Moore-Penrose-invertible normal and Hermitian elements in rings

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Abstract

In this paper we present several new characterizations of normal and Hermitian elements in rings with involution in purely algebraic terms, and considerably simplify proofs of already existing characterizations.

Key words and phrases: Normal elements, Hermitian elements, Moore–Penrose inverse, group inverse, ring with involution.

2000 *Mathematics subject classification*: 16W10, 15A09, 46L05.

1 Introduction

Normal and Hermitian matrices, as well as normal and Hermitian linear operators on Banach or Hilbert spaces have been investigated by many authors (see, for example, [1, 2, 3, 5, 6, 7, 9, 10, 12, 16]). In this paper we use a different approach, exploiting the structure of rings with involution to investigate normal and Hermitian elements which are also Moore-Penrose invertible. We give new characterizations, and the proofs are based on ring theory only.

Let $\mathcal{R}$ be an associative ring, and let $a \in \mathcal{R}$. Then a is *group invertible* if there is $a^\# \in \mathcal{R}$ such that

$$aa^\# a = a, \quad a^\# aa^\# = a^\#, \quad aa^\# = a^\# a;$$

$a^\#$ is a group inverse of a and it is uniquely determined by these equations [4]. We provide a short proof: if b, c are two candidates for a group inverse of a, then $b = b^2a = b^2a^2c = bac = ba^2c^2 = ac^2 = c$. Thus, the group inverse of a is unique if it exists. The group inverse $a^\#$ double commutes with a, that is,

*The authors are supported by the Ministry of Science, Republic of Serbia, grant no. 144003.

$ax = xa$ implies $a^\# x = xa^\#$ [4, 7]. To see that $a^\#$ double commutes with a, we assume that $ax = xa$. Then $a^\# x = (a^\#)^2 ax = (a^\#)^2 xa = (a^\#)^2 xa^2 a^\# = (a^\#)^2 axaa^\# = a^\# xaa^\#$. Similarly $xa^\# = a^\# axa^\#$, and $a^\# x = xa^\#$.

We use $\mathcal{R}^\#$ to denote the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

An element $a \in \mathcal{R}$ satisfying $aa^* = a^* a$ is called *normal.* An element $a \in \mathcal{R}$ satisfying $a = a^*$ is called *Hermitian* (or *symmetric*).

We say that $b = a^\dagger$ is the *Moore–Penrose inverse* (or *MP-inverse*) of a, if the following hold [15]:

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [11, 13]). The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

Definition 1.1. An element $a \in \mathcal{R}$ is $*$-cancellable if

$$a^* ax = 0 \Rightarrow ax = 0 \quad \text{and} \quad xaa^* = 0 \Rightarrow xa = 0. \tag{1}$$

Applying the involution to (1), we observe that a is *-cancellable if and only if a^* is *-cancellable. In C^*-algebras all elements are $*$-cancellable.

Theorem 1.1. [14] *Let $a \in \mathcal{R}$. Then $a \in \mathcal{R}^\dagger$ if and only if a is *-cancellable and $a^* a$ is group invertible.*

Theorem 1.2. [7] *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^* a)^\dagger = a^\dagger (a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^* a^\dagger$;

(f) $a^* = a^\dagger aa^* = a^* aa^\dagger$;

(g) $a^\dagger = (a^* a)^\dagger a^* = a^* (aa^*)^\dagger = (a^* a)^\# a^* = a^* (aa^*)^\#$;

(h) $(a^*)^\dagger = a(a^* a)^\dagger = (aa^*)^\dagger a$.

Proof. (a) It is easy to check that a is the Moore–Penrose inverse of $a^\dagger$, by direct computation.

(b) From $(a^\dagger)^*a^*(a^\dagger)^* = (a^\dagger aa^\dagger)^* = (a^\dagger)^*$, $a^*(a^\dagger)^*a^* = (aa^\dagger a)^* = a^*$, $(a^*(a^\dagger)^*)^* = a^\dagger a = (a^\dagger a)^* = a^*(a^\dagger)^*$ and $((a^\dagger)^*a^*)^* = aa^\dagger = (aa^\dagger)^* = (a^\dagger)^*a^*$, we see that $(a^\dagger)^*$ is the Moore–Penrose inverse of a^*.

(c) Since $a^*aa^\dagger(a^\dagger)^*a^*a = a^*aa^\dagger(aa^\dagger)^*a = a^*aa^\dagger aa^\dagger a = a^*a$, $a^\dagger(a^\dagger)^*a^*aa^\dagger(a^\dagger)^* = a^\dagger(aa^\dagger)^*aa^\dagger(a^\dagger)^* = a^\dagger aa^\dagger aa^\dagger(a^\dagger)^* = a^\dagger(a^\dagger)^*$, $(a^*aa^\dagger(a^\dagger)^*)^* = ((a^\dagger)^*)^*(aa^\dagger)^*(a^*)^* = a^\dagger aa^\dagger a = a^\dagger a = (a^\dagger a)^* = a^*(a^\dagger)^* = (aa^\dagger a)^*(a^\dagger)^* = a^*aa^\dagger(a^\dagger)^*$ and $(a^\dagger(a^\dagger)^*a^*a)^* = (a^\dagger aa^\dagger a)^* = (a^\dagger a)^* = a^\dagger a = a^\dagger aa^\dagger a = a^\dagger(a^\dagger)^*a^*a$, we conclude that $a^\dagger(a^\dagger)^*$ is the Moore–Penrose inverse of a^*a.

(d) This part can be proved in a similar way as (c).

(f) The first equality follows from $a^* = (aa^\dagger a)^* = (a^\dagger a)^*a^* = a^\dagger aa^*$. The second equality can be obtain in the same manner.

