Outer generalized inverses in rings and related idempotents

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Abstract. In this paper we investigate outer generalized inverses of elements in rings, and related idempotents. Among other things, if \( a'aa' = a' \) and \( bb' = b' \), we consider the relations \( b'b = a'a + u \) and \( bb' = aa' + v \) for a suitable choice of \( u \) and \( v \).

1. Introduction and preliminaries

Let \( R \) be ring with the unit 1. We use \( R^{-1} \) and \( R^\bullet \), respectively, to denote the set of all invertible elements of \( R \) and the set of all idempotents of \( R \). An element \( a \in R \) is outer generalized invertible, if there exists some \( a' \in R \) satisfying \( a' = a'aa' \). Such an \( a' \) is called the outer generalized inverse of \( a \). In this case \( a'a \) and \( 1 - aa' \) are idempotents corresponding to \( a \) and \( a' \).

For example, the ordinary and generalized Drazin inverse, as well as the Moore–Penrose inverse in rings with involution, are special cases of outer generalized inverses.

Recently, CASTRO-GONZALEZ and VÉLEZ-CERRADA [3] considered generalized Drazin invertible elements \( a, b \in R \), such that the corresponding idempotents \( a^\pi = 1 - aa^D \) and \( b^\pi = 1 - bb^D \) satisfy \( 1 - (b^\pi - a^\pi)^2 \in R^{-1} \).

Generalized inverses in rings have been studied in [4], [5], [6] and [8]. Related results concerning the perturbation of the generalized inverse or related idempotents can be found in [1], [2], [3], [9], [10].

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In this paper we extend some results from [3] to idempotents related to outer generalized inverses of \(a\) and \(b\). Particularly, we investigate outer generalized inverses with prescribed idempotents, which are introduced in [4].

2. Idempotents in rings

In this section we prove some statements concerning idempotents in rings. Let \(R\) be a ring with the unit 1. First we investigate the relations between idempotents \(p\) and \(p + u\) for \(u \in R\) and \(1 - u^2 \in R^{-1}\).

**Theorem 2.1.** Let \(u \in R\) be such that \(1 - u^2 \in R^{-1}\), and let \(p \in R^*\). Then the following conditions are equivalent:

1. \(p + u \in R^*\);
2. \(p + u = (1 - u)^{-1}p(1 + u) = (1 + u)p(1 - u)^{-1}\);
3. \(1 - p - u = (1 + u)^{-1}(1 - p)(1 - u) = (1 - u)(1 - p)(1 + u)^{-1}\).

Moreover, if previous conditions hold, and \(r = (1 + u)p + (1 - u)(1 - p)\), then \(r \in R^{-1}\), where

\[
r^{-1} = p(1 - u)^{-1} + (1 - p)(1 + u)^{-1},
\]

and \(p + u = rpr^{-1}\).

**Proof.** (i) \(\Leftrightarrow\) (ii): Since \(p \in R^*\), we have the following:

\[
p + u \in R^* \iff (p + u)^2 = p + u \iff p^2 + pu + up + u^2 = p + u
\]

\[
\iff p(1 + u) = (1 - u)(p + u) \iff (1 - u)^{-1}p(1 + u) = p + u.
\]

In the same way the second equality can be proved.

(i) \(\Leftrightarrow\) (iii) Since \(p \in R^*\), \(1 - p \in R^*\), we have

\[
p + u \in R^* \iff (1 - p) + (-u) \in R^*.
\]

Now we use (i) \(\Leftrightarrow\) (ii) for \(1 - p\) and \(-u\) instead of \(p\) and \(u\), respectively, and the result follows.

Now, suppose that (i), (ii) and (iii) hold. Let \(r = (1 + u)p + (1 - u)(1 - p)\).

