Additive results for the $Wg$-Drazin inverse

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Abstract

In this paper we prove the formula for the expression $(A + B)^{d,W}$ in terms of $A, B, W, A^{d,W}, B^{d,W}$, assuming some conditions for $A, B$ and $W$. Here $S^{d,W}$ denotes the generalized $W$-weighted Drazin inverse of a linear bounded operator $S$ on a Banach space.

Key words and phrases: $Wg$-Drazin inverse, additive result, explicit formula.


1 Introduction

Let $X$ and $Y$ denote arbitrary Banach spaces. We use $B(X, Y)$ to denote the set of all linear bounded operators from $X$ to $Y$. Set $B(X) = B(X, X)$. Let $A \in B(X, Y)$ and $W \in B(Y, X)$ be nonzero operators. If there exists some $S \in B(X, Y)$ satisfying

$$(AW)^{k+1}SW = (AW)^k, \quad SWAWS = S, \quad AWS = SWA,$$

for some nonnegative integer $k$, then $S$ is called the $W$-weighted Drazin inverse of $A$ and denoted by $S = A^{D,W}$ [12], [13], [15]. If there exists $A^{D,W}$, then we say that $A$ is $W$-Drazin invertible and $A^{D,W}$ must be unique [12]. If $X = Y$, $A \in B(X)$ and $W = I$, then $S = A^D$, the ordinary Drazin inverse of $A$. Further related results can also be found in [3, 4, 7, 11, 14, 16, 17].

Let $B_W(X, Y)$ be the space $B(X, Y)$ equipped with the multiplication $A \ast B = AWB$ and the norm $\|A\|_W = \|A\|\|W\|$. Then $B_W(X, Y)$ becomes a Banach algebra [6]. $B_W(X, Y)$ has the unit if and only if $W$ is invertible, in which case $W^{-1}$ is that unit.

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Let \( A \) be a Banach algebra. Then \( a \in A \) is quasipolar if and only if there exists \( b \in A \) such that

\[
ab = ba, \quad bab = b, \quad a - aba \text{ is quasinilpotent.}
\]

The element \( b \), if exists, is unique [9] (Theorem 7.5.3), [10]. Such \( b \) is the generalized Drazin inverse, or Koliha-Drazin inverse of \( a \), and it is denoted by \( a^d \).

Let \( W \in B(Y, X) \) be a fixed nonzero operator. An operator \( A \in B(X, Y) \) is called \( Wg \)-Drazin invertible if \( A \) is quasipolar in the Banach algebra \( B_W(X, Y) \). The \( Wg \)-Drazin inverse \( A^{d,W} \) of \( A \) is defined as the \( g \)-Drazin inverse of \( A \) in the Banach algebra \( B_W(X, Y) \) [6].

Let us recall that if \( A \in B(X, Y) \) and \( W \in B(Y, X) \) then the following conditions are equivalent [6]:

1. \( A \) is \( Wg \)-Drazin invertible,
2. \( AW \) is quasipolar in \( B(Y) \) with \((AW)^d = A^{d,W}W\),
3. \( WA \) is quasipolar in \( B(X) \) with \((WA)^d = WA^{d,W}\).

Then, the \( Wg \)-Drazin inverse \( A^{d,W} \) of \( A \) satisfies

\[
A^{d,W} = ((AW)^d)^2 A = A((WA)^d)^2.
\]

**Lemma 1.1** [6] Let \( A \in B(X, Y) \) and \( W \in B(Y, X) \setminus \{0\} \). Then \( A \) is \( Wg \)-Drazin invertible if and only if there exist topological direct sums \( X = X_1 \oplus X_2, Y = Y_1 \oplus Y_2 \) such that

\[
A = A_1 \oplus A_2, \quad W = W_1 \oplus W_2,
\]

where \( A_i \in B(X_i, Y_i) \), \( W_i \in B(Y_i, X_i) \), with \( A_1, W_1 \) invertible, and \( W_2A_2 \) and \( A_2W_2 \) quasinilpotent in \( B(X_2) \) and \( B(Y_2) \), respectively. The \( Wg \)-Drazin inverse of \( A \) is given by

\[
A^{d,W} = (W_1A_1W_1)^{-1} \oplus 0
\]

with \((W_1A_1W_1)^{-1} \in B(X_1, Y_1)\) and \(0 \in B(X_2, Y_2)\).

Recall that if \( A^D \) and \( B^D \) exist, it is possible that \((A+B)^D\) does not exist. Moreover, if \((A+B)^D\) exists, then we do not always know how to calculate \((A+B)^D\) in terms of \( A, B, A^D, B^D \). In this paper we investigate some
special cases of this phenomenon. In [5] Hartwig, Wang and Wei obtained a formula for the Drazin inverse of a sum of two matrices, when one of the products of these matrices vanishes. Djordjević and Wei generalized their results to bounded linear operators on Banach spaces [8]. In [1], Castro Gonzalez extended these additive Drazin inverse results to complex matrices using weaker conditions. Finally, Castro-Gonzalez and Koliha extended the results for the generalized Drazin inverse of Banach algebra elements [2]. In this paper we extend previous results to linear bounded operators on Banach spaces, and give a formula for computing the $Wg$-Drazin inverse of a sum of two operators.

We state one lemma concerning $g$–Drazin inverse of a partitioned matrix that will be needed later (see Djordjević and Wei [8]).