(g) From the part (c), we get $a^\dagger = a^\dagger aa^\dagger = a^\dagger(aa^\dagger)^* = a^\dagger(a^\dagger)^*a^* = (a^*a)^\dagger a^*$. In the same way the equality $a^\dagger = a^*(aa^*)^\dagger$ follows by (d). The equalities $a^\dagger = (a^*a)^\# a^* = a^*(aa^*)^\#$ are proved in [14].

(h) Applying involution to the first equality $a^\dagger = (a^*a)^\dagger a^*$ in the part (g), we get $(a^\dagger)^* = a((a^*a)^\dagger)^*$. Then, by (b), $(a^*)^\dagger = a((a^*a)^*)^\dagger = a(a^*a)^\dagger$. The second equality follows analogously. □

In this paper we will use the following definition of EP elements [14].

Definition 1.2. An element a of a ring $\mathcal{R}$ with involution is said to be EP if $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ and $a^\# = a^\dagger$. An element a is *generalized EP* (or *gEP* for short) if there exists $k \in \mathbb{N}$ such that a^k is EP.

The following result is well known for matrices, Hilbert space operators and elements of C^*-algebras, and it is equally true in rings with involution:

Lemma 1.1. *Let $a \in \mathcal{R}^\dagger$ and $b \in \mathcal{R}$. If $ab = ba$ and $a^*b = ba^*$, then $a^\dagger b = ba^\dagger$.*

Proof. Suppose that b commutes with a and a^*. Since $a \in \mathcal{R}^\dagger$, we get $a^\dagger = a^*(aa^*)^\dagger = a^*(aa^*)^\#$. Now, aa^* commutes with b. The group inverse $(aa^*)^\#$ double commutes with aa^*, so $(aa^*)^\#$ commutes with b. It follows that $a^\dagger$ commutes with b. □

The next result is also well known for matrices, Hilbert space operators and elements of C^*-algebras, and we prove that it is true in rings with involution:

Lemma 1.2. *Let $a \in \mathcal{R}^\dagger$. Then a is normal if and only if $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$.*

Proof. If a is normal, then, from Lemma 1.1, it follows that $a^\dagger$ commutes with a and with a^*.

Conversely, suppose that $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$. Now, we obtain

$$aa^* = aa^*(aa^\dagger) = a(a^*a^\dagger)a = (aa^\dagger)a^*a = a^\dagger aa^*a = a^*a.$$

Hence, a is normal. $\square$

Notice that the condition $aa^\dagger = a^\dagger a$ generalizes the notion of EP matrices, and the condition $a^*a^\dagger = a^\dagger a^*$ generalizes the notion of star-dagger matrices [12].

From Lemma 1.2, we obtain the following result.

Lemma 1.3. *If $a \in \mathcal{R}^\dagger$ is normal, then a is EP.*

The following result is proved in [14].

Theorem 1.3. *An element $a \in \mathcal{R}$ is EP if and only if a is group invertible and $a^\# a$ is symmetric.*

The paper is organized as follows. In Section 2 characterizations of MP-invertible normal elements in rings with involution are given. In Section 3, MP-invertible Hermitian elements in rings with involution are investigated. Some of these results are proved for complex square matrices in [3], using the rank of a matrix, or in [1], using an elegant representation of square matrices as the main technique. Moreover, the operator analogues of these results are proved in [5] and [6] for linear bounded operators on Hilbert spaces, using the operator matrices as the main tool. In this paper we show that neither the rank (in the finite dimensional case) nor the properties of operator matrices (in the infinite dimensional case) are necessary for the characterization of normal and Hermitian elements.

2 Normal elements

In this section MP-invertible normal elements in rings with involution are characterized by conditions involving their group and Moore–Penrose inverse.

We mention that the following result is a consequence of a direct computation.

Lemma 2.1. *If $a \in \mathcal{R}^\dagger$, then $aa^*a \in \mathcal{R}^\dagger$ and $(aa^*a)^\dagger = a^\dagger(a^*)^\dagger a^\dagger$.*

We start with the following necessary and sufficient conditions for an element a of a ring with involution to be normal.

Theorem 2.1. *Suppose that $a \in \mathcal{R}^\dagger$. The following statements are equivalent:*

(i) *a is normal;*

(ii) *$a(aa^*a)^\dagger = (aa^*a)^\dagger a$;*

(iii) *$a^\dagger(a + a^*) = (a + a^*)a^\dagger$.*

Proof. (i) ⇒ (ii): If a is normal, by Lemma 1.2, we get $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$. Applying involution to the second equality, we get $(a^\dagger)^*a = a(a^\dagger)^*$. Then

$$aa^\dagger(a^\dagger)^*a^\dagger = a^\dagger a(a^\dagger)^*a^\dagger = a^\dagger(a^\dagger)^*aa^\dagger = a^\dagger(a^\dagger)^*a^\dagger a. \tag{2}$$

By Lemma 2.1, we have $(aa^*a)^\dagger = a^\dagger(a^\dagger)^*a^\dagger$. Using this equality in (2), we get $a(aa^*a)^\dagger = (aa^*a)^\dagger a$. So, the condition (ii) holds.

(ii) ⇒ (iii): Let $a(aa^*a)^\dagger = (aa^*a)^\dagger a$. Applying Lemma 2.1, we know that $(aa^*a)^\dagger = a^\dagger(a^\dagger)^*a^\dagger$ holds. Thus, the assumption (ii) is equivalent to

$$aa^\dagger(a^\dagger)^*a^\dagger = a^\dagger(a^\dagger)^*a^\dagger a,$$

which gives, using Theorem 1.2,

$$(a^\dagger)^*a^\dagger = a^\dagger(a^\dagger)^*. \tag{3}$$

Multiplying (3) by a^* from the left and from the right side, we obtain

$$a^\dagger a^* = a^*a^\dagger. \tag{4}$$

Now, by (3) and (4), we have

$$aa^\dagger = a(a^\dagger a)^*a^\dagger = aa^*((a^\dagger)^*a^\dagger) = a(a^*a^\dagger)(a^\dagger)^* = aa^\dagger a^*(a^\dagger)^* = aa^\dagger a^\dagger a,$$

and

$$a^\dagger a = a^\dagger(aa^\dagger)^*a = (a^\dagger(a^\dagger)^*)a^*a = (a^\dagger)^*(a^\dagger a^*)a = (a^\dagger)^*a^*a^\dagger a = aa^\dagger a^\dagger a.$$

Hence,

$$aa^\dagger = a^\dagger a. \tag{5}$$

From (4) and (5), we deduce that the condition (iii) is satisfied.