If we take

\[
r' = p(1 - u)^{-1} + (1 - p)(1 + u)^{-1},
\]

then we get

\[
rr' = (1 + u)p(1 - u)^{-1} + (1 - u)(1 - p)(1 + u)^{-1} = (p + u) + (1 - p - u) = 1,
\]
because of (i) and (ii). From the same reason we have
\[ r' r = p(1-u)^{-1}(1+u)p + (1-p)(1+u)^{-1}(1-u)(1-p) \]
\[ = (1+u)^{-1}(p+u)(1-u) + (1-u)^{-1}(1-p-u)(1-p-u)(1+u) \]
\[ = p + (1-p) = 1. \]

Consequently \( r^{-1} = r' \). Moreover, we have \( r pr^{-1} = (1+u)p(1-u)^{-1} + 0 = p+u \), because of (i) and (ii). □

We state two auxiliary results, and prove the second one.

**Lemma 2.2.** If \( p, p+u \in R^\bullet \) hold, then \( p^2 = -pu^2 = -u^2p \). If \( m, m-u \in R^\bullet \) is satisfied, then \( m^2 = mu^2 = u^2m \).

**Theorem 2.3.** Let \( u \in R \) be such that \( 1-u^2 \in R^{-1} \), and let \( p, m, p+u \in R^\bullet \).

Then the following conditions are equivalent:

(i) \( m = p + u \);

(ii) \( p(1+u)(1-m) = (1-p)(1-u)m \);

(iii) \( m(1-u)(1-p) = (1-m)(1+u)p \).

**Proof.** (i) \(\Rightarrow\) (ii) and (iii): Suppose that (i) holds. Using the results from Theorem 2.1 we have \( p(1+u)(1-m) = (1-u)(p+u)(1-(p+u)) = 0 \) and \( (1-p)(1-u)m = (1+u)(1-(p+u))(p+u) = 0 \). Similarly, it is straightforward to prove that (iii) holds.

(ii) \(\Rightarrow\) (i): Now, let \( p(1+u)(1-m) = (1-p)(1-u)m \). Multiplying this equality from the right side with \( m \) and \( 1-m \) respectively, we get
\[ (1-p)(1-u)m = 0 \] and \( p(1+u)(1-m) = 0 \).

Using Theorem 2.1 again, it follows that \( m = (p+u)m \) and \( p + u = (p+u)m \)
so (i) follows.

In the same manner (iii) \(\Rightarrow\) (i) can be proven, taking \( p, m-u \) and \(-u\) instead of \( m, m+u \) and \( u \), respectively. □
3. Outer generalized inverses and idempotents

In this section we prove some results concerning the outer generalized inverses with prescribed idempotents.

**Definition 3.1.** Let \( a \in \mathcal{R} \) and \( p, q \in \mathcal{R}^\bullet \). An element \( a' \in \mathcal{R} \) satisfying
\[
a'a = a', \quad a'a = p, \quad 1 - aa' = q
\]is called a \((p, q)\)-outer generalized inverse of \( a \), denoted by \( a'(2)_{p,q} \). It is proved in [4] that if \( a'(2)_{p,q} \) exists, then it is unique. The set of all \((p, q)\)-outer invertible elements of \( \mathcal{R} \) is denoted by \( \mathcal{R}(2)_{p,q} \).

Now, as our main result, we characterize elements \( a \) and \( b \) such that
\[
b'b = a'a + u \quad \text{and} \quad bb' = aa' + v
\]such that \( 1 - u^2 \in \mathcal{R}^{-1} \) and \( 1 - v^2 \in \mathcal{R}^{-1} \).

**Theorem 3.2.** Let \( a, b, u, v \in \mathcal{R} \) such that \( a \) and \( b \) are outer invertible and \( 1 - u^2, 1 - v^2 \in \mathcal{R}^{-1} \). Then the following conditions are equivalent

(i) \( b'b = a'a + u \) and \( bb' = aa' + v \);

(ii) \( ub' + a'v = b' - a' - a(a - b)b' \) and \( au + vb = bb'b - aa'a - a(a' - b)b \);

(iii) \( ua' + bv = b' - a' - b'(a - b)a' \) and \( bu + va = bb'b - aa'a - b(a' - b)a \).