**Lemma 1.2** If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ are $g$–Drazin invertible, $C \in \mathcal{B}(Y,X)$ and $D \in \mathcal{B}(X,Y)$, then

$$
M = \begin{bmatrix}
A & C \\
0 & B
\end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix}
A & 0 \\
D & B
\end{bmatrix}
$$

are also $g$–Drazin invertible and

$$
M^d = \begin{bmatrix}
A^d & S \\
0 & B^d
\end{bmatrix}, \quad N^d = \begin{bmatrix}
A^d & 0 \\
R & B^d
\end{bmatrix},
$$

where

$$
S = (A^d)^2 \sum_{n=0}^{\infty} (A^d)^n CB^n (I - BB^d) + (I - AA^d) \sum_{n=0}^{\infty} A^n C (B^d)^n (B^d)^2 - A^d CB^d
$$

and

$$
R = (B^d)^2 \sum_{n=0}^{\infty} (B^d)^n DA^n (I - AA^d) + (I - BB^d) \sum_{n=0}^{\infty} B^n D (A^d)^n (A^d)^2 - B^d DA^d.
$$

We also need the following important results from [8].
Lemma 1.3 If \( P, Q \in \mathcal{B}(X) \) are quasinilpotent and \( PQ = 0 \) or \( PQ =QP \), then \( P+Q \) is also quasinilpotent. Hence, \((P+Q)^d = 0\).

Lemma 1.4 If \( P \in \mathcal{B}(X) \) is \( g\)-Drazin invertible, \( Q \in \mathcal{B}(X) \) is quasinilpotent and \( PQ = 0 \), then \( P+Q \) is \( g\)-Drazin invertible and

\[
(P + Q)^d = \sum_{i=0}^{\infty} Q^i(P^d)^{i+1}.
\]

We also state the following useful result.

Lemma 1.5 Let \( A \) be a complex Banach algebra with the unit \( 1 \), and let \( p \) be an idempotent of \( A \). If \( x \in pA_p \), then \( \sigma_{pA_p}(x) = \sigma_A(x) \), where \( \sigma_A(x) \) denotes the spectrum of \( x \) in the algebra \( A \), and \( \sigma_{pA_p}(x) \) denotes the spectrum of \( x \) in the algebra \( pA_p \).

2 Wg–Drazin inverse of a sum of two operators

First we state one particular case of our main result.

Theorem 2.1 Let \( W \in \mathcal{B}(Y, X) \), and let \( B \in \mathcal{B}(X, Y) \) be \( Wg\)-Drazin invertible and \( N \in \mathcal{B}(X, Y) \) such that \( WN \in \mathcal{B}(X) \) is quasinilpotent. If \( NWB^d,W = 0 \) and \((I - WBWB^d,W)WNW = 0\), then

\[
\begin{align*}
(1) \quad (WN + WB)^d &= (WB)^d + ((WB)^d)^2 \left( \sum_{i=0}^{\infty} ((WB)^d)^i WNS(i) \right), \\
(2) \quad S(i) &= (I - WBWB^d,W)(WN + WB)^i \\
&= (I - WBWB^d,W) \left( \sum_{j=0}^{i} (WB)^{i-j}(WN)^j \right).
\end{align*}
\]

Moreover, for all \( i \geq l \geq 1 \), we have

\[
S(i) = (WB)^{i-l+1}S(l - 1) = S(l - 1)(WN)^{i-l+1}.
\]

Proof. Since \( B \) is \( Wg\)-Drazin invertible, by Lemma 1.1, we conclude that \( B \) and \( W \) have the matrix forms

\[
B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},
\]


where $B_1, W_1$ are invertible, and $W_2 B_2$ is quasinilpotent. From $NW B^{d,W} = 0$ it follows that $N$ has the matrix form

$$N = \begin{bmatrix} 0 & N_1 \\ 0 & N_2 \end{bmatrix}.$$  

Since $WN = \begin{bmatrix} 0 & W_1 N_1 \\ 0 & W_2 N_2 \end{bmatrix}$ is quasinilpotent, from Lemma 1.5 we conclude that $W_2 N_2$ is quasinilpotent. From $(I - WB B^{d,W}) WN = 0$ it follows that $W_2 N_2 W_2 B_2 = 0$. Thus, for any $i \geq 0$,

$$(W_2 N_2 + W_2 B_2)^i = \sum_{j=0}^{i} (W_2 B_2)^{i-j} (W_2 N_2)^j = \sum_{j=0}^{i} (W_2 B_2)^{j} (W_2 N_2)^{i-j}.$$  

From Lemma 1.4, we see that $W_2 N_2 + W_2 B_2$ is quasinilpotent. Now, from Lemma 1.2, we get

$$(WN + WB)^d = \left( \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} 0 & N_1 \\ 0 & N_2 \end{bmatrix} + \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right)^d$$

$$= \begin{bmatrix} W_1 B_1 & W_1 N_1 \\ 0 & W_2 N_2 + W_2 B_2 \end{bmatrix}^d = \begin{bmatrix} (W_1 B_1)^{-1} X \\ 0 & 0 \end{bmatrix}$$

where

$$X = (W_1 B_1)^{-2} \sum_{i=0}^{\infty} (W_1 B_1)^{-i} W_1 N_1 (W_2 N_2 + W_2 B_2)^i$$

$$= (W_1 B_1)^{-2} \sum_{i=0}^{\infty} (W_1 B_1)^{-i} W_1 N_1 \left( \sum_{j=0}^{i} (W_2 B_2)^{i-j} (W_2 N_2)^j \right).$$