(iii) $\Rightarrow$ (i): The condition $a^\dagger(a+a^*) = (a+a^*)a^\dagger$ can be written as

$$a^\dagger a + a^\dagger a^* = aa^\dagger + a^* a^\dagger. \tag{6}$$

Multiplying (6) by a from the left and from the right side, we get

$$aa + aa^\dagger a^* a = aa + aa^* a^\dagger a,$$

i.e.

$$aa^\dagger a^* a = aa^* a^\dagger a.$$

Multiplying the previous equality by $a^\dagger$ from the left and from the right side, we obtain, using Theorem 1.2,

$$a^\dagger a^* = a^* a^\dagger. \tag{7}$$

Now, from (6), we obtain

$$a^\dagger a = aa^\dagger. \tag{8}$$

Thus, by (8), (7) and Lemma 1.2, a is normal. $\square$

In the following theorem we again assume that the element a is Moore–Penrose invertible, and study 22 conditions involving $a^\dagger$, $a^\#$ and a^* to ensure that a is normal. The following result is inspired by Theorems 2, 5 and 6 in [1].

Theorem 2.2. *Suppose that $a \in \mathcal{R}^\dagger$. Then a is normal if and only if $a \in \mathcal{R}^\#$ and one of the following equivalent conditions holds:*

(i) $aa^* a^\# = a^\# aa^*$;

(ii) $aa^\# a^* = a^\# a^* a$;

(iii) $a^* aa^\# = a^\# a^* a$;

(iv) $aa^* a^\# = a^* a^\# a$;

(v) $aaa^* = aa^* a$;

(vi) $aa^* a = a^* aa$;

(vii) $a^* a^\# = a^\# a^*$;

(viii) $a^* a^\dagger = a^\# a^*$;

(ix) $a^*a^\# = a^\dagger a^*$;

(x) $aa^*a^\dagger = a^*$;

(xi) $a^\dagger a^*a = a^*$;

(xii) $aa^*a^\# = a^*$;

(xiii) $a^\# a^*a = a^*$;

(xiv) $a^*a^\# a^\# = a^\# a^*a^\#$;

(xv) $a^\# a^*a^\# = a^\# a^\# a^*$;

(xvi) $a^*a^*a^\# = a^*a^\# a^*$;

(xvii) $a^*a^\# a^* = a^\# a^*a^*$;

(xviii) $a^*a^\dagger a^\# = a^\# a^*a^\dagger$;

(xix) $a^*a^\# a^\dagger = a^\dagger a^*a^\#$;

(xx) $a^\dagger a^*a^\# = a^\# a^\dagger a^*$;

(xxi) $a^\dagger a^\# a^* = a^\# a^*a^\dagger$;

(xxii) *There exists some* $x \in \mathcal{R}$ *such that* $ax = a^*$ *and* $(a^\dagger)^*x = a^\dagger$.

Proof. If a is normal, then it commutes with $a^\dagger$ and a^* and $a^\# = a^\dagger$. It is not difficult to verify that conditions (i)-(xxii) hold.

Conversely, we assume that $a \in \mathcal{R}^\#$. To conclude that a is normal, we show that the condition $aa^* = a^*a$ is satisfied, or that the element is subject to one of the preceding already established conditions of this theorem.

(i) Suppose that $aa^*a^\# = a^\# aa^*$. Then we get

$$\begin{aligned} a^\dagger aaa^* &= a^\dagger aa(aa^\dagger a)^* = a^\dagger aaa^*aa^\dagger = a^\dagger aaa^*aa^\# aa^\dagger \\ &= a^\dagger a(aa^*a^\#)aaa^\dagger = a^\dagger aa^\# aa^*aaa^\dagger \\ &= a^\dagger aa^*aaa^\dagger = a^*aaa^\dagger. \end{aligned}$$

Now, from the previous equality and (i), it follows

$$\begin{aligned} a^*aa^\dagger &= a^* = a^\dagger aa^* = a^\dagger a(a^\# aa^*) = (a^\dagger aaa^*)a^\# \\ &= a^*aaa^\dagger a^\# = a^*aaa^\dagger a(a^\#)^2 = a^*aa(a^\#)^2 \\ &= a^*aa^\#, \end{aligned}$$

i.e.

$$a^*a(a^\dagger - a^\#) = 0. \tag{9}$$

Since $a \in \mathcal{R}^\dagger$, a is *-cancellable by Theorem 1.1. Hence from (9) and *-cancellation, we obtain $a(a^\dagger - a^\#) = 0$, i.e. $aa^\dagger = aa^\#$. Hence

$$a^* = (aaa^\#)^* = (aaa^\dagger)^* = aa^\dagger a^* = aa^\# a^*. \tag{10}$$

By (10), (i) and $a^* = a^*aa^\dagger = a^*aa^\# = a^*a^\# a$, it follows

$$a^*a = aa^\# a^*a = (a^\# aa^*)a = a(a^*a^\# a) = aa^*.$$

(ii) Using the assumption $aa^\# a^* = a^\# a^*a$, we have

$$\begin{aligned} aa^*a^\# &= a(aa^\# a^*)a^\# = (aa^\# a^*)aa^\# = a^\# a^*aaa^\# \\ &= (a^\# a^*a) = aa^\# a^* = a^\# aa^*, \end{aligned}$$

i.e. the condition (i) is satisfied.

