EP elements in rings

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Abstract

In this paper we present a number of new characterizations of EP elements in rings with involution in purely algebraic terms, and considerably simplify proofs of already existing characterizations.

Key words and phrases: EP elements, Moore–Penrose inverse, group inverse, ring with involution.
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1 Introduction

The EP matrices and EP linear operators on Banach or Hilbert spaces have been investigated by many authors (see, for example, [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 15, 17, 18, 21]). In this paper we use the setting of rings with involution to investigate EP elements, give new characterizations, and provide simpler and more transparent proofs to already existing characterizations.

Let $\mathcal{R}$ be an associative ring, and let $a \in \mathcal{R}$. Then a is *group invertible* if there is $a^\# \in \mathcal{R}$ such that

$$aa^\#a = a, \quad a^\#aa^\# = a^\#, \quad aa^\# = a^\#a;$$

$a^\#$ is uniquely determined by these equations. We use $\mathcal{R}^\#$ to denote the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ satisfying $a^* = a$ is called *symmetric*.

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We say that $b = a^\dagger$ is the *Moore–Penrose inverse* (or *MP-inverse*) of a, if the following hold [23]:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [6, 12, 14, 16]). The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

Definition 1.1. An element $a \in \mathcal{R}$ is *-cancellable if

$$(1) \qquad a^*ax = 0 \Rightarrow ax = 0 \quad \text{and} \quad xaa^* = 0 \Rightarrow xa = 0.$$

Applying the involution to (1), we observe that a is *-cancellable if and only if a^* is *-cancellable. In C^*-algebras all elements are *-cancellable.

Theorem 1.1. [19] *Let $a \in \mathcal{R}$. Then $a \in \mathcal{R}^\dagger$ if and only if a is *-cancellable and a^*a is group invertible.*

In this paper we will use the following definition of EP elements [19].

Definition 1.2. An element a of a ring $\mathcal{R}$ with involution is said to be EP if $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and $a^\# = a^\dagger$.

These elements are important since they are characterized by commutativity with their Moore–Penrose inverse. The following result is well known for matrices, Hilbert space operators and elements of C^*-algebras, and it is equally true in rings with involution:

Lemma 1.1. *An element $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\dagger$ and $aa^\dagger = a^\dagger a$.*

We observe that $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ if and only if $a^* \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ (see [19]) and a is EP if and only if a^* is EP. For further comments on the definition of EP elements see the last section of this paper. The following result is proved in [19].

Theorem 1.2. *An element $a \in \mathcal{R}$ is EP if and only if a is group invertible and one of the following equivalent conditions holds:*

(a) *$a^\# a$ is symmetric;*

(b) *$(a^\#)^* = aa^\#(a^\#)^*$;*

(c) *$(a^\#)^* = (a^\#)^* a^\# a$;*

(d) *$a^\#(a^\pi)^* = a^\pi(a^\#)^*$, where a^π is the spectral idempotent of the element a ([19]).*

2 EP elements in rings with involution

In this section EP elements in rings with involution are characterized by conditions involving their group and Moore–Penrose inverse. Some of these results are proved in [5] or in [1] for complex matrices, using mostly the rank of a matrix, or other finite dimensional methods. Moreover, the operator analogues of these results are proved in [8] and [9] for linear operators on Hilbert spaces, using the operator matrices. In this paper we show that neither the rank (in the finite dimensional case) nor the properties of operator matrices (in the infinite dimensional case) are necessary for the characterization of EP elements.

In the following theorem we present 34 necessary and sufficient conditions for an element a of a ring with involution to be EP. These conditions are known in special cases such as matrices and operators on Hilbert spaces ([1], [5], [8] and [9]).