Write $S(i) = (I - WB B^{d,W}) \left( \sum_{j=0}^{i} (WB)^{i-j} (WN)^j \right)$, for all $i \geq 0$. Now, for all $i \geq 1$, we have

$$S(i) = \begin{bmatrix} 0 & 0 \\ 0 & (W_2 B_2)^i \end{bmatrix} + \sum_{j=1}^{i} \begin{bmatrix} 0 & 0 \\ 0 & (W_2 B_2)^{i-j} \end{bmatrix} \begin{bmatrix} 0 & W_1 N_1 (W_2 N_2)^{j-1} \\ 0 & (W_2 N_2)^j \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \sum_{j=0}^{i} (W_2 B_2)^{i-j} (W_2 N_2)^j \end{bmatrix}.$$
Hence,

\[(WB)^d + (WB)^d)^2 \left( \sum_{i=0}^{\infty} ((WB)^d)^i WNS(i) \right) =
\]
\[= \left[ (W_1 B_1)^{-1} \sum_{i=0}^{\infty} (W_1 B_1)^{-(i+2)} W_1 N_1 \left( \sum_{j=0}^{i} (W_2 B_2)^{i-j} (W_2 N_2)^j \right) \right]
\]
\[= \left[ (W_1 B_1)^{-1} X \right] = (WN + WB)^d.
\]

The second statement of the theorem are easily verified. □

As corollaries we obtain the following results.

**Corollary 2.1** Let \(B, N \in \mathcal{B}(X, Y)\) satisfy conditions of Theorem 2.1. Then we have

\[(WN + WB)^d(WN + WB) = (WB)^d WB + \left( \sum_{i=0}^{\infty} ((WB)^d)^i WNS(i) \right), \]

where \(S(i)\) is defined in (2).

**Corollary 2.2** Let \(B, N \in \mathcal{B}(X, Y)\) satisfy conditions of Theorem 2.1.

(i) If \((WN)^2 = 0\), then

\[(WN + WB)^d = (WB)^d + ((WB)^d)^2 \left( \sum_{i=0}^{\infty} ((WB)^d)^i WNS(WB)^i \right) + ((WB)^d)^3 \left( \sum_{i=1}^{\infty} ((WB)^d)^i WNS(WB)^i \right) WNS(WB).
\]

(ii) If \(WNWR = 0\), for all \(R \in \mathcal{B}(X, Y)\), then

\[(WN + WB)^d WR = (WB)^d WR + ((WB)^d)^2 \left( \sum_{i=1}^{\infty} ((WB)^d)^i WNS(WB)^i \right) WR.
\]

(iii) If \((WB)^2 = WB\), then

\[(WN + WB)^d = (I - WN)^{-1} WB.
\]
Proof. Each of these cases follows directly from Theorem 2.1 and the following simplification.

Write \( S(i) = (I - WBWB^{d,W}) \left( \sum_{j=0}^{i} (WB)^{i-j}(WN)^j \right) \), for all \( i \geq 0 \).

(i) Since \( (WN)^2 = 0 \), \( WNS(i) = WN(WB)^i + WN(WB)^{i-1}WN \) for all \( i \geq 1 \).

(ii) Since \( WNWR = 0 \), \( WNS(i)WR = WN(WB)^iWR \).

(iii) Since \( (WB)^2 = WB \), \( (WB)^d = WB \) and then the hypothesis \( NWB^{d,W} = 0 \) implies \( NWB = N(WB)^d = NWB^{d,W} = 0 \). Then from Lemma 1.4 it follows

\[
(WN + WB)^d = \sum_{i=0}^{\infty} (WN)^i ((WB)^d)^{i+1}
= \sum_{i=0}^{\infty} (WN)^i (WB)^{i+1}
= \left( \sum_{i=0}^{\infty} (WN)^i \right) WB
= (I - WN)^{-1}WB.
\]

\[\square\]

Now, we state and prove the main result.

**Theorem 2.2** Let \( W \in \mathcal{B}(Y, X) \), and let \( A, B \in \mathcal{B}(X, Y) \) be Wg-Drazin invertible. If \( A^{d,W}WB = 0 \), \( AWB^{d,W} = 0 \) and \( (I - WBWB^{d,W})WAWB(I - WAWA^{d,W}) = 0 \), then \( A + B \) is Wg-Drazin invertible and

\[
(A + B)^{d,W} = (A + B) \left[ (WB)^d \left( I + \sum_{i=0}^{\infty} \left( (WB)^d \right)^{i+1} WAZ(i) \right) (I - WAWA^{d,W}) \right]^2
+ (A + B)(I - WBWB^{d,W}) \left( I + \sum_{i=0}^{\infty} Z(i)WB \left( (WA)^d \right)^{i+1} \right) (WA)^2
- (A + B) \left( (WB)^d \right)^2 \sum_{i=0}^{\infty} \left( (WB)^d \right)^i WAZ(i)WB \left( (WA)^d \right)^2
\]