(iii) Applying the equality $a^*aa^\# = a^\# a^*a$, we obtain

$$\begin{aligned} a^\dagger aaa^* &= a^\dagger aaa^*aa^\dagger = a^\dagger aa(a^*aa^\#)aa^\dagger = a^\dagger aaa^\# a^*aaa^\dagger \\ &= a^\dagger aa^*aaa^\dagger = a^*aaa^\dagger. \end{aligned}$$

Using the previous equality and (iii), we get

$$\begin{aligned} a^\dagger aa^* &= a^* = a^*aa^\dagger = (a^*aa^\#)aa^\dagger = a^\#(a^*aaa^\dagger) \\ &= a^\# a^\dagger aaa^* = (a^\#)^2aa^\dagger aaa^* = (a^\#)^2aaa^* = a^\# aa^*, \end{aligned}$$

i.e.

$$(a^\dagger - a^\#)aa^* = 0. \tag{11}$$

From $a \in \mathcal{R}^\dagger$, by Theorem 1.1, we know that a is *-cancellable. Then, by (11) and *-cancellation, we get $(a^\dagger - a^\#)a = 0$, i.e. $a^\dagger a = a^\# a$. So

$$a^* = (a^\# aa)^* = (a^\dagger aa)^* = a^*a^\dagger a = a^*a^\# a. \tag{12}$$

Thus, from $a^* = a^\# aa^* = aa^\# a^*$, (iii) and (12), we have

$$a^*a = a(a^\# a^*a) = a(a^*aa^\#) = aa^*.$$

(iv) The equality $aa^*a^\# = a^*a^\# a$ gives

$$\begin{aligned} a^*aa^\# &= a^*a^\# a = aa^*a^\# = a^\# a(aa^*a^\#) \\ &= a^\#(aa^*a^\#)a = a^\# a^*a^\# aa = a^\# a^*a. \end{aligned}$$

Hence, the condition (iii) is satisfied.

(v) If $aaa^* = aa^*a$, we get

$$aa^{\#}a^* = a^{\#}a^{\#}(aaa^*) = a^{\#}a^{\#}aa^*a = a^{\#}a^*a.$$

Therefore, the condition (ii) holds.

(vi) Suppose that $aa^*a = a^*aa$, then we have

$$aa^*a^{\#} = (aa^*a)a^{\#}a^{\#} = a^*aaa^{\#}a^{\#} = a^*aa^{\#} = a^*a^{\#}a.$$

So, the equality (iv) holds.

(vii) From the equality $a^*a^{\#} = a^{\#}a^*$, we obtain

$$aa^*a^{\#} = aa^{\#}a^* = a^{\#}aa^*.$$

Then, we deduce that the condition (i) is satisfied.

(viii) The assumption $a^*a^{\dagger} = a^{\#}a^*$ implies

$$(a^{\#})^2aa^* = a^{\#}a^* = a^*a^{\dagger} = a^{\dagger}a(a^*a^{\dagger}) = a^{\dagger}aa^{\#}a^*,$$

i.e.

$$((a^{\#})^2 - a^{\dagger}a^{\#})aa^* = 0. \tag{13}$$

By the condition $a \in \mathcal{R}^{\dagger}$ and Theorem 1.1, we conclude that a is *-cancellable. Using the equality (13) and *-cancellation, we have $((a^{\#})^2 - a^{\dagger}a^{\#})a = 0$, i.e.

$$a^{\#} = a^{\dagger}a^{\#}a. \tag{14}$$

From this equality, we get $a^{\#}a = a^{\dagger}a$ and

$$a^* = (a^{\#}aa)^* = (a^{\dagger}aa)^* = a^*a^{\dagger}a = a^*a^{\#}a. \tag{15}$$

The equalities (14), (viii) and (15) give

$$a^*a^{\#} = (a^*a^{\dagger})a^{\#}a = a^{\#}(a^*a^{\#}a) = a^{\#}a^*.$$

Now condition (viii) is obtained from (vii).

(ix) Assume that $a^*a^{\#} = a^{\dagger}a^*$. Now, it follows

$$a^*a(a^{\#})^2 = a^*a^{\#} = a^{\dagger}a^* = (a^{\dagger}a^*)aa^{\dagger} = a^*a^{\#}aa^{\dagger},$$

i.e.

$$a^*a((a^{\#})^2 - a^{\#}a^{\dagger}) = 0. \tag{16}$$

Since $a \in \mathcal{R}^{\dagger}$, a is *-cancellable by Theorem 1.1. From (16) and *-cancellation, we obtain $a((a^{\#})^2 - a^{\#}a^{\dagger}) = 0$, i.e.

$$a^{\#} = aa^{\#}a^{\dagger}. \tag{17}$$

By (17), we have $aa^{\#} = aa^{\dagger}$ and

$$a^* = (aaa^{\#})^* = (aaa^{\dagger})^* = aa^{\dagger}a^* = aa^{\#}a^*. \tag{18}$$

Then, from (18), (ix) and (17), we get

$$a^*a^{\#} = aa^{\#}(a^*a^{\#}) = (aa^{\#}a^{\dagger})a^* = a^{\#}a^*.$$

Hence, the condition (vii) is satisfied.

(x) Using $aa^*a^{\dagger} = a^*$, we have

$$a^*a^{\dagger} = aa^*a^{\dagger}a^{\dagger} = a^{\#}a(aa^*a^{\dagger})a^{\dagger} = a^{\#}(aa^*a^{\dagger}) = a^{\#}a^*.$$

Thus, the equality (viii) holds.