Theorem 2.1. *An element $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$ and one of the following equivalent conditions holds:*

(i) $aa^{\dagger}a^{\#} = a^{\dagger}a^{\#}a$;

(ii) $aa^{\dagger}a^{\#} = a^{\#}aa^{\dagger}$;

(iii) $a^*aa^{\#} = a^*$;

(iv) $aa^{\#}a^* = a^*aa^{\#}$;

(v) $aa^{\#}a^{\dagger} = a^{\dagger}aa^{\#}$;

(vi) $aa^{\#}a^{\dagger} = a^{\#}a^{\dagger}a$;

(vii) $a^{\dagger}aa^{\#} = a^{\#}a^{\dagger}a$;

(viii) $(a^{\dagger})^2a^{\#} = a^{\dagger}a^{\#}a^{\dagger}$;

(ix) $a^{\dagger}a^{\#}a^{\dagger} = a^{\#}(a^{\dagger})^2$;

(x) $a^{\dagger}(a^{\#})^2 = a^{\#}a^{\dagger}a^{\#}$;

(xi) $a^{\dagger}(a^{\#})^2 = (a^{\#})^2a^{\dagger}$;

(xii) $(a^{\#})^2a^{\dagger} = a^{\#}a^{\dagger}a^{\#}$;

(xiii) $a(a^{\dagger})^2 = a^{\#}$;

(xiv) $a^*a^\dagger = a^*a^\#$;

(xv) $a^\dagger a^* = a^\# a^*$;

(xvi) $a^\dagger a^\dagger = a^\# a^\dagger$;

(xvii) $a^\dagger a^\dagger = a^\dagger a^\#$;

(xviii) $(a^\dagger)^2 = (a^\#)^2$;

(xix) $aa^\# a^\dagger = a^\#$;

(xx) $a^\# a^\dagger = (a^\#)^2$;

(xxi) $a^\dagger a^\# = (a^\#)^2$;

(xxii) $a^\dagger aa^\# = a^\dagger$;

(xxiii) $a^\# a^\dagger a = a^\dagger$;

(xxiv) $aa^\dagger a^*a = a^*aaa^\dagger$;

(xxv) $a^\dagger aaa^* = aa^*a^\dagger a$;

(xxvi) $aa^\dagger(aa^* - a^*a) = (aa^* - a^*a)aa^\dagger$;

(xxvii) $a^\dagger a(aa^* - a^*a) = (aa^* - a^*a)a^\dagger a$;

(xxviii) $a^*a^\# a + aa^\# a^* = 2a^*$;

(xxix) $a^\dagger a^\# a + aa^\# a^\dagger = 2a^\dagger$;

(xxx) $aaa^\dagger + a^\dagger aa = 2a$;

(xxxi) $aaa^\dagger + (aaa^\dagger)^* = a + a^*$;

(xxxii) $a^\dagger aa + (a^\dagger aa)^* = a + a^*$;

(xxxiii) $aa^\dagger a^* = a^*aa^\dagger$;

(xxxiv) $a^*a^\dagger a = a^\dagger aa^*$.

Proof. If a is EP, then it commutes with its Moore–Penrose inverse and $a^\# = a^\dagger$. It is not difficult to verify that conditions (i)-(xxxiv) hold.

Conversely, we assume that $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$. We known that $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ if and only if $a^* \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and a is EP if and only if a^* is EP. To conclude that a is EP, we show that one of the conditions of Theorem 1.2 is satisfied, or that the element a or the element a^* is subject to one of the preceding already established conditions of this theorem. If a^* satisfies one of the preceding already established conditions of this theorem, then a^* is EP and so a is EP.

(i) Using the assumption $aa^\dagger a^\# = a^\dagger a^\# a$, we get the equality

$$aa^\# = aa^\dagger aa^\# = (aa^\dagger a^\#)a = (a^\dagger a^\# a)a = a^\dagger aa^\# a = a^\dagger a.$$

Since $a^\dagger a$ is symmetric, we get that $aa^\#$ is also symmetric.

(ii) From $aa^\dagger a^\# = a^\# aa^\dagger$, we obtain

$$aa^\# = aa^\# aa^\# = a^2(a^\#)^2 = aaa^\dagger a(a^\#)^2 = a(aa^\dagger a^\#) = a(a^\# aa^\dagger) = aa^\dagger.$$

Since $aa^\dagger$ is symmetric, $aa^\#$ is also symmetric.

(iii) If $a^*aa^\# = a^*$, then we have

$$(aa^\#)^* = (a^\#)^*a^* = (a^\#)^*(a^*aa^\#) = (aa^\#)^*aa^\#.$$

Since $(aa^\#)^*aa^\#$ is symmetric, so is $aa^\#$.

