IDEMPOTENTS RELATED TO
THE WEIGHTED MOORE–PENROSE INVERSE

Dijana Mosić and Dragan S. Djordjević

Abstract

We investigate necessary and sufficient conditions for $aa^\dagger_{e,f} = bb^\dagger_{e,f}$ to hold in rings with involution. Here, $a^\dagger_{e,f}$ denotes the weighted Moore-Penrose inverse of $a$, related to invertible and Hermitian elements $e, f \in R$. Thus, some recent results from [7] are extended to the weighted Moore-Penrose inverse.

1 Introduction

Let $R$ be an associative ring with the unit 1. An involution $a \mapsto a^*$ in a ring $R$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$ 

An element $a \in R$ is selfadjoint (or Hermitian) if $a^* = a$. An element $a \in R$ is regular if there exists some inner inverse (or 1-inverse) $a^{-1} \in R$ satisfying $aa^{-1}a = a$. The set of all inner inverses (or 1-inverses) is denoted by $a\{1\}$. Hence, $a$ is regular if $a\{1\} \neq \emptyset$. A reflexive inverse $a^+_{e,f}$ of $a$ is a 1-inverse of $a$ such that $a^+_{e,f}aa^+_{e,f} = a^+_{e,f}.$

**Definition 1.1.** Let $R$ be a ring with involution, and let $e, f$ be invertible Hermitian elements in $R$. The element $a \in R$ has the weighted Moore-Penrose inverse (weighted MP-inverse) with weights $e, f$ if there exists $b \in R$ such that

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$ 

The unique weighted MP-inverse with weights $e, f$, will be denoted by $a^\dagger_{e,f}$ if it exists [4]. The set of all weighted MP-invertible elements of $R$ with weights $e, f$, will be denoted by $R^\dagger_{e,f}$. If $e = f = 1$, then the weighted MP-inverse reduces to the ordinary MP-inverse of $a$, denoted by $a^\dagger$.

If $a \in R^\dagger_{e,f}$, then $aa^\dagger_{e,f}$ and $a^\dagger_{e,f}a$ are idempotents related to $a$ and $a^\dagger_{e,f}$.
Notice that if $\mathcal{R}$ is a $C^*$-algebra, if $e, f$ are selfadjoint, invertible and positive elements in a $C^*$-algebra $\mathcal{R}$, and if $a \in \mathcal{R}$ is regular, then the following formula holds:

$$
(a^\dagger_{e,f})^\dagger = f^{-1/2}(a^{1/2} a f^{-1/2})^e 1/2.
$$

Hence, the existence of an inner inverse of $a$ implies the existence of the MP-inverse and the weighted MP-inverse of $a$.

However, if $\mathcal{R}$ is a general ring with involution, then we do not have the existence of a square root of a positive element. Hence, in this case we always have to assume that the weighted MP-inverse of $a$ exists.

Define the mapping $(\ast, e, f) : x \mapsto x^{\ast, e, f} = e^{-1} x^* f$, for all $x \in \mathcal{R}$. Notice that $(\ast, e, f) : \mathcal{R} \rightarrow \mathcal{R}$ is not an involution, because in general $(xy)^{\ast, e, f} \neq y^{\ast, e, f} x^{\ast, e, f}$.

Now, we formulate the following result which can be proved directly by the definition of the weighted MP-inverse.

**Theorem 1.1.** Let $\mathcal{R}$ be a ring with involution and let $e, f$ be invertible Hermitian elements in $\mathcal{R}$. For any $a \in \mathcal{R}^{e,f}$, the following is satisfied:

(a) $(a^\dagger_{e,f})^\dagger = a$;

(b) $(a^{\ast, f, e})^\dagger = (a_{e,f})^{\ast, e, f}$;

(c) $a^{\ast, f, e} = a_{e,f}^*a a^{\ast, e, f} = a^{\ast, f,e} a a_{e,f}^\dagger$;