[7]
\[-(A + B)(WB)^d \left( \sum_{i=0}^{\infty} WAZ(i)WB \left( (WA)^d \right)^i \right) \left( (WA)^d \right)^3 \]
\[-(A + B) \left( (WB)^d \right)^2 \times \]
\[\times \left( \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left( (WB)^d \right)^i WAZ(i + k + 1)WB \left( (WA)^d \right)^k \right) \left( (WA)^d \right)^3 \]
\[-(A + B) \times \]
\[\times \left[ (WB)^d \left( I + \sum_{i=0}^{\infty} \left( (WB)^d \right)^{i+1} WAZ(i) \right) (I - WAWA^{d,W}) \right]^2 \]
\[\times WB(WA)^d, \]
(3)

where

(4) \(Z(i) = (I - WBWB^{d,W}) \left( \sum_{j=0}^{i} (WB)^{i-j} (WA)^j \right) (I - WAWA^{d,W}).\)

Moreover, for all \(i \geq l \geq 1\), we have
\[Z(i) = (WB)^{i-l+1}Z(l-1) = Z(l-1)(WA)^{i-l+1}.\]

**Proof.** Since \(A\) is \(W\)-\(g\)-Drazin invertible, by Lemma 1.1, we conclude that \(A\) and \(W\) have the matrix forms
\[A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \]
where \(A_1, W_1\) are invertible and \(W_2A_2\) is quasinilpotent. From \(A^{d,W}WB = 0\) it follows that \(B\) can be written as
\[B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \end{bmatrix}. \]

We use Lemma 1.2 to compute \((WB)^d\) which in turn equals \(WB^{d,W}\). From the assumptions \(AWB^{d,W} = 0\) and \((I - WBWB^{d,W})WAWB(I - WAWA^{d,W}) = 0\), we get that \(A_2W_2B_2^{d,W_2} = 0\) and \((I - W_2B_2W_2B_2^{d,W_2})W_2A_2W_2 = 0\). We see that the conditions of Theorem 2.1 are satisfied with: \(B_2, W_2, A_2\), respectively, instead of \(B, W, N\).
From Lemma 1.2 we have that
\[(A + B)^{d,W} = (A + B)((W(A + B))^d)^2 = (A + B)((WA + WB)^d)^2 \]
\[= (A + B) \left( \begin{bmatrix} W_1 A_1 & 0 \\ W_2 B_1 & W_2 A_2 + W_2 B_2 \end{bmatrix} \right)^d \]
\[= (A + B) \left[ \begin{bmatrix} (W_1 A_1)^{-1} & 0 \\ X & (W_2 A_2 + W_2 B_2)^d \end{bmatrix} \right]^2 \]
\[= \begin{bmatrix} A_1 & 0 \\ B_1 & A_2 + B_2 \end{bmatrix} \times \]
\[\times \begin{bmatrix} (W_1 A_1)^{-2} & 0 \\ X(W_1 A_1)^{-1} + (W_2 A_2 + W_2 B_2)^d X & ((W_2 A_2 + W_2 B_2)^d)^2 \end{bmatrix} = \begin{bmatrix} A_1(W_1 A_1)^{-2} & 0 \\ X' & (A_2 + B_2)((W_2 A_2 + W_2 B_2)^d)^2 \end{bmatrix}, \]
where
\[X = (I - (W_2 A_2 + W_2 B_2)(W_2 A_2 + W_2 B_2)^d) \times \]
\[\times \left( \sum_{i=0}^{\infty} (W_2 A_2 + W_2 B_2)^i W_2 B_1(W_1 A_1)^{-1} \right)(W_1 A_1)^{-2} \]
\[- (W_2 A_2 + W_2 B_2)^d W_2 B_1(W_1 A_1)^{-1} \]
and
\[X' = B_1(W_1 A_1)^{-2} + (A_2 + B_2)[X(W_1 A_1)^{-1} + (W_2 A_2 + W_2 B_2)^d X]. \]

Using Theorem 2.1 we get
\[(W_2 A_2 + W_2 B_2)^d = (W_2 B_2)^d + ((W_2 B_2)^d)^2 \left( \sum_{i=0}^{\infty} ((W_2 B_2)^d)^i W_2 A_2 S(i) \right), \]
where \(S(i) = (I - W_2 B_2 W_2 B_2^{d,W_2}) \left( \sum_{j=0}^{i} (W_2 B_2)^j (W_2 A_2)^{i-j} \right) \) for all \(i \geq 0. \)