(iv) Suppose that $aa^\# a^* = a^*aa^\#$. Then we get

$$a^* = (aa^\dagger a)^* = a^\dagger aa^* = a^\dagger a(aa^\# a^*) = (a^\dagger a)^*(a^*aa^\#) = a^*aa^\#.$$

Now the condition (iii) holds, so a is EP.

(v) Applying $aa^\# a^\dagger = a^\dagger aa^\#$, we have

$$a^\# a = a^\# aa^\dagger a = (aa^\# a^\dagger)a = (a^\dagger aa^\#)a = a^\dagger a.$$

Therefore $a^\# a$ is symmetric.

(vi) By the equality $aa^\# a^\dagger = a^\# a^\dagger a$, we get

$$a^\# a = a^\# aa^\dagger a = a(a^\# a^\dagger a) = a(aa^\# a^\dagger) = aa^\# aa^\dagger = aa^\dagger.$$

Then $a^\# a$ is symmetric.

(vii) When we use the equality $a^\dagger aa^\# = a^\# a^\dagger a$, we obtain

$$a^\# a = (a^\#)^2a^2 = (a^\#)^2aa^\dagger aa = (a^\# a^\dagger a)a = (a^\dagger aa^\#)a = a^\dagger a.$$

So $a^\# a$ is symmetric.

(viii) The equality $(a^\dagger)^2a^\# = a^\dagger a^\# a^\dagger$ gives

$$\begin{aligned}(a^\dagger)^2a^\# &= ((a^\dagger)^2a^\#)aa^\# = (a^\dagger a^\# a^\dagger)aa^\# = a^\dagger(a^\#)^2aa^\dagger aa^\# \\ &= a^\dagger(a^\#)^2aa^\# = a^\dagger(a^\#)^2.\end{aligned}$$

Now, from the previous equality and (viii), it follows

$$\begin{aligned} aa^{\#}a^{\dagger} &= aa(a^{\#})^2a^{\dagger} = aaa^{\dagger}a(a^{\#})^2a^{\dagger} = aaa^{\dagger}a^{\#}a^{\dagger} = aa((a^{\dagger})^2a^{\#}) \\ &= aa(a^{\dagger}(a^{\#})^2) = aaa^{\dagger}a(a^{\#})^2a^{\#} = aa(a^{\#})^2a^{\#} \\ &= a(a^{\#})^2 = a^{\#} = (a^{\#})^2a = (a^{\#})^2aa^{\dagger}a = a^{\#}a^{\dagger}a. \end{aligned}$$

The condition (vi) is obtained from (viii).

(ix) Suppose that $a^{\dagger}a^{\#}a^{\dagger} = a^{\#}(a^{\dagger})^2$. Applying the involution to (ix), we get

$$(a^{\dagger})^*(a^{\#})^*(a^{\dagger})^* = (a^{\dagger})^*(a^{\dagger})^*(a^{\#})^*.$$

Since $(a^{\dagger})^* = (a^*)^{\dagger}$ and $(a^{\#})^* = (a^*)^{\#}$ (see [19]), we have

$$(a^*)^{\dagger}(a^*)^{\#}(a^*)^{\dagger} = (a^*)^{\dagger}(a^*)^{\dagger}(a^*)^{\#},$$

Therefore, the condition (viii) holds for a^*.

(x) Using the equality $a^{\dagger}(a^{\#})^2 = a^{\#}a^{\dagger}a^{\#}$, we get

$$\begin{aligned} a^{\#}a &= (a^{\#})^2aa = (a^{\#})^2aa^{\dagger}aa = a^{\#}a^{\dagger}aa = (a^{\#}a^{\dagger}a^{\#})a^2a \\ &= (a^{\dagger}(a^{\#})^2)a^3 = a^{\dagger}a^{\#}a^2 = a^{\dagger}a. \end{aligned}$$

Hence, $a^{\#}a$ is symmetric.