(d) $a^{\ast, f,e}(a_{e,f})^{\ast, e, f} = a_{e,f}^\dagger a$;

(e) $(a_{e,f}^\dagger)^{\ast, e, f} a^{\ast, f, e} = a a_{e,f}^\dagger$;

(f) $(a^{\ast, f, e} a_{e,f}^\dagger)^\dagger = (a_{e,f} a_{e,f}^{\ast, e, f})^\dagger$;

(g) $(a a^{\ast, f, e})_{e,e} = (a_{e,f})^{\ast, e, f} a a_{e,f}^\dagger$;

(h) $a_{e,f}^\dagger = (a^{\ast, f, e})_{e,f}^\dagger a^{\ast, f, e} = a^{\ast, f, e} (a a^{\ast, f, e})_{e,e}^\dagger$;

(i) $(a^{\ast, e, f})_{e,f}^\dagger = a (a^{\ast, f, e})_{e,f}^\dagger = (a a^{\ast, e, f})_{e,e}^\dagger a$.

For $a \in \mathcal{R}$ consider two annihilators

$$
\mathcal{o} = \{x \in \mathcal{R} : ax = 0\}, \quad \mathcal{a} = \{x \in \mathcal{R} : xa = 0\}.
$$

Notice that,

$$(a^\circ)^\circ = a^\circ \iff \mathcal{o}(a^\circ) = \mathcal{a}, \quad \mathcal{a} = a^* \mathcal{R} \iff \mathcal{R} a = \mathcal{R} a^*.$$

**Lemma 1.1.** Let $a \in \mathcal{A}^\ast$, and let $e, f$ be invertible positive elements in $\mathcal{A}$. Then

$$
(a_{e,f}^\dagger)^\circ = (a^{\ast, f, e} a + 1 - a_{e,f}^\dagger a)^{-1} a^{\ast, f, e} = a^{\ast, f, e} (aa^{\ast, f, e} + 1 - aa_{e,f}^\dagger)^{-1},
$$

$$
a^{\ast, f, e} a^\circ = a_{e,f}^\dagger a^{\ast, f,e} a^\circ = a_{e,f}^\dagger a^{\ast, f,e} A^{-1} a^{\ast, f,e} = A^{-1} a_{e,f}^\dagger a^{\ast, f,e},
$$

and

$$
(a^{\ast, f, e})^\circ = (a_{e,f}^\dagger)^\circ \quad \text{and} \quad (a^{\ast, f, e})^\circ = (a_{e,f}^\dagger)^\circ.
$$
Proof. By Theorem 1.1, we can verify
\[
\begin{align*}
(a^*, f, e) &= (a^*, f, e) + a^* f e = (a^*, f, e) + 1 - a^{-1} e f a,
\end{align*}
\]
and
\[
\begin{align*}
(a^*, f, e) &= (a^*(a^*) + 1 - a^{-1} e f a) = (a^*(a^*) + 1 - a^{-1} e f a).
\end{align*}
\]
Thus, the part (1) holds and it implies the equalities (2) and (3).

Now, we state an useful result from [7].

Lemma 1.2. [7, Lemma 2.1] Let \(a, b \in R\) be regular elements.

1. There exist \(a^{-} \in a\{1\}, b^{-} \in b\{1\}\) for which \((1 - bb^{-}) aa^{-} = 0\) if and only if \((1 - bb^{-}) aa^{-} = 0\) for all \(a^{-} \in a\{1\}, b^{-} \in b\{1\}\).

2. There exist \(a^{-} \in a\{1\}, b^{-} \in b\{1\}\) for which \((1 - bb^{-})(1 - a^{-} a) = 0\) if and only if \((1 - bb^{-})(1 - a^{-} a) = 0\) for all \(a^{-} \in a\{1\}, b^{-} \in b\{1\}\).

In [7], necessary and sufficient conditions for \(aa^* = bb^*\) in ring with involution are investigated. In this paper we generalized this results to the weighted Moore-Penrose in rings with involution.