(xi) By $a^{\dagger}a^*a = a^*$, it follows

$$a^*a^{\#} = a^{\dagger}a^*aa^{\#} = a^{\dagger}a^{\dagger}a^*aaa^{\#} = a^{\dagger}(a^{\dagger}a^*a) = a^{\dagger}a^*.$$

We obtain that the condition (ix) is satisfied.

(xii) The condition $aa^*a^{\#} = a^*$ implies

$$a^*a^{\#} = a^{\dagger}(aa^*a^{\#}) = a^{\dagger}a^*,$$

i.e. the equality (ix) holds.

(xiii) If $a^{\#}a^*a = a^*$, then we get

$$a^*a^{\dagger} = a^{\#}a^*aa^{\dagger} = a^{\#}a^*.$$

The equality (xiii) is obtained from (viii).

(xiv) Applying $a^*a^{\#}a^{\#} = a^{\#}a^*a^{\#}$, we have

$$a^*aa^{\#} = (a^*a^{\#}a^{\#})aa = a^{\#}a^*a^{\#}aa = a^{\#}a^*a.$$

Now, the condition (iii) holds.

(xv) From the equality $a^{\#}a^*a^{\#} = a^{\#}a^{\#}a^*$, we obtain

$$aa^*a^{\#} = aa(a^{\#}a^*a^{\#}) = aaa^{\#}a^{\#}a^* = aa^{\#}a^* = a^{\#}aa^*.$$

Therefore, the equality (i) is satisfied.

(xvi) Suppose that $a^*a^*a^\# = a^*a^\# a^*$. Then we obtain

$$\begin{aligned} a^*a(a^\#)^2a^* &= a^*a^\#a^* = a^*a^*a^\# = (a^*a^*a^\#)aa^\# \\ &= a^*a^\#a^*aa^\# = a^*a(a^\#)^2a^*aa^\#, \end{aligned}$$

i.e.

$$a^*a((a^\#)^2a^* - (a^\#)^2a^*aa^\#) = 0. \tag{19}$$

From $a \in \mathcal{R}^\dagger$, by Theorem 1.1, we deduce that a is *-cancellable. Then, by (19) and *-cancellation, we get $a((a^\#)^2a^* - (a^\#)^2a^*aa^\#) = 0$, i.e. $a^\#a^* = a^\#a^*aa^\#$. The last equality gives

$$\begin{aligned} a^*aa^\dagger &= a^* = a^\dagger aa^* = a^\dagger a^2(a^\#a^*) = a^\dagger a^2a^\#a^*aa^\# \\ &= a^\dagger aa^*aa^\# = a^*aa^\#. \end{aligned}$$

Thus, $a^*a(a^\dagger - a^\#) = 0$ and, using *-cancellation again,

$$aa^\dagger = aa^\#. \tag{20}$$

Multiplying (20) by a from the left side, we obtain

$$aaa^\dagger = aaa^\# = a. \tag{21}$$

Applying the equality (21) and (xvi), we have

$$\begin{aligned} a^* &= (aaa^\dagger)^* = aa^\dagger a^* = aa^\dagger aa^\dagger a^* = aa^\dagger(aa^\dagger)^*a^*aa^\dagger \\ &= aa^\dagger(a^\dagger)^*(a^*a^*a^\#)a^2a^\dagger = aa^\dagger(a^\dagger)^*a^*a^\#a^*(a^2a^\dagger) \\ &= aa^\dagger aa^\dagger a^\#a^*a = aa^\dagger a^\#a^*a = aa^\dagger a(a^\#)^2a^*a \\ &= a(a^\#)^2a^*a = a^\#a^*a. \end{aligned}$$

Therefore, the condition (xiii) is satisfied.

(xvii) Assume that $a^*a^\#a^* = a^\#a^*a^*$. Then, it follows

$$\begin{aligned} a^*(a^\#)^2aa^* &= a^*a^\#a^* = a^\#a^*a^* = a^\#a(a^\#a^*a^*) \\ &= a^\#aa^*a^\#a^* = a^\#aa^*(a^\#)^2aa^*. \end{aligned}$$

So

$$(a^*(a^\#)^2 - a^\#aa^*(a^\#)^2)aa^* = 0. \tag{22}$$

The assumption $a \in \mathcal{R}^\dagger$ and Theorem 1.1 imply that a is *-cancellable. Thus, from (22) and *-cancellation, we have $(a^*(a^\#)^2 - a^\# aa^*(a^\#)^2)a = 0$, i.e. $a^*a^\# = a^\# aa^*a^\#$. Now, by the previous equality, we obtain

$$\begin{aligned} a^\dagger aa^* &= a^* = a^*aa^\dagger = (a^*a^\#)a^2a^\dagger = a^\# aa^*a^\# a^2a^\dagger \\ &= a^\# aa^*aa^\dagger = a^\# aa^*. \end{aligned}$$

Hence, $(a^\dagger - a^\#)aa^* = 0$ and, using *-cancellation,

$$a^\dagger a = a^\# a. \tag{23}$$

Multiplying (23) by a from the right side, we get

$$a^\dagger aa = a^\# aa = a. \tag{24}$$

From (24) and (xvii), we have

$$\begin{aligned} a^* &= (a^\dagger aa)^* = a^*a^\dagger a = a^\dagger aa^*a^\dagger aa^\dagger a = a^\dagger a^2(a^\# a^*a^*)(a^\dagger)^*a^\dagger a \\ &= (a^\dagger a^2)a^*a^\# a^*(a^\dagger)^*a^\dagger a = aa^*a^\# a^\dagger aa^\dagger a \\ &= aa^*a^\# a^\dagger a = aa^*(a^\#)^2aa^\dagger a = aa^*(a^\#)^2a \\ &= aa^*a^\#. \end{aligned}$$

The equality (xii) holds.