(xi) Using the assumption $a^{\dagger}(a^{\#})^2 = (a^{\#})^2a^{\dagger}$, we have

$$\begin{aligned} a^{\#}a &= (a^{\#})^3a^3 = (a^{\#})^3aa^{\dagger}aa^2 = ((a^{\#})^2a^{\dagger})a^3 = (a^{\dagger}(a^{\#})^2)a^3 \\ &= a^{\dagger}a^{\#}a^2 = a^{\dagger}a. \end{aligned}$$

Then $a^{\#}a$ is symmetric.

(xii) Assume that $(a^{\#})^2a^{\dagger} = a^{\#}a^{\dagger}a^{\#}$. Applying the involution to (xii), we get

$$(a^*)^{\dagger}(a^*)^{\#}(a^*)^{\#} = (a^*)^{\#}(a^*)^{\dagger}(a^*)^{\#}.$$

Hence, the condition (x) holds for a^*.

(xiii) From the condition $a(a^{\dagger})^2 = a^{\#}$ follows

$$aa^{\#} = aa(a^{\dagger})^2 = aa(a^{\dagger})^2aa^{\dagger} = aa^{\#}aa^{\dagger} = aa^{\dagger}.$$

Hence, $aa^{\#}$ is symmetric.

(xiv) By $a^*a^{\dagger} = a^*a^{\#}$, it follows

$$\begin{aligned} a(a^{\dagger})^2 &= (aa^{\dagger})^*a^{\dagger} = (a^{\dagger})^*(a^*a^{\dagger}) = (a^{\dagger})^*(a^*a^{\#}) = (aa^{\dagger})^*a^{\#} = aa^{\dagger}a^{\#} \\ &= aa^{\dagger}a(a^{\#})^2 = a(a^{\#})^2 = a^{\#}, \end{aligned}$$

i.e. the condition (xiii) is satisfied.

(xv) Applying the involution to $a^\dagger a^* = a^\# a^*$, we obtain

$$(a^*)^*(a^*)^\dagger = (a^*)^*(a^*)^\#,$$

i.e. the condition (xvi) holds for a^*.

(xvi) The condition $a^\dagger a^\dagger = a^\# a^\dagger$ implies

$$\begin{aligned} a^\dagger a^* &= a^\dagger(aa^\dagger a)^* = a^\dagger(a^\dagger a)^* a^* = (a^\dagger a^\dagger)aa^* = (a^\# a^\dagger)aa^* \\ &= a^\#(a^\dagger a)^* a^* = a^\#(aa^\dagger a)^* = a^\# a^*, \end{aligned}$$

i.e. the equality (xv) is satisfied.

(xvii) Applying the involution to $a^\dagger a^\dagger = a^\dagger a^\#$, we get

$$(a^*)^\dagger(a^*)^\dagger = (a^*)^\#(a^*)^\dagger.$$

So, the equality (xvi) holds for a^*.

(xviii) Suppose that $(a^\dagger)^2 = (a^\#)^2$, then we have

$$a(a^\dagger)^2 = a(a^\#)^2 = a^\#.$$

Therefore, the condition (xiii) is satisfied.

(xix) Using $aa^\# a^\dagger = a^\#$, we get the equality

$$aa^\# = a(aa^\# a^\dagger) = aa^\dagger.$$

Hence, $aa^\#$ is symmetric.

(xx) Assume that $a^\# a^\dagger = (a^\#)^2$. Then it follows

$$aa^\# a^\dagger = a(a^\#)^2 = a^\#.$$

Hence, the condition (xix) holds.

(xxi) Applying the involution to $a^\dagger a^\# = (a^\#)^2$, we obtain

$$(a^*)^\#(a^*)^\dagger = (a^*)^\#(a^*)^\#.$$

Now, a^* satisfies the condition (xx).

(xxii) Applying $a^\dagger aa^\# = a^\dagger$, we have

$$aa^\# = a(a^\dagger aa^\#) = aa^\dagger.$$

So, $aa^\#$ is symmetric.

(xxiii) The equality $a^{\#}a^{\dagger}a = a^{\dagger}$ gives

$$a^{\#}a = a^{\#}aa^{\dagger}a = a(a^{\#}a^{\dagger}a) = aa^{\dagger}.$$

Therefore $a^{\#}a$ is symmetric.