2 Results

A semigroup is a regular, if every elements of that semigroup has an inner generalized inverse. The notion extends to rings also.

In a regular semigroup, the natural partial order is defined by ([2], [5], [6])

\[
a \leq_\ast b \text{ if } aa^* = ba^* \text{ and } a^* a = a^{-} b \text{ for some inner inverse } a^{-} \text{ of } a.
\]

See also [3] for intuitionistic fuzzy matrices. Notice that \(\leq_\ast\) is a partial order in regular rings.

A semigroup with involution \(x \mapsto x^\ast\) is proper, if the following implication holds:

\[
a^* a = a^* b = b^* a = b^* b \implies a = b.
\]

Notice that if the semigroup has the zero element 0, then a semigroup is a proper with respect to the involution \(x \mapsto x^\ast\), if and only if \(a^* a = 0 \implies a = 0\). The last implication is called \(*\)-cancellability. For example, every element of a \(C^*\)-algebra is \(*\)-cancellable, so every \(C^*\)-algebra is proper (with respect to multiplication).

Drazin [1] presented a partial order on a proper \(*\)-semigroup in the following way

\[
a \leq_\ast b \text{ if } aa^* = ba^* \text{ and } a^* a = a^* b.
\]

If \(a \in R\) is MP invertible, then “\(\leq_\ast\)” implies “\(\leq_\ast\)”. Indeed, \(aa^* = ba^* \Rightarrow a^* a = a^* a \Rightarrow a^* a = a^* a \Rightarrow a^* a = a^* a\) and similarly \(a^* a = a^* a \Rightarrow a^* a = a^* a\).
In this paper we introduce the "\(\leq_{*,e,f}\)" as follows:

\[ a \leq_{*,e,f} b \text{ if } aa^{*,e,f} = ba^{*,e,f} \text{ and } a^{*,e,f}a = a^{*,e,f}b. \]

Here \(e, f\) are Hermitian invertible elements in a ring \(R\) with involution \(x \mapsto x^*\). We like to see that \(\leq_{*,e,f}\) is a partial ordering in \(R\).

If \(a \in R_{e,f}^+\), then "\(\leq_{*,e,f}\)" implies "\(\leq_{-}\)." Indeed, from \(aa^{*,e,f} = ba^{*,e,f}\) we get \(aa^{*,e,f}_{1} = aa^{*,e,f}(a^{*,e,f}_{1}) = ba^{*,e,f}(a^{*,e,f}_{1}) = ba^{*,e,f}_{1}\). Similarly, \(a^{*,e,f}a = a^{*,e,f}b\) gives \(a^{*,e,f}_{1}a = a^{*,e,f}_{1}b\).

In the rest of the paper we assume that \(e, f \in R\) are Hermitian end invertible.

The ring \(R\) is \((*, e, f)\)-proper if the following implication holds:

\[ a^{*,e,f}a = a^{*,e,f}b = b^{*,e,f}a = b^{*,e,f}b \implies a = b. \]

If \(R\) is a \(C^*\)-algebra and \(e, f\) are positive Hermitian elements in \(R\), then \(R\) is \((*, e, f)\)-proper. Indeed, \(a^{*,e,f}a = a^{*,e,f}b = b^{*,e,f}a = b^{*,e,f}b\) gives \((a-b)^{*,e,f}(a-b) = 0\) which gives that \([f^{1/2}(a-b)]^* f^{1/2}(a-b) = 0\). Since every element in \(C^*\)-algebra is \(-\)cancellable, then \(f^{1/2}(a-b) = 0\), that is \(a = b\).

**Theorem 2.1.** Let \(R\) be: \((*, e, f)\)-proper, \((*, e, f)\)-proper and \((*, f, e)\)-proper. Then \(\leq_{*,e,f}\) is a partial ordering in \(R\).