(xviii) The equality $a^*a^\dagger a^\# = a^\# a^*a^\dagger$ gives

$$a^*a^\dagger = a^*a^\dagger aa^\dagger = (a^*a^\dagger a^\#)aaa^\dagger = a^\# a^*a^\dagger aaa^\dagger. \tag{25}$$

Using (xviii) and (25), we see that

$$\begin{aligned} a^\# a^*a^\dagger &= a^*a^\dagger a^\# = a^\dagger a(a^*a^\dagger a^\#) = a^\dagger aa^\#(a^*a^\dagger) \\ &= a^\dagger aa^\# a^\# a^*a^\dagger aaa^\dagger = a^\dagger(a^\# a^*a^\dagger aaa^\dagger) \\ &= a^\dagger a^*a^\dagger. \end{aligned} \tag{26}$$

Then, from (26), we get

$$aa^*a^* = a^2a^\# a^*a^* = a^2(a^\# a^*a^\dagger)aa^* = a^2a^\dagger a^*a^\dagger aa^* = aaa^\dagger a^*a^*. \tag{27}$$

Applying the equality (27), we obtain

$$\begin{aligned} a^\# aa^* &= a^\# a^\# aaa^* = a^\# a^\#(aa^*a^*)^* = a^\# a^\#(aaa^\dagger a^*a^*)^* \\ &= a^\# a^\# aaaa^\dagger a^* = aa^\dagger a^* = aa^\dagger a^\dagger aa^* \end{aligned}$$

and consequently

$$(a^{\#} - aa^{\dagger}a^{\dagger})aa^* = 0. \tag{28}$$

The condition $a \in \mathcal{R}^{\dagger}$ implies that a is *-cancellable, by Theorem 1.1. Now, from (28) and *-cancellation, we have $(a^{\#} - aa^{\dagger}a^{\dagger})a = 0$ which gives

$$a^{\#}a = aa^{\dagger}a^{\dagger}a. \tag{29}$$

Thus, by (25) and (29),

$$a^*a^{\dagger} = a^{\#}a^*a^{\dagger}aaa^{\dagger} = a^{\#}(aa^{\dagger}a^{\dagger}aa)^* = a^{\#}(a^{\#}aa)^* = a^{\#}a^*.$$

Therefore, the condition (viii) is satisfied.

(xix) If $a^*a^{\#}a^{\dagger} = a^{\dagger}a^*a^{\#}$, then we get

$$\begin{aligned} a^*a(a^{\#})^2a^{\#} &= a^*(a^{\#})^2aa^{\dagger}aa^{\#} = (a^*a^{\#}a^{\dagger})aa^{\#} = a^{\dagger}a^*a^{\#}aa^{\#} \\ &= a^{\dagger}a^*a^{\#} = a^*a^{\#}a^{\dagger} = a^*a(a^{\#})^2a^{\dagger} \end{aligned}$$

which yields

$$a^*a((a^{\#})^2a^{\#} - (a^{\#})^2a^{\dagger}) = 0. \tag{30}$$

From $a \in \mathcal{R}^{\dagger}$, by Theorem 1.1, we deduce that a is *-cancellable. Now, by (30) and *-cancellation, we obtain $a((a^{\#})^2a^{\#} - (a^{\#})^2a^{\dagger}) = 0$, i.e.

$$a^{\#}a^{\#} = a^{\#}a^{\dagger}. \tag{31}$$

When we use the equality (31), we get

$$a^*aa^{\#} = a^*a^2(a^{\#}a^{\#}) = a^*a^2a^{\#}a^{\dagger} = a^*aa^{\dagger} = a^*. \tag{32}$$

Then, by (xix) and (32), we have

$$\begin{aligned} a^*a^{\#} &= a^*(a^{\#})^2a = a^*(a^{\#})^2aa^{\dagger}a = (a^*a^{\#}a^{\dagger})a \\ &= a^{\dagger}a^*a^{\#}a = a^{\dagger}(a^*aa^{\#}) = a^{\dagger}a^*, \end{aligned}$$

i.e. the condition (ix) holds.

(xx) Since $a^{\dagger}a^*a^{\#} = a^{\#}a^{\dagger}a^*$, we get

$$a^{\dagger}a^* = a^{\dagger}aa^{\dagger}a^* = a^{\dagger}aa(a^{\#}a^{\dagger}a^*) = a^{\dagger}aaa^{\dagger}a^*a^{\#}. \tag{33}$$

By (xx) and (33), we have

$$\begin{aligned}
a^\dagger a^* a^\# &= a^\# a^\dagger a^* = (a^\# a^\dagger a^*) a a^\dagger = (a^\dagger a^*) a^\# a a^\dagger \\
&= a^\dagger a a a^\dagger a^* a^\# a^\# a a^\dagger = (a^\dagger a a a^\dagger a^* a^\#) a^\dagger \\
&= a^\dagger a^* a^\dagger.
\end{aligned}$$

From the previous equality, we obtain

$$a^* a^* a = a^* a^* a^\# a^2 = a^* a (a^\dagger a^* a^\#) a^2 = a^* a a^\dagger a^* a^\dagger a^2 = a^* a^* a^\dagger a a. \tag{34}$$

Using (34), we get

$$\begin{aligned}
a^* a a^\# &= a^* a a a^\# a^\# = (a^* a^* a)^* a^\# a^\# = (a^* a^* a^\dagger a a)^* a^\# a^\# \\
&= a^* a^\dagger a a a a^\# a^\# = a^* a^\dagger a = a^* a a^\dagger a^\dagger a
\end{aligned}$$

which gives

$$a^* a (a^\# - a^\dagger a^\dagger a) = 0. \tag{35}$$

Then a is *-cancellable, by $a \in \mathcal{R}^\dagger$ and Theorem 1.1, and thus, from (35),

$$a a^\# = a a^\dagger a^\dagger a. \tag{36}$$

Applying (36) and (33), we see that

$$a^* a^\# = (a a a^\#)^* a^\# = (a a a^\dagger a^\dagger a)^* a^\# = a^\dagger a a a^\dagger a^* a^\# = a^\dagger a^*.$$

So the condition (ix) is satisfied.

