} A_1^* D^{-1} A_1 B_1 &\text{ is Hermitian,} \\ \text{since } [B_1 B_1^*, A_1^* D^{-1} A_1] &= 0. \end{aligned}$$

(c) Follows from (a) and (b).

□

Theorem 2.5. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $B^\dagger = (AB)^\dagger A \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(A^* AB)$.

(b) $A^\dagger = B(AB)^\dagger \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^* A^*)$.

Proof. (a) We keep the matrix forms of A and B as in previous theorems.

1. It is easy to obtain that $B^\dagger = (AB)^\dagger A$ if and only if $I = (A_1B_1)^\dagger A_1B_1$ and $(A_1B_1)^\dagger A_2 = 0$. Hence, 1. is equivalent to the following two conditions: A_1 is one-one with closed range, and $(A_1B_1)^\dagger A_2 = 0$.
2. $\mathcal{R}(B) = \mathcal{R}(A^*AB)$ if and only if $\mathcal{R}(A_1^*A_1B_1) = \mathcal{R}(B)$ and $A_2^*A_1B_1 = 0$. Hence, 2. is equivalent to the following two conditions: A_1 is one-one with closed range and $A_1^*A_2 = 0$.

To prove the equivalence 1. $\iff$ 2., we have the following:

$$\begin{aligned}&(A_1B_1)^\dagger A_2 = 0 \iff \mathcal{R}(A_2) \subset \mathcal{N}((A_1B_1)^\dagger) = \mathcal{N}((A_1B_1)^*)\\ &\iff B_1^*A_1^*A_2 = 0 \iff A_1^*A_2 = 0.\end{aligned}$$

(b) From the part (a) it follows that $(B^*)^\dagger = A^*(B^*A^*)^\dagger$ if and only if $\mathcal{R}(B) = \mathcal{R}(A^*AB)$. Now, we change A^* by B' and B^* by A', to obtain that (b) holds.

□

We need the following auxiliary result.

Lemma 2.1. *Let X, Y be Hilbert spaces, let $C \in \mathcal{L}(X, Y)$ have a closed range, and let $D \in \mathcal{L}(Y)$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $[D, CC^\dagger] = 0$.*

Proof. $\Longrightarrow$: We consider the orthogonal decompositions $X = \mathcal{R}(C^*) \oplus \mathcal{N}(C)$ and $Y = \mathcal{R}(C) \oplus \mathcal{N}(C^*)$. Then the operators C and D have the corresponding matrix forms as follows:

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where C_1 is invertible, and

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where $D_3 = D_2^*$. It follows that

$$DC = \begin{bmatrix} D_1C_1 & 0 \\ D_3C_1 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^* \end{bmatrix}.$$

Hence, $\mathcal{R}(DC) = \mathcal{R}(C)$ implies $D_3 = 0$ and $D_2 = 0$, so $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix}$. Since D is Hermitian and invertible, we obtain that D_1 and D_4 are also

Hermitian and invertible. Since $C^\dagger = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, we obtain that $DCC^\dagger = CC^\dagger D$ holds.

$\Longleftarrow$: If D is invertible and $DCC^\dagger = CC^\dagger D$, then

$$\mathcal{R}(DC) = \mathcal{R}(DCC^\dagger) = \mathcal{R}(CC^\dagger D) = \mathcal{R}(CC^\dagger) = \mathcal{R}(C).$$

□

Finally, we prove the following results.

Theorem 2.6. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AA^*AB) = \mathcal{R}(AB)$;

(b) $(AB)^\dagger = B^\dagger(ABB^\dagger)^\dagger \Leftrightarrow \mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$.

Notice that $A^\dagger AB$ and $ABB^\dagger$ have closed ranges.

Proof. (a) Notice that

$$\mathcal{R}((A^\dagger AB)^*) = \mathcal{R}(B^*A^\dagger A) = B^*\mathcal{R}(A^\dagger A) = B^*\mathcal{R}(A^*) = \mathcal{R}((AB)^*)$$

is closed, so $\mathcal{R}(A^\dagger AB)$ is closed. First, let we see how our conditions looks like in the terms of their components.

1. Let us denote $T = A^\dagger AB$. We find $T^\dagger$ as follows

$$\begin{aligned} T^\dagger &= (T^*T)^\dagger T^* \\ &= \begin{bmatrix} (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_1 & (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Now, it is easy to see that $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ is equivalent with

$$(A_1B_1)^\dagger = (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1} = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}.$$

2. It is obvious that $AA^*AB = \begin{bmatrix} DA_1B_1 & 0 \\ 0 & 0 \end{bmatrix}$, so 2. holds if and only if $\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1)$.

1. $\Rightarrow$ 2. From the third Penrose equation for $(A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}$, we see that $A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}$ is Hermitian. So, we have the following equivalences:

$$\begin{aligned} & A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2} \text{ is Hermitian} \\ \Longleftrightarrow \quad & D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1} \text{ is Hermitian} \\ \Longleftrightarrow \quad & [D, D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger] = 0 \\ \Longleftrightarrow \quad & D^{1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger = D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger D \\ \Longleftrightarrow \quad & DA_1B_1(D^{-1/2}A_1B_1)^\dagger = A_1B_1(D^{-1/2}A_1B_1)^\dagger D. \end{aligned}$$

Now,

$$\mathcal{R}(DA_1B_1) = \mathcal{R}(DA_1B_1(A_1B_1)^\dagger) = \mathcal{R}(A_1B_1(A_1B_1)^\dagger D) = \mathcal{R}(A_1B_1).$$

2. $\Rightarrow$ 1. If $\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1)$, then we apply Lemma 2.1 to obtain $[D, A_1B_1(A_1B_1)^\dagger] = 0$. Now, from the previous implication it follows that $A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}$ is Hermitian. Notice that $D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1B_1$ is the orthogonal projection onto

$$\mathcal{R}((A_1B_1)^* D^{-1/2}) \subset \mathcal{R}((A_1B_1)^*),$$

so $A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1B_1 = A_1B_1$. Finally, it is not difficult to verify that $(A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}$ holds.

(b) According to (a), we have the following equivalences:

$$\begin{aligned} & (AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AA^*AB) = \mathcal{R}(AB) \\ & (B^*A^*)^\dagger = (A^*)^\dagger (B^*A^\dagger A)^\dagger \iff \mathcal{R}(AA^*AB) = \mathcal{R}(A) \\ & (\text{now take } A' = B^* \text{ and } B' = A^*) \\ & (A'B')^\dagger = B'^\dagger (ABB'^\dagger)^\dagger \iff \mathcal{R}(BB'^*B'^*A'^*) = \mathcal{R}(B'^*A'^*). \end{aligned}$$

□

Finally, it is interesting to see how much of this extends to C^*-algebras. This will be a subject of further investigations.

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