**Proof.** Since \(a \leq_{*,e,f} a\), then "\(\leq_{*,e,f}\)" is reflexive.

From \(a \leq_{*,e,f} b\) and \(b \leq_{*,e,f} a\), we get \(a^{*,e,f}a = a^{*,e,f}b\) and \(b^{*,e,f}a = b^{*,e,f}b\). Observe that

\[ a^{*,e,f}a = (a^{*,e,f}a)^e.e = (a^{*,e,f}b)^e.e = b^{*,e,f}a \]

(4)

So, we deduce \(a^{*,e,f}a = a^{*,e,f}b = b^{*,e,f}a = b^{*,e,f}b\) which gives \(a = b\).

If \(a \leq_{*,e,f} b\) and \(b \leq_{*,e,f} c\), we obtain (4) and, applying (4) for \(b\) and \(c\) instead of \(a\) and \(b\), we have \(b^{*,e,f}b = c^{*,e,f}b\). Further,

\[ (a^{*,e,f}a)a^{*,e,f}c = (a^{*,e,f}f(a^{*,e,f}a))^{e.e}c = (b^{*,e,f}b)(a^{*,e,f}c) = (b^{*,e,f}b) a^{*,e,f}c = (b^{*,e,f}b) a^{*,e,f}c = a^{*,e,f}b^{*,e,f}a. \]

\[ (a^{*,e,f}a)a^{*,e,f}c = (b^{*,e,f}b)(a^{*,e,f}c) = (b^{*,e,f}b) a^{*,e,f}c = a^{*,e,f}b^{*,e,f}c. \]

Since \((a^{*,e,f}a)^{*,e.e} = a^{*,e,f}a\) and \((a^{*,e,f}c)^{*,e.e} = c^{*,e,f}\), by the previous tree equalities, we conclude

\[ (a^{*,e,f}a)^{*,e.e}a^{*,e,f}a = (a^{*,e,f}a)^{*,e.e}a^{*,e,f} = (a^{*,e,f}c)^{*,e.e}a^{*,e,f}a = (a^{*,e,f}c)^{*,e.e}a^{*,e,f}c. \]

which implies \(a^{*,e,f}a = a^{*,e,f}c\), because ring \(R\) is \((*, e, f)\)-proper. Similarly, by \((*, f, e)\)-proper of \(R\), we can verify that \(aa^{*,e,f} = (ca^{*,e,f})^{*,f}f\) which yields \(aa^{*,e,f} = (aa^{*,e,f})^{*,f}f = (ca^{*,e,f})^{*,f}f = ca^{*,e,f}\). Thus, \(a^{*,e,f}a = a^{*,e,f}c\) and \(aa^{*,e,f} = ca^{*,e,f}\) give that \(a \leq_{*,e,f} c\).
In the following theorem, we present some equivalent conditions for $aa_{e,f}^\dagger = bb_{e,f}^\dagger aa_{e,f}^\dagger$ to hold.

**Theorem 2.2.** Let $\mathcal{R}$ be a ring with involution, and let $e$, $f$ be invertible Hermitian elements in $\mathcal{R}$. If $a, b \in \mathcal{R}_{e,f}^\dagger$, then the following conditions are equivalent:

1. $aa_{e,f}^\dagger = bb_{e,f}^\dagger aa_{e,f}^\dagger$;
2. $aa_{e,f}^\dagger = aa_{e,f}^\dagger bb_{e,f}^\dagger$;
3. $a = bb_{e,f}^\dagger a$;
4. $a_{e,f}^\dagger = a_{e,f}^\dagger bb_{e,f}^\dagger$;
5. $aa^{*,f,e}_{e,f} = bb_{e,f}^\dagger aa^{*,f,e}_{e,f}$;
6. $aa^{*,f,e}_{e,f} = aa^{*,f,e}_{e,f} bb_{e,f}^\dagger$;
7. $a^{*,f,e}_{e,f} = a^{*,f,e}_{e,f} bb_{e,f}^\dagger$;
8. $aa^- = bb^- aa^-$ for all choices $a^- \in a\{1\}$, $b^- \in b\{1\}$;
9. $aa^- = bb^- aa^-$ for some $a^- \in a\{1\}$, $b^- \in b\{1\}$;
10. $a = bb^- a$ for all $b^- \in b\{1\}$;
11. $a = bb^- a$ for some $b^- \in b\{1\}$;
12. $aa^{*,f,e}_{e,f} = bb^- aa^{*,f,e}_{e,f}$ for all $b^- \in b\{1\}$;
13. $aa^{*,f,e}_{e,f} = bb^- aa^{*,f,e}_{e,f}$ for some $b^- \in b\{1\}$;
14. $aa_{e,f}^\dagger \leq bb_{e,f}^\dagger$;
15. $aa_{e,f}^\dagger \leq_{*,e,e} bb_{e,f}^\dagger$;
16. $a \leq bb^- a$ for all $b^- \in b\{1\}$;
17. $a \leq bb^- a$ for some $b^- \in b\{1\}$;
18. $a\mathcal{R} \subseteq bb_{e,f}^\dagger a\mathcal{R}$;
19. $a\mathcal{R} \subseteq b\mathcal{R}$;
20. $\mathcal{R}a_{e,f}^\dagger \subseteq \mathcal{R}a_{e,f}^\dagger bb_{e,f}^\dagger$;
21. $\mathcal{R}a_{e,f}^\dagger \subseteq \mathcal{R}b_{e,f}^\dagger$.
Proof. (1) ⇔ (2): Applying the involution, the equality \( a_{e,f}^\dagger = b_{e,f}^\dagger a_{e,f}^\dagger \) is equivalent to \( (e^{-1}eaa_{e,f}^\dagger b_{e,f}^\dagger e^{-1})^\ast = (e^{-1}eb_{e,f}^\dagger e^{-1}eaa_{e,f}^\dagger b_{e,f}^\dagger e^{-1})^\ast \) which is \( eaa_{e,f}^\dagger e^{-1} = eaa_{e,f}^\dagger e^{-1}eb_{e,f}^\dagger e^{-1} \), i.e. \( aa_{e,f}^\dagger = aa_{e,f}^\dagger b_{e,f}^\dagger \).

(1) ⇔ (3): Multiplying (1) by a from the right side we get (3), and multiplying (3) by \( a_{e,f}^\dagger \) from the right side we obtain (1).

(2) ⇔ (4): This part can be verified in the same way as (1) ⇔ (3).

(3) ⇔ (5): If we multiply (3) by \( a^\ast_{f,e} \) from the right side we obtain (5), and if we multiply (5) by \( (a_{e,f}^\dagger)^{\ast e,f} \) from the right side, by Theorem 1.1(d), we have (3).

(2) ⇔ (6): By Theorem 1.1, multiplying (2) by \( a^\ast_{f,e} \) from the left side, we obtain (6). Conversely, multiplying (6) by \( (a_{e,f}^\dagger)^{\ast e,f} a_{e,f}^\dagger \) from the left side, we get (2).

(6) ⇔ (7): Multiplying (6) by \( a_{e,f}^\dagger \) from the left side, we obtain (7) and multiplying (7) by \( a \) from the left side, we get (6).

(1) ⇔ (8): The assumption \( aa_{e,f}^\dagger = bb_{e,f}^\dagger aa_{e,f}^\dagger \) is equivalent to \( (1−bb_{e,f}^\dagger)aa_{e,f}^\dagger = 0 \). Applying Lemma 1.2, we obtain this equivalence.

(8) ⇔ (9): By Lemma 1.2.

(8) ⇔ (10), (9) ⇔ (11): Obviously.

(10) ⇔ (12): Multiplying (10) by \( a^\ast_{f,e} \) from the right side, we obtain (12). On the other hand, multiplying (12) from the right side by \( (a_{e,f}^\dagger)^{\ast e,f} \), we get (10).

(11) ⇔ (13): See the previous part.