(xxi) Assume that $a^\dagger a^\# a^* = a^\# a^* a^\dagger$. Then we obtain

$$\begin{aligned}
a^\# (a^\#)^2 a a^* &= a^\# a a^\dagger a (a^\#)^2 a^* = a^\# a (a^\dagger a^\# a^*) = a^\# a a^\# a^* a^\dagger \\
&= a^\# a^* a^\dagger = a^\dagger a^\# a^* = a^\dagger (a^\#)^2 a a^*
\end{aligned}$$

such that

$$(a^\# (a^\#)^2 - a^\dagger (a^\#)^2) a a^* = 0. \tag{37}$$

From $a \in \mathcal{R}^\dagger$ and Theorem 1.1, a is *-cancellable and thus, by (37),

$$(a^\# (a^\#)^2 - a^\dagger (a^\#)^2) a = 0,$$

i.e.

$$a^\# a^\# = a^\dagger a^\#. \tag{38}$$

Using (38), we get

$$a^{\#}aa^* = (a^{\#}a^{\#})a^2a^* = a^{\dagger}a^{\#}a^2a^* = a^{\dagger}aa^* = a^*. \tag{39}$$

Hence, by (39) and (xxi), it follows

$$\begin{aligned} a^*a^{\dagger} &= a^{\#}aa^*a^{\dagger} = a(a^{\#}a^*a^{\dagger}) = aa^{\dagger}a^{\#}a^* \\ &= aa^{\dagger}a(a^{\#})^2a^* = a(a^{\#})^2a^* = a^{\#}a^*. \end{aligned}$$

Therefore, the condition (xxi) implies the equality (viii).

(xxii) Suppose that there exists some $x \in \mathcal{R}$ such that $ax = a^*$ and $(a^{\dagger})^*x = a^{\dagger}$. From Theorem 1.2, we have

$$(a^{\dagger})^* = (a^*)^{\dagger} = (aa^*)^{\dagger}a = (a^{\dagger})^*a^{\dagger}a.$$

Then, by $ax = a^*$ and $(a^{\dagger})^*x = a^{\dagger}$, we get

$$a^{\dagger} = (a^{\dagger})^*x = (a^{\dagger})^*a^{\dagger}(ax) = (a^{\dagger})^*a^{\dagger}a^*.$$

Now, this equality implies

$$a^*a^{\dagger} = a^*(a^{\dagger})^*a^{\dagger}a^* = a^{\dagger}aa^{\dagger}a^* = a^{\dagger}a^* \tag{40}$$

and

$$a^{\dagger} = (a^{\dagger})^*(a^{\dagger}a^*) = (a^{\dagger})^*a^*a^{\dagger} = aa^{\dagger}a^{\dagger}. \tag{41}$$

Using (41), we obtain

$$a^{\#}aa^* = a^{\#}aa^{\dagger}aa^* = a^{\#}aaa^{\dagger}a^{\dagger}aa^* = (aa^{\dagger}a^{\dagger})aa^* = a^{\dagger}aa^*, \tag{42}$$

i.e.

$$(a^{\dagger} - a^{\#})aa^* = 0. \tag{43}$$

By $a \in \mathcal{R}^{\dagger}$ and Theorem 1.1, a is *-cancellable and thus, from (43),

$$a^{\dagger}a = a^{\#}a.$$

Hence, by the last equality and (40),

$$a^*a^{\dagger} = a^{\dagger}a^* = a^{\dagger}(a^{\dagger}a)a^* = a^{\dagger}a^{\#}aa^* = (a^{\dagger}a)a^{\#}a^* = a^{\#}aa^{\#}a^* = a^{\#}a^*.$$

The condition (viii) holds. $\square$

Finally, we set some open problems for the characterization of normal elements in rings with involutions. Notice that the following result holds for linear bounded operators on Hilbert spaces.

Conjecture 2.1. *Let $a \in \mathcal{R}^\dagger$. Then a is normal if and only if one of the following conditions hold:*

(i) $a(a^* + a^\dagger) = (a^* + a^\dagger)a$;

(ii) $a \in \mathcal{R}^\#$ *and* $a^*a(aa^*)^\dagger a^*a = aa^*$;

(iii) $a \in \mathcal{R}^\#$ *and* $aa^*(a^*a)^\dagger aa^* = a^*a$;

(iv) *there exists some* $x \in \mathcal{R}$ *such that* $aa^*x = a^*a$ *and* $a^*ax = aa^*$;

(v) $aa^\dagger a^*aaa^\dagger = aa^*$.

3 Hermitian elements

In this section we characterize Hermitian elements in rings with involution which are Moore-Penrose invertible. In the next theorem we present some equivalent conditions for an element a of a ring with involution to be Hermitian.

Theorem 3.1. *Suppose that $a \in \mathcal{R}^\dagger$. Then a is Hermitian if and only if one of the following equivalent conditions holds:*

(i) $aaa^\dagger = a^*$;

(ii) $aa = a^*a$;

(iii) $aa^\dagger = a^*a^\dagger$.

Proof. If a is Hermitian, then it commutes with its Moore–Penrose inverse and $a^* = a$. It is not difficult to verify that conditions (i)-(iii) hold.

Conversely, to conclude that a is Hermitian, we show that the condition $a = a^*$ is satisfied, or that the element is subject to one of the preceding already established conditions of this theorem.