(xxiv) From $aa^{\dagger}a^*a = a^*aaa^{\dagger}$, we have

$$a^*aaa^{\#} = (a^*aaa^{\dagger})aa^{\#} = (aa^{\dagger}a^*a)aa^{\#} = aa^{\dagger}a^*a = a^*aaa^{\dagger},$$

i.e.

$$a^*a(aa^{\#} - aa^{\dagger}) = 0. \tag{2}$$

Using the assumption $a \in \mathcal{R}^{\dagger}$, by Theorem 1.1, we know that a is *-cancellable. Then, by (2) and *-cancellation, we get

$$a(aa^{\#} - aa^{\dagger}) = 0,$$

i.e.

$$a = aaa^{\dagger}.$$

Now

$$a^{\#}a = a^{\#}(aaa^{\dagger}) = aa^{\dagger}.$$

So $a^{\#}a$ is symmetric.

(xxv) The equality $a^{\dagger}aaa^* = aa^*a^{\dagger}a$ gives

$$a^{\#}aaa^* = a^{\#}a(a^{\dagger}aaa^*) = a^{\#}a(aa^*a^{\dagger}a) = aa^*a^{\dagger}a = a^{\dagger}aaa^*,$$

i.e.

$$(a^{\#}a - a^{\dagger}a)aa^* = 0. \tag{3}$$

Since $a \in \mathcal{R}^{\dagger}$, a is *-cancellable by Theorem 1.1. From (3) and *-cancellation, we obtain

$$(a^{\#}a - a^{\dagger}a)a = 0,$$

i.e.

$$a = a^{\dagger}aa.$$

Using the previous equality, we get

$$aa^{\#} = (a^{\dagger}aa)a^{\#} = a^{\dagger}a.$$

Hence $a^{\#}a$ is symmetric.

(xxvi) The assumption

$$aa^\dagger(aa^* - a^*a) = (aa^* - a^*a)aa^\dagger$$

is equivalent to

$$aa^* - aa^\dagger a^*a = aa^* - a^*aaa^\dagger.$$

Then, we deduce that $aa^\dagger a^*a = a^*aaa^\dagger$, i.e. the condition (xxiv) is satisfied.

(xxvii) The equality

$$a^\dagger a(aa^* - a^*a) = (aa^* - a^*a)a^\dagger a$$

is equivalent to

$$a^\dagger aaa^* - a^*a = aa^*a^\dagger a - a^*a$$

which implies

$$a^\dagger aaa^* = aa^*a^\dagger a.$$

Thus, the equality (xxv) holds.

(xxviii) Using $a^*a^\# a + aa^\# a^* = 2a^*$, we get

$$\begin{aligned} 2a^* &= 2(aa^\dagger a)^* = 2(a^\dagger a)^*a^* = a^\dagger a(2a^*) = a^\dagger a(a^*a^\# a + aa^\# a^*) \\ &= a^*a^\# a + a^\dagger aa^* = a^*a^\# a + a^*. \end{aligned}$$

This gives

$$a^* = a^*a^\# a = a^*aa^\#,$$

and then a satisfies condition (iii).

(xxix) If $a^\dagger a^\# a + aa^\# a^\dagger = 2a^\dagger$, then we obtain

$$2aa^\dagger = a(2a^\dagger) = a(a^\dagger a^\# a + aa^\# a^\dagger) = aa^\dagger aa^\# + aa^\dagger = aa^\# + aa^\dagger,$$

i.e. $aa^\dagger = aa^\#$. Therefore, $aa^\#$ is symmetric.

(xxx) Suppose that $aaa^\dagger + a^\dagger aa = 2a$, then we get

$$2aa^\# = (aaa^\dagger + a^\dagger aa)a^\# = aaa^\dagger a(a^\#)^2 + a^\dagger a = aa(a^\#)^2 + a^\dagger a = aa^\# + a^\dagger a,$$

i.e. $aa^\# = a^\dagger a$. So $aa^\#$ is symmetric.