(1) ⇔ (14): We can easily verify that \( (aa_{e,f}^\dagger)_{e,e}^\dagger = aa_{e,f}^\dagger \). Now, for \( (aa_{e,f}^\dagger)^{\ast e,e} \) we have \( aa_{e,f}^\dagger \leq bb_{e,f}^\dagger \) if and only if \( aa_{e,f}^\dagger (aa_{e,f}^\dagger)^{\ast e,e} = bb_{e,f}^\dagger (aa_{e,f}^\dagger)^{\ast e,e} \) and \( (aa_{e,f}^\dagger)^{\ast e,e} aa_{e,f}^\dagger = (aa_{e,f}^\dagger)^{\ast e,e} bb_{e,f}^\dagger \), which is equivalent to \( aa_{e,f}^\dagger = bb_{e,f}^\dagger \) and \( aa_{e,f}^\dagger = aa_{e,f}^\dagger bb_{e,f}^\dagger \).

(1) ⇔ (15): Since \( (aa_{e,f}^\dagger)^{\ast e,e} = e^{-1}eaa_{e,f}^\dagger e^{-1} \), \( e = aa_{e,f}^\dagger \), we show this equivalence in the same way as (1) ⇔ (14).

(10) ⇒ (16): For \( a^\dagger = a_{e,f}^\dagger \), we already proved this part.

(16) ⇒ (17): Obviously.

(17) ⇒ (11): Suppose that \( a \leq bb^\ast_a \) for some \( b^\ast \in b\{1\} \). Then, for some \( a^\dagger \), we have \( aa^\dagger = bb^\ast aa^\dagger \), so \( a = bb^\ast a \).

(3) ⇒ (18) ⇒ (19): Obviously.

(19) ⇒ (3): The hypothesis \( aR ⊆ bR \) gives \( a = bx \), for some \( x \in R \). Therefore, \( a = bb_{e,f}^\dagger (bx) = bb_{e,f}^\dagger a \).

(2) ⇒ (20) ⇒ (21) ⇒ (4): Similarly as (3) ⇒ (18) ⇒ (19) ⇒ (3).

\[ \square \]

Theorem 2.3. Let \( R \) be a ring with involution, and let \( e, f \) be invertible Hermitian elements in \( R \). If \( a, b \in R_{e,f}^\dagger \), then the following conditions are equivalent:

1. \( aa_{e,f}^\dagger = bb_{e,f}^\dagger \);
2. \( aa_{e,f}^\dagger = aa_{e,f}^\dagger bb_{e,f}^\dagger \) and \( u = aa_{e,f}^\dagger + 1 − bb_{e,f}^\dagger \in R_{e,f}^{-1} \);
3. \( aa_{e,f}^\dagger = aa_{e,f}^\dagger bb_{e,f}^\dagger \) and \( v = aa_{e,f}^\ast_{f,e} + 1 − bb_{e,f}^\dagger \in R_{e,f}^{-1} \).
Idempotents related to the weighted Moore–Penrose inverse

(4) \( a a_{e,f}^\dagger = a a_{e,f}^\dagger b b_{e,f}^\dagger \) and \( \forall b^- \in b[1] \) \( w = a a^{*,f,e} + 1 - b b^- \in \mathcal{R}^{-1}; \)

(5) \( a a_{e,f}^\dagger = a a_{e,f}^\dagger b b_{e,f}^\dagger \) and \( \exists b^- \in b[1] \) \( w = a a^{*,f,e} + 1 - b b^- \in \mathcal{R}^{-1}; \)

(6) \( a a_{e,f}^\dagger b b_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger, u = a a_{e,f}^\dagger + 1 - b b_{e,f}^\dagger \in \mathcal{R}^{-1} \) and \( l = b b_{e,f}^\dagger + 1 - a a_{e,f}^\dagger \in \mathcal{R}^{-1}; \)

(7) \( a a_{e,f}^\dagger b b_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger, v = a a^{*,f,e} + 1 - b b_{e,f}^\dagger \in \mathcal{R}^{-1} \) and \( k = b b^{*,f,e} + 1 - a a_{e,f}^\dagger \in \mathcal{R}^{-1}; \)

Proof. (1) \( \Leftrightarrow \) (2): It is easy to check.