Now, we have
\[I - (W_2 A_2 + W_2 B_2)(W_2 A_2 + W_2 B_2)^d \]
\[= I - W_2 B_2(W_2 B_2)^d - (W_2 B_2)^d \left( \sum_{i=0}^{\infty} ((W_2 B_2)^d)^i W_2 A_2 S(i) \right). \]
Since
\[(W_2 A_2 + W_2 B_2)^d X = - \left( (W_2 A_2 + W_2 B_2)^d \right)^2 W_2 B_1 (W_1 A_1)^{-1},\]
we get
\[
X' = B_1 (W_1 A_1)^{-2} + (A_2 + B_2) \left[ \left( I - W_2 B_2 (W_2 B_2)^d \right) \right.
\]
\[
- (W_2 B_2)^d \sum_{i=0}^{\infty} \left( (W_2 B_2)^d \right)^i W_2 A_2 S(i) \times
\]
\[
\times \left( \sum_{i=0}^{\infty} (W_2 A_2 + W_2 B_2)^i W_2 B_1 (W_1 A_1)^{-i} \right) (W_1 A_1)^{-3}
\]
\[
- (W_2 A_2 + W_2 B_2)^d W_2 B_1 (W_1 A_1)^{-2}
\]
\[
- \left( (W_2 A_2 + W_2 B_2)^d \right)^2 W_2 B_1 (W_1 A_1)^{-1} \right]
\]
\[
= B_1 \left( (W_1 A_1)^{-1} \right)^2 + X_1 + X_2 + X_3 + X_4,
\]
where \( X_1, X_2, X_3 \) and \( X_4 \) are the following terms:
\[
X_1 = (A_2 + B_2) (I - W_2 B_2 (W_2 B_2)^d) \times
\]
\[
\times \left( \sum_{i=0}^{\infty} (W_2 A_2 + W_2 B_2)^i W_2 B_1 (W_1 A_1)^{-i} \right) (W_1 A_1)^{-3}
\]
\[
= (A_2 + B_2) (I - W_2 B_2 (W_2 B_2)^d) \times
\]
\[
\times \left( \sum_{i=0}^{\infty} S(i) W_2 B_1 (W_1 A_1)^{-i} \right) (W_1 A_1)^{-3}
\]
and the last equality follows by using (2) in Theorem 2.1. Moreover,
\[
X_2 = -(A_2 + B_2) (W_2 B_2)^d \left( \sum_{i=0}^{\infty} \left( (W_2 B_2)^d \right)^i W_2 A_2 S(i) \right) \times
\]
\[
\times \left( \sum_{i=0}^{\infty} (W_2 A_2 + W_2 B_2)^i W_2 B_1 (W_1 A_1)^{-i} \right) (W_1 A_1)^{-3}
\]
\[
= -(A_2 + B_2) (W_2 B_2)^d \left( \sum_{k=0}^{\infty} W_2 A_2 S(k) W_2 B_1 (W_1 A_1)^{-(k+3)} \right)
\]
\[-(A_2 + B_2)(W_2B_2)^d \times \left( \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} ((W_2B_2)^d)^{i+1} W_2A_2S(i + k + 1)W_2B_1(W_1A_1)^{-(k+3)} \right) \]

and the last equality follows by using (2) to obtain that $S(i)(W_2A_2 + W_2B_2)^k = (I - W_2B_2W_2^d(W_2A_2 + W_2B_2)^{i+k} = S(i + k)$ and after we change $i$ by $i - 1$ in the last sum. Also

\[
X_3 = -(A_2 + B_2)(W_2A_2 + W_2B_2)^dW_2B_1(W_1A_1)^{-2} = -(A_2 + B_2)(W_2B_2)^dW_2B_1(W_1A_1)^{-2} - (A_2 + B_2) \left( (W_2B_2)^d \right)^2 \times \\
\times \left( \sum_{i=0}^{\infty} ((W_2B_2)^d)^i W_2A_2S(i)W_2B_1 \right)(W_1A_1)^{-2}.
\]

Finally,

\[
X_4 = -(A_2 + B_2) \left( (W_2A_2 + W_2B_2)^d \right)^2 W_2B_1(W_1A_1)^{-1}.
\]

Write $Z(i) = (I - BWB^dW) \left( \sum_{j=0}^{i} (WB)^{i-j}(WA)^j \right)(I - WAWA^dW)$. By direct computations, for all $i \geq 1$ we have,

\[
Z(i) = \left[ \begin{array}{c}
I \\
-(W_2B_2)^dW_2B_1 & I - W_2B_2(W_2B_2)^d
\end{array} \right] \times \\
\times \left\{ \sum_{j=0}^{i-1} \left[ \begin{array}{c}
(W_2B_2)^{i-j-1}W_2B_1 & 0 \\
0 & (W_2B_2)^{i-j}
\end{array} \right] \left[ \begin{array}{c}
0 \\
0
\end{array} \right] \left[ \begin{array}{c}
0 \\
(W_2A_2)^j
\end{array} \right] \right\} \\
+ \left[ \begin{array}{c}
0 \\
0
\end{array} \right] \\
+ \left[ \begin{array}{c}
0 \\
0
\end{array} \right] \\
= \left[ \begin{array}{c}
0 \\
0
\end{array} \right]
\]

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\[ WAZ(i)WB \left( (WA)^d \right)^q = \begin{bmatrix} 0 & 0 \\ W_2 A_2 S(i) W_2 B_1 (W_1 A_1)^{-q} & 0 \end{bmatrix}, \quad \text{for all } q \geq 1. \]

Now, we compute the terms of the expressions (3) for \((A + B)^d W\) using the block decomposition:

\[ \Sigma_1 = (A + B) \left[ (WB)^d \left( I + \sum_{i=0}^{\infty} \left( (WB)^d \right)^{i+1} WAZ(i) \right) (I - WAWA^{d,W}) \right]^2 \]

\[ = (A + B) \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}^2 \]

\[ + \sum_{i=0}^{\infty} \left[ \left( (WB)^2 \right)^{i+3} W_2 B_1 \left( (WB)^2 \right)^{i+2} \right] \left[ \begin{bmatrix} 0 & 0 \\ 0 & W_2 A_2 S(i) \end{bmatrix} \right]^2 \]

\[ = (A + B) \left[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right] \left[ \begin{bmatrix} 0 & 0 \\ 0 & W_2 A_2 S(i) \end{bmatrix} \right]^2 \]