(xxxi) Multiplying $aaa^\dagger + (aaa^\dagger)^* = a + a^*$ by a from the right side, we get $aa^\dagger a^*a = a^*a$. Thus $aa^\dagger a^*a$ is symmetric and

$$aa^\dagger a^*a = (aa^\dagger a^*a)^* = a^*aaa^\dagger.$$

Then the condition (xxiv) holds.

(xxxii) Multiplying $a^\dagger aa + (a^\dagger aa)^* = a + a^*$ by a from the left side, we obtain $aa^*a^\dagger a = aa^*$. Hence $aa^*a^\dagger a$ is symmetric and

$$aa^*a^\dagger a = (aa^*a^\dagger a)^* = a^\dagger aaa^*.$$

Now the equality (xxv) is satisfied.

(xxxiii) From $aa^\dagger a^* = a^*aa^\dagger$, we have

$$aa^\# a^* = aa^\#(a^*aa^\dagger) = aa^\#(aa^\dagger a^*) = aa^\dagger a^* = a^*aa^\dagger = a^*.$$

Then, from the previous equality, we get

$$(a^\# a)^* = a^*(a^\#)^* = aa^\# a^*(a^\#)^* = aa^\#(a^\# a)^* = aa^\#(aa^\#)^*.$$

So $a^\# a$ is symmetric.

(xxxiv) Applying $a^*a^\dagger a = a^\dagger aa^*$, we obtain

$$a^*aa^\# = (a^\dagger aa^*)aa^\# = (a^*a^\dagger a)aa^\# = a^*a^\dagger a = a^\dagger aa^* = a^*.$$

Thus a satisfies condition (iii).

□

The following result is well-known for complex matrices and for linear operators on Hilbert spaces (see [1], [5], [8] and [9]). However, we are not in a position to prove this result for elements of a ring with involution, so we state it as a conjecture.

Conjecture. An element $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and one of the following equivalent conditions holds:

(i) $(a^\dagger)^2 a^\# = a^\#(a^\dagger)^2$;

(ii) $aa^\dagger = a^2(a^\dagger)^2$;

(iii) $a^\dagger a = (a^\dagger)^2 a^2$.

3 Comparisons with other results

The original definition of a complex EP matrix A requires the equality of the ranges $R(A)$ and $R(A^*)$ (hence EP for equal projections onto $R(A)$ and $R(A^*)$). For complex matrices this is equivalent to $A^\dagger = A^\#$. However, for Hilbert space operators and elements of C^*-algebras this is no longer true as only regular operators and elements of C^*-algebras possess the Moore–Penrose inverse. (Here the regularity of a is understood in the sense of von Neumann as the existence of b such that $aba = a$.) In rings with involution the regularity is not enough to ensure the existence of a Moore–Penrose inverse (see the discussion in [20, Section 3]).

In order to carefully distinguish between various conditions on an element of a ring with involution, Patrício and Puystjens in [22] introduced a whole new terminology. In [22], an element is called *-EP if $a^*\mathcal{R} = a\mathcal{R}$. The elements we call EP are called *-gMP elements in [22].

Adhering to our terminology, we can reproduce some of the results of [22] as follows:

Theorem 3.1. [22, Corollary 3] *If a is an element of a ring with involution, then*

$$a \text{ is EP} \iff a\mathcal{R} = a^*\mathcal{R} \text{ and } a \in \mathcal{R}^\# \iff a\mathcal{R} = a^*\mathcal{R} \text{ and } a \in \mathcal{R}^\dagger.$$

Patrício and Puystjens in [22] further discuss the necessary and sufficient conditions for a regular element a to be EP.

Theorem 3.2. [22, Theorem 4] *A regular element a of $\mathcal{R}$ is EP if and only if $a^* = v^{-1}au$, where*

$$u = (aa^-)^*(aa^* - 1) + 1, \qquad v = (a^2 - 1)aa^- + 1$$

are invertible for some (any) inner inverse a^-.

Boasso in [3] made further inroads into the theory when he gave a definition of EP elements of a Banach algebra in the absence of involution. His definition relies on the characterization of Hermitian elements using the topology of the underlying algebra due to Palmer and Vidav. However, there are no obvious candidates for Hermitian elements in a ring (or algebra) without involution, and so this avenue does not seem to be accessible from the purely algebraic point of view.

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