**Proof.** From the fact that \( a \) and \( b \) are outer invertible it follows that there exist \( p, q, m, n \in \mathcal{R}^\bullet \) and there exist \( a', b' \in \mathcal{R} \) such that \( a' = a'(2)_{p,q} \) and \( b' = b'(2)_{m,n} \). That is \( a'a = p, 1 - aa' = q, b'b = m \) and \( 1 - bb' = n \).

(i) \( \Rightarrow \) (ii): Using direct computations from \( u = b'b - a'a \) and \( v = bb' - aa' \) we have that (ii) is satisfied.

(ii) \( \Rightarrow \) (i): Suppose that (ii) holds. If we multiply first equality with \( 1 - a'a \) from the left side we get
\[
(1 - a'a)(1 - u)b' = 0,
\]
and then multiplying the last equality by \( b \) from the right side we get
\[
(1 - a'a)(1 - u)b'b = 0
\]
that is
\[
(1 - p)(1 - u)m = 0 \quad (3.2)
\]holds.

In the same manner, if we multiply first equality in (ii) with \( 1 - bb' \) from the right side we get
\[
a'(1 + v)(1 - bb') = 0, \quad (3.3)
\]
and than multiplying the last equality with a from the left side we have

\[ aa'(1 + v)(1 - bb') = 0, \]

that is

\[ (1 - q)(1 + v)n = 0 \quad (3.4) \]

holds.

Similarly, if we multiply second equality in (ii) with \(1 - b' b\) from the right side we get

\[ au(1 - b' b) + vb(1 - b' b) = -aa'(a + b)(1 - b' b), \]

which is the same as

\[ au(1 - b' b) + aa' a(1 - b' b) = -v(1 - bb')b - aa'(1 - bb')b. \]

Multiplying the last equality with \(a'\) from the left side we get

\[ a' a(1 + u)(1 - b' b) = -a'(1 + v)(1 - bb')b. \quad (3.5) \]

The right-hand side of (3.5) is equal to zero because of (3.3). So, we have

\[ a' a(1 + u)(1 - b' b) = 0 \text{ or } \]

\[ p(1 + u)(1 - m) = 0. \quad (3.6) \]

Now, using Theorem 2.3 together with (3.2) and (3.6) we have the result that \(b' b = a' a + u\).

Now, multiplying the second equality of (ii) with \(1 - aa'\) from the left side, we get

\[ (1 - aa')(au + vb) = (1 - aa')(bb' + ab'b), \]

which is the same as

\[ a(1 - a'a)(u - b' b) = (1 - aa')(bb' - v)b. \]

Multiplying the last equality with \(b'\) from the right side we get

\[ (1 - aa')(1 - v)bb' = -a(1 - a'a)(1 - u)b'. \quad (3.7) \]

Because of (3.1), it follows that \((1 - aa')(1 - v)bb' = 0\), or

\[ q(1 - v)(1 - n) = 0. \quad (3.8) \]

Finally, using again Theorem 2.3 together with (3.4) and (3.8) it follows that \(bb' = aa' + v\).

The proof of (i) ⇔ (iii) is just the same as (i) ⇔ (ii), replacing the role of \(a\) and \(b\) and taking \(-u\) and \(-v\) instead of \(u\) and \(v\), respectively. Or, in other words we prove the result taking (i) to be \(a' a = b' b - u\) and \(aa' = bb' - v\). \(\square\)
Theorem 3.2 gives a characterization of the elements in a ring which have related idempotents differing by a suitable choice of \( u \) and \( v \). If \( a \) is generalized Drazin invertible element in \( R \) and if \( a^\pi \) is the spectral idempotent of \( a \) then \( a^D = a_{p,1-p}^{(2)} \) for \( p = 1 - a^\pi \). See [6] and [7] for the definition of quasinilpotent elements and the generalized Drazin inverse in rings.

Now, as a corollary we obtain one partial result from the main Theorem 3.2 from [3].

**Corollary 3.3.** Let \( a \) and \( b \) are generalized Drazin invertible elements in \( R \) and \( s \in R \) such that \( 1 - s^2 \in R^\bullet \). If \( a^\pi + s \in R^\bullet \) then the following conditions are equivalent:

(i) \( b^\pi = a^\pi + s \);
(ii) \( (1 + s)b^D - a^D(1 - s) = a^D(a - b)b^D \);
(iii) \( b^D(1 + s) - (1 - s)a^D = b^D(a - b)a^D \).