\[ = \left[ \begin{bmatrix} A_1 & 0 \\ B_1 & A_2 + B_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (W_2 A_2 + W_2 B_2)^2 \end{bmatrix} \right] = \left[ \begin{bmatrix} 0 & 0 \\ 0 & (A_2 + B_2) (W_2 A_2 + W_2 B_2)^2 \end{bmatrix} \right], \]

\[ \Sigma_2 = (A + B)(I - WBWB^{d,W}) \left( I + \sum_{k=0}^{\infty} Z(k)WB \left( (WA)^d \right)^{k+1} \right) \left( (WA)^d \right)^2 \]

\[ = \left[ \begin{bmatrix} A_1 & 0 \\ B_1 & A_2 + B_2 \end{bmatrix} \begin{bmatrix} I \\ -(W_2 B_2)^d W_2 B_1 \end{bmatrix} \right] \times \left[ \begin{bmatrix} 0 & 0 \\ W_2 B_2 (W_2 B_2)^d \end{bmatrix} \right] \times \]

\[ \sum_{k=0}^{\infty} \left[ \begin{bmatrix} S(k) W_2 B_2 (W_1 A_1)^{-k+3} \\ 0 \end{bmatrix} \right] \times \]

\[ = \left[ \begin{bmatrix} A_1 (W_1 A_1)^{-2} & 0 \\ X'' & 0 \end{bmatrix} \right], \]
\[ X'' = B_1(W_1A_1)^{-2} \]
\[ \quad - (A_2 + B_2) \left[ (W_2B_2)^d W_2B_1(W_1A_1)^{-2} \right. \]
\[ \quad + \left. (I - W_2B_2(W_2B_2)^d) \left( \sum_{k=0}^{\infty} S(k)W_2B_1(W_1A_1)^{-(k+3)} \right) \right], \]

\[ \Sigma_3 = - (A + B) \left[ (WB)^d \right]^2 \left( \sum_{i=0}^{\infty} (WB)^d \right)^i WAZ(i)WB \left( (WA)^d \right)^3 \]
\[ = - (A + B) \left[ \sum_{i=0}^{\infty} (WB)^d \right]^{i+2} \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \]
\[ = - \left[ (A_2 + B_2) \sum_{i=0}^{\infty} (WB)^d \right]^{i+2} \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \]

\[ \Sigma_4 = - (A + B)(WB)^d \left( \sum_{i=0}^{\infty} WAZ(i)WB \left( (WA)^d \right)^i \right) \left( (WA)^d \right)^3 \]
\[ = - (A + B) \left[ \sum_{i=0}^{\infty} W_2A_2S(i)W_2B_1(W_1A_1)^{-(i+3)} \right] \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \]
\[ = - \left[ (A_2 + B_2)(WB)^d \sum_{i=0}^{\infty} W_2A_2S(i)W_2B_1(W_1A_1)^{-(i+3)} \right] \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \]

\[ \Sigma_5 = - (A + B) \left( (WB)^d \right)^2 \times \]
\[ \times \left( \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (WB)^d \right)^i WAZ(i + k + 1)WB \left( (WA)^d \right)^k \left( (WA)^d \right)^3 \]
\[ = - \left[ \begin{array}{ccc} A_1 & 0 & 0 \\ B_1 & A_2 + B_2 & X'' \\ 0 & X'' & 0 \end{array} \right], \]
\[ = - \left[ \begin{array}{ccc} (A_2 + B_2)X'' & 0 & 0 \\ 0 & X'' & 0 \end{array} \right], \]
where
\[ X''' = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left( (W_2 B_2)^d \right)^{i+2} W_2 A_2 S(i + k + 1) W_2 B_1 (W_1 A_1)^{-(k+3)}, \]

\[
\begin{align*}
\Sigma_6 &= -(A + B) \times \\
&\times \left[ (WB)^d \left( I + \sum_{i=0}^{\infty} \left( (WB)^d \right)^{i+1} WAZ(i) \right) (I - W A W A^{d,W}) \right]^2 \times \\
&\times WB(WA)^d \\
&= -(A + B) \left[ \begin{array}{cc} 0 & 0 \\
0 & \left( (W_2 A_2 + W_2 B_2)^d \right)^2 \end{array} \right] \left[ \begin{array}{cc} 0 & 0 \\
W_2 B_1 (W_1 A_1)^{-1} & 0 \end{array} \right] \\
&= - \left[ \begin{array}{cc} 0 & 0 \\
(A_2 + B_2) \left( (W_2 A_2 + W_2 B_2)^d \right)^2 W_2 B_1 (W_1 A_1)^{-1} & 0 \end{array} \right].
\end{align*}
\]

Thus,
\[
\begin{align*}
\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 &= \left[ \begin{array}{cc} A_1 (W_1 A_1)^{-2} & 0 \\
X' & (A_2 + B_2) \left( (W_2 A_2 + W_2 B_2)^d \right)^2 \end{array} \right],
\end{align*}
\]

completing the proof of (3). The second statement of the theorem can easily be verified. \(\square\)

We obtain some corollaries as follows.