(2) \( \Leftrightarrow \) (3): Using Theorem 2.2, \( (a a_{e,f}^\dagger + 1 - b b_{e,f}^\dagger)(a a^{*,f,e} + 1 - a a_{e,f}^\dagger) = a a^{*,f,e} + 1 - b b_{e,f}^\dagger. \) By Lemma 1.1, \( a a^{*,f,e} + 1 - a a_{e,f}^\dagger \in \mathcal{R}^{-1} \) and then \( u \in \mathcal{R}^{-1} \Leftrightarrow v \in \mathcal{R}^{-1}. \)

(3) \( \Rightarrow \) (1): Observe that, by Theorem 2.2, \( v a a_{e,f}^\dagger = a a^{*,f,e} = v b b_{e,f}^\dagger. \) Since \( v \in \mathcal{R}^{-1}, \) we have \( a a_{e,f}^\dagger = b b_{e,f}^\dagger. \)

(3) \( \Rightarrow \) (4): By Theorem 2.2, we have \( a a^{*,f,e} = b b_{e,f}^\dagger a a^{*,f,e} = b b_{e,f}^\dagger a a^{*,f,e} b b_{e,f}^\dagger. \) Now, by [8, Proposition 3], \( v = a a^{*,f,e} + 1 - b b_{e,f}^\dagger = b b_{e,f}^\dagger a a^{*,f,e} b b_{e,f}^\dagger + 1 - b b_{e,f}^\dagger \in \mathcal{R}^{-1} \)

(5) \( \Rightarrow \) (4): From \( w = a a^{*,f,e} + 1 - b b^- = 1 - b b^- (-aa^{*,f,e} + 1) \in \mathcal{R}^{-1}, \) we deduce that \( 1 - (-aa^{*,f,e} + 1)bb^- = bb^- aa^{*,f,e} bb^- + 1 - b b^- \in \mathcal{R}^{-1}. \) Then, by [8, Proposition 3], \( bb^- aa^{*,f,e} bb^- + 1 - b b^- = 1 - (-aa^{*,f,e} + 1)bb^- \in \mathcal{R}^{-1}, \) for all \( b^- \in \{1\}, \) which gives \( 1 - b b^- (-aa^{*,f,e} + 1) = bb^- aa^{*,f,e} + 1 - b b^- = aa^{*,f,e} + 1 - b b^- \in \mathcal{R}^{-1}. \)

(1) \( \Rightarrow \) (6): Obviously.

(6) \( \Rightarrow \) (1): Since, by \( a a_{e,f}^\dagger b b_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger, b b_{e,f}^\dagger u = b b_{e,f}^\dagger a a_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger u \)

and \( u \in \mathcal{R}^{-1}, \) then \( b b_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger. \) Similarly, \( l a a_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger \)

and \( l \in \mathcal{R}^{-1} \) give \( a a_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger. \) Thus, \( a a_{e,f}^\dagger = b b_{e,f}^\dagger. \)

(1) \( \Rightarrow \) (7): By Lemma 1.1.

(7) \( \Rightarrow \) (3): The equality \( a a_{e,f}^\dagger b b_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger \) implies \( a a_{e,f}^\dagger k = a a_{e,f}^\dagger b b^{*,f,e} = b b_{e,f}^\dagger a a_{e,f}^\dagger k. \) Because \( k \in \mathcal{R}^{-1}, \) then \( a a_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger \) and the condition (3) holds.

References


Address:
Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, P.O. Box 224, 18000 Niš, Serbia

E-mail:
Dijana Mosić: dijana@pmf.ni.ac.rs
Dragan S. Djordjević: dragan@pmf.ni.ac.rs