**Proof.** Let \( a' = a^D \) and \( b' = b^D \). Using Theorem 3.2 with \( u = v = -s \) the result follows. \( \square \)

Also, as a corollary we obtain the first result from Theorem 4.2 in [4].

**Corollary 3.4.** Let \( a, b \in R \) and let \( p, q \in R^\bullet \) be such that \( a^{(2)}_{p,q} \) and \( b^{(2)}_{p,q} \) exist. Then the following hold

\[
a^{(2)}_{p,q} - b^{(2)}_{p,q} = b^{(2)}_{p,q}(b - a)a^{(2)}_{p,q} = a^{(2)}_{p,q}(b - a)b^{(2)}_{p,q}.
\]

**Proof.** With \( a' = a^{(2)}_{p,q} \) and \( b' = b^{(2)}_{p,q} \) and \( u = v = 0 \) from (ii) and (iii) in Theorem 3.2 the result follows. \( \square \)

4. Perturbation of outer generalized invertible elements in Banach algebras

In this section we assume that \( R \) is a Banach algebra with the unit 1. Results from Theorem 3.2 are also valid in complex Banach algebras. Now we state the following upper bound for \( \|b' - a'\|/\|a'\| \).

**Theorem 4.1.** Let \( a, b, u, v \in R \), \( p, q, m, n \in R^\bullet \), \( a' = a^{(2)}_{p,q} \) and \( b' = b^{(2)}_{m,n} \). Let \( b'b = a'a + u \) and \( bb' = aa' + v \).

If \( \|a'(a - b)\| + \|u\| < 1 \) and \( \|v\| < 1 \), then

\[
\frac{\|b' - a'\|}{\|a'\|} \leq \frac{\|a'(a - b)\| + \|u\| + \|v\|}{1 - \|u\| - \|a'(a - b)\|}.
\]
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Proof. From \(|u|, |v| < 1| \) it follows that \(1 - u^2, 1 - v^2 \in R^{-1}\). Then using the first equation from (ii) in Theorem 3.2 we have \(b^\prime - a^\prime = ub^\prime + a^\prime v + a^\prime (a - b)b^\prime = (a^\prime(a - b) + u)(b^\prime - a^\prime) + a^\prime(a - b)a^\prime + ua^\prime + a^\prime v\). Applying the norm here we get

\[
\|b^\prime - a^\prime\| \leq (\|a^\prime(a - b)\| + \|u\|)\|b^\prime - a^\prime\| + (\|a^\prime(a - b)\| + \|u\| + \|v\|)\|a^\prime\|
\]

and the result follows. \(\square\)

As a corollary we obtain Theorem 5.3 in [3].

Corollary 4.2. Let \(a\) and \(b\) are generalized Drazin invertible elements in \(R\). If \(\|b^\tau - a^\tau\| + \|a^D(b - a)\| < 1\), then

\[
\frac{\|b^D - a^D\|}{\|a^D\|} \leq \frac{\|a^D(b - a)\| + 2\|b^\tau - a^\tau\|}{1 - \|b^\tau - a^\tau\| - \|a^D(b - a)\|}.
\]

Again, as a corollary we obtain the second result in Theorem 4.2 in [4].

Corollary 4.3. Let \(a\) and \(b\) are elements in a Banach algebra \(R\), and \(p, q \in R^\bullet\) be such that \(a^{(2)}_{p,q}\) and \(b^{(2)}_{p,q}\) exist. Then if \(\|a^{(2)}_{p,q}\| \|b - a\| < 1\) then

\[
\frac{\|b^{(2)}_{p,q} - a^{(2)}_{p,q}\|}{\|a^{(2)}_{p,q}\|} \leq \frac{\|a^{(2)}_{p,q}(b - a)\|}{1 - \|a^{(2)}_{p,q}(b - a)\|}.
\]

References