**Corollary 2.3** Let \( W \in \mathcal{B}(Y, X) \), and let \( A, B \in \mathcal{B}(X, Y) \) be \( W \)-\( W \)-Drazin invertible. If \( A^{d,W} WB = 0 \) and \( AWB(I - W A W A^{d,W}) = 0 \), then

\[
\begin{align*}
(A + B)^{d,W} &= (A + B) \left[ \left( \sum_{i=0}^{\infty} \left( (WB)^d \right)^{i+1} (WA)^i \right) (I - W A W A^{d,W}) \right]^2 \\
&+ (A + B)(I - WBW B^{d,W}) \left( \sum_{i=0}^{\infty} (WB)^i \left( (WA)^d \right)^{i+2} \right) \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{i} (WB)^{i-j} (WA)^j WB \left( (WA)^d \right)^{i+3}
\end{align*}
\]
\[-(A + B) \left( (WB)^d \right)^2 \left( \sum_{i=0}^{\infty} (WB)^i (WA)^{i+1} WB \right) \left( (WA)^d \right)^2 \]
\[-(A + B)(WB)^d \left( \sum_{i=0}^{\infty} (WA)^{i+1} WB \left( (WA)^d \right)^i \right) \left( (WA)^d \right)^3 \]
\[-(A + B) \left( (WB)^d \right)^2 \times \left( \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (WB)^d \right)^i (WA)^{i+k+2} WB \left( (WA)^d \right)^k \left( (WA)^d \right)^3 \]
\[-(A + B) \left[ \left( \sum_{i=0}^{\infty} (WB)^d \right)^{i+1} (WA)^i \right] (I - WA WA^{d,W}) \right]^2 WB(WA)^d.\]

**Proof.** From \( A^{d,W}WB = 0 \) and \( AWB(I - WA^{d,W}) = 0 \) it follows that
\[
A(WB)^2 = AWB(I - WA^{d,W})WB + AWBWAW^{d,W} WB
\]
\[
= AWBWAW^{d,W} WB
\]
\[
= 0
\]
and thus
\[
AWB^{d,W} = A(WB)^d = AWB \left( (WB)^d \right)^2 = A(WB)^2 \left( (WB)^d \right)^3 = 0.
\]

Then we apply Theorem 2.2, together with the simplification \( WAZ(i) = (WA)^{i+1}(I - WA^{d,W}) \) for all \( i \geq 0 \), to get the statement of this corollary. \( \square \)

**Corollary 2.4** Let \( W \in \mathcal{B}(Y, X) \), and let \( A, B \in \mathcal{B}(X, Y) \) be Wg–Drazin invertible. Suppose that \( A^{d,W}WB = 0 \) and \( AWB(I - WA^{d,W}) = 0 \).

(i) If \( (WB)^2 = WB \), then
\[
(A + B)^{d,W}
\]
\[
= (A + B) \left[ \left( WB \sum_{i=0}^{\infty} (WA)^i \right) (I - WA^{d,W}) \right]^2
\]
\[
+ (A + B)(I - WB) \left( (WA)^d \right)^2 + \sum_{i=1}^{\infty} (WA)^i WB \left( (WA)^d \right)^{i+3}
\]

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\[ -(A + B)WB \left( \sum_{i=0}^{\infty} (WA)^{i+1}WB \right) \left( (WA)^d \right)^2 \]
\[ -(A + B)WB \left( \sum_{i=0}^{\infty} (WA)^{i+1}WB \left( \left( (WA)^d \right)^i \right) \left( (WA)^d \right)^3 \right) \]
\[ -(A + B)WB \left( \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (WA)^{i+k+2}WB \left( \left( (WA)^d \right)^k \right) \left( (WA)^d \right)^3 \right) \]
\[ -(A + B) \left[ \left( WB \sum_{i=0}^{\infty} (WA)^i \right) (I - WAWA^{d,W}) \right]^2 WB(WA)^d. \]

(ii) If \( WB \) is quasinilpotent, then
\[
(A + B)^{d,W} = (A + B) \left[ \left( (WA)^d \right)^2 \right. \\
+ \left. \left( \sum_{i=0}^{\infty} (WB)^{i-j} (WA)^jWB \left( (WA)^d \right)^i \right) \left( (WA)^d \right)^3 \right].
\]

(iii) If \( (WB)^2 = 0 \), then
\[
(A + B)^{d,W} = (A + B) \left[ \left( (WA)^d \right)^2 \right. \\
+ \left. WB \left( \sum_{i=0}^{\infty} (WA)^iWB \left( (WA)^d \right)^i \right) \left( (WA)^d \right)^4 \right. \\
+ \left. \left( \sum_{i=0}^{\infty} (WA)^iWB \left( (WA)^d \right)^i \right) \left( (WA)^d \right)^3 \right].
\]

Proof. Each of these cases follows directly from Corollary 2.3 and the following simplifications:

(i) Since \( (WB)^2 = WB \), we have \( WB^{d,W} = (WB)^d = WB \) and \( (I - WBWB^{d,W})WB = 0 \).

(ii) Since \( WB \) is quasinilpotent, we get \( (WB)^d = 0 \).

(iii) Since \( (WB)^2 = 0 \), it follows that
\[
(WB)^d = WB \left( (WB)^d \right)^2 = (WB)^2 \left( (WB)^d \right)^3 = 0. \]
Corollary 2.5 Let $W \in \mathcal{B}(Y, X)$, and let $A, B \in \mathcal{B}(X, Y)$ be $Wg$–Drazin invertible. If $AWB^{d,W} = 0$ and $(I - WBW B^{d,W})WAWB = 0$, then

$$(A + B)^{d,W}$$

$$= (A + B) \left[ \left( \sum_{i=0}^{\infty} (WB)^{d}_{i+1} (WA)^{i} \right) + \sum_{i=1}^{\infty} \sum_{j=1}^{i} (WB)^{d}_{i+j} WA(WB)^{j}(WA)^{i-j} \right] (I - WAWA^{d,W})^{2}$$

$$+ (A + B)(I - WBW B^{d,W}) \left( \sum_{i=0}^{\infty} (WB)^{d}_{i} (WA)^{i+2} \right)$$

$$- (A + B) \left( (WB)^{d}_{i} \right) (I - WBW B^{d,W})^{2}$$

$$- (A + B)(WB)^{d} \left( \sum_{i=0}^{\infty} WA(WB)^{i} \right) (WA)^{i+3}$$

$$- (A + B) \left( (WB)^{d}_{i} \right) \left( \sum_{i=0}^{\infty} \sum_{k=0}^{i} WA(WB)^{i+k} (WA)^{k+3} \right)$$

$$- (A + B) \left[ \left( \sum_{i=0}^{\infty} (WB)^{d}_{i+1} (WA)^{i} \right) + \sum_{i=1}^{\infty} \sum_{j=1}^{i} (WB)^{d}_{i+j} WA(WB)^{j}(WA)^{i-j} \right] (I - WAWA^{d,W})^{2}$$

$$\times WB(WA)^{d}. $$

Proof. From $AWB^{d,W} = 0$ and $(I - WBW B^{d,W})WAWB = 0$ it follows that

$$(AW)^{2}B = A(I - WBW B^{d,W})WAWB + AWBW B^{d,W}WAWB$$

$$= AWB^{d,W}WBWAWB$$

$$= 0$$

and thus

$$A^{d,W}WB = (AW)^{d}B = (AW)^{d}_{i} AWB = (AW)^{d}_{2} (AW)^{2}B = 0.$$
Corollary 2.6 Let $W \in \mathcal{B}(Y, X)$, and let $A, B \in \mathcal{B}(X, Y)$ be Wg–Drazin invertible. Suppose that $AWB_{d,W} = 0$ and $(I - BWB_{d,W})WAWB = 0$.

(i) If $(WA)^2 = WA$, then

$$(A + B)^{d,W} = (A + B) \left[ (WB)^d + \sum_{i=1}^{\infty} \left( (WB)^d \right)^{i+2} W A (WB)^i \right] (I - WA)^2$$

$$+ (A + B)(I - BWB_{d,W}) \left( \sum_{i=0}^{\infty} (WB)^i \right) WA$$

$$- (A + B) \left( (WB)^d \right)^2 \left( \sum_{i=0}^{\infty} \left( (WB)^d \right)^{i} WA (WB)^{i+1} \right) WA$$

$$- (A + B)(WB)^d \left( \sum_{i=0}^{\infty} WA (WB)^{i+1} \right) WA$$

$$- (A + B) \left( (WB)^d \right)^2 \left( \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left( (WB)^d \right)^{i} WA (WB)^{i+k+2} \right) WA$$

$$- (A + B) \left[ \left( (WB)^d + \sum_{i=1}^{\infty} \left( (WB)^d \right)^{i+2} WA (WB)^i \right) (I - WA)^2 \right]$$

$$\times WB(WA)^d.$$ 

(ii) If $WA$ is quasinilpotent, then

$$(A + B)^{d,W} = (A + B) \left[ (WB)^d + \sum_{i=0}^{\infty} \sum_{j=0}^{i} \left( (WB)^d \right)^{i+j} WA (WB)^j (WA)^{i-j} \right]^2.$$ 

Proof. We apply Corollary 2.5 and the following simplifications:

(i) Since $(WA)^2 = WA$, we have $WA_{d,W} = (WA)^d = WA$ and $(WA)^j (I - WAWA_{d,W}) = 0$ for all $j \geq 1$.

(ii) Since $WA$ is quasinilpotent, we get $(WA)^d = 0$. □

Corollary 2.7 Let $A, B \in \mathcal{B}(X, Y)$ be Wg–Drazin invertible. If $AWB = 0$, then

$$(A + B)^{d,W}$$
\[ = (A + B) \left[ (WB)^d \left( \sum_{i=0}^{\infty} (WB)^i (WA)^i \right) (I - WAWA^{d,W}) \right]^2 \]
\[ + (A + B)(I - WBWB^{d,W}) \left( \sum_{i=0}^{\infty} (WB)^i (WA)^i \right) (WA)^2 \]
\[ - (A + B) \left[ (WB)^d \left( \sum_{i=0}^{\infty} (WB)^i (WA)^i \right) (I - WAWA^{d,W}) \right]^2 \]
\[ \times WB(WA)^d. \]

**Proof.** Since \( AWB = 0 \), then it follows that
\[ A^{d,W}WB = (A^{d,W}WA^{d,W}WB)^2 AWB = 0, \]
\( (I - WBWB^{d,W})WAWB = 0, \) \( AWB(I - WAWA^{d,W}) = 0 \) and then \( A^{d,W}WB = 0. \) Thus, we apply Corollary 2.3, or Corollary 2.5, to get the above result. \( \square \)

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