(i) Suppose that $aaa^\dagger = a^*$. Then

$$a = (a^*)^* = (aaa^\dagger)^* = aa^\dagger a^* = aa^\dagger aaa^\dagger = aaa^\dagger = a^*.$$

(ii) From $aa = a^*a$, we get

$$aaa^\dagger = a^*aa^\dagger = a^*.$$

Thus, the condition (i) is satisfied.

(iii) Multiplying $aa^\dagger = a^*a^\dagger$ by a from the right side, we obtain $a = a^*a^\dagger a$. Hence

$$a^* = (a^*a^\dagger a)^* = a^\dagger aa = a^\dagger aa^*a^\dagger a = a^*a^\dagger a = a.$$

□

The following theorem imply that Hermitian element in ring with involution can be characterized by some equalities involving the Moore-Penrose inverse and group inverse.

Theorem 3.2. *Suppose that $a \in \mathcal{R}^\dagger$. Then a is Hermitian if and only if $a \in \mathcal{R}^\#$ and one of the following equivalent conditions holds:*

(i) $aa^\# = a^*a^\dagger$;

(ii) $aa^\# = a^*a^\#$;

(iii) $aa^\# = a^\dagger a^*$;

(iv) $a^\dagger a = a^\# a^*$;

(v) $a^*aa^\# = a$;

(vi) $a^*a^*a^\# = a^*$;

(vii) $a^*a^\dagger a^\dagger = a^\#$;

(viii) $a^*a^\dagger a^\# = a^\dagger$;

(ix) $a^*a^\dagger a^\# = a^\#$;

(x) $a^*a^\# a^\# = a^\#$;

(xi) $a^\# a^*a^\# = a^\dagger$;

(xii) $aa^*a^\dagger = a$.

Proof. If a is Hermitian, then it commutes with its Moore–Penrose inverse and $a^\# = a^\dagger$. It is not difficult to verify that conditions (i)-(xii) hold.

Conversely, we assume that $a \in \mathcal{R}^\#$, and show that a satisfies the equality $a = a^*$ or one of the conditions of Theorem 3.1, or one of the preceding, already established condition of this theorem.

(i) By $aa^{\#} = a^*a^\dagger$, we have

$$aa^\dagger = (aa^{\#})aa^\dagger = a^*a^\dagger aa^\dagger = a^*a^\dagger.$$

So, a satisfies condition (iii) of Theorem 3.1.

(ii) If $aa^{\#} = a^*a^{\#}$, then

$$aa = (aa^{\#})aa = a^*a^{\#}aa = a^*a.$$

The condition (ii) of Theorem 3.1 is satisfied.

(iii) Multiplying $aa^{\#} = a^\dagger a^*$ by a from the left side, we get $a = aa^\dagger a^*$. Now,

$$a^* = (aa^\dagger a^*)^* = aaa^\dagger.$$

Therefore, the condition (i) of Theorem 3.1 holds.

(iv) Applying $a^\dagger a = a^{\#}a^*$, we have

$$a^\dagger aa = a^\dagger aa(a^\dagger a) = a^\dagger aaa^{\#}a^* = a^\dagger aa^* = a^*$$

and

$$a = a(a^\dagger a) = aa^{\#}a^*.$$

Then, by these equalities,

$$a^* = (a^\dagger a)a = a^{\#}a^*a = a^{\#}(aa^{\#}a^*)a = a^{\#}aa = a.$$

(v) The assumption $a^*aa^{\#} = a$ implies

$$aa = a^*aa^{\#}a = a^*a.$$

The condition (ii) of Theorem 3.1 is satisfied.

(vi) Assume that $a^*a^*a^{\#} = a^*$. Now

$$aa^\dagger = (a^\dagger)^*a^* = (a^\dagger)^*a^*a^*a^{\#} = aa^\dagger a^*a^{\#}. \tag{44}$$

Using (44), we get

$$aa^{\#} = (aa^\dagger)aa^{\#} = aa^\dagger a^*a^{\#}aa^{\#} = aa^\dagger a^*a^{\#}. \tag{45}$$

From (44) and (45), we deduce that $aa^{\#} = aa^\dagger$. Since $aa^\dagger$ is symmetric, $aa^{\#}$ is symmetric too. Then, by Theorem 1.3, a is EP and $aa^\dagger = a^\dagger a$. Therefore,

$$a = (aa^{\#})a = (aa^\dagger)a^*(a^{\#}a) = a^\dagger aa^*aa^\dagger = a^*.$$

(vii) The equality $a^*a^\dagger a^\dagger = a^\#$ gives

$$aa^\# = aa^*a^\dagger a^\dagger = a(a^*a^\dagger a^\dagger)aa^\dagger = aa^\# aa^\dagger = aa^\dagger.$$

Now, we conclude that $aa^\#$ is symmetric. Then, from Theorem 1.3, a is EP and $a^\# = a^\dagger$, by definition. Hence, from the previous equality and (vii), we obtain the condition (v):

$$a = a^\# aa = a^*a^\dagger a^\dagger aa = a^*a^\# a^\# aa = a^*aa^\#.$$

(viii) When we use the equality $a^*a^\dagger a^\# = a^\dagger$, we have

$$aa^\# = aa^\dagger aa^\# = aa^*a^\dagger a^\# aa^\# = a(a^*a^\dagger a^\#) = aa^\dagger.$$

So, $aa^\#$ is symmetric and, by Theorem 1.3, a is EP. Thus, by $a^\# = a^\dagger$ and (viii),

$$aa^\# = a^\# a = a^\dagger a = a^*a^\dagger a^\# a = a^*a^\# a^\# a = a^*a^\#.$$

The condition (ii) holds.

(ix) From the equality $a^*a^\dagger a^\# = a^\#$, we obtain

$$aa^\dagger = a^\# aaa^\dagger = a^*a^\dagger a^\# aaa^\dagger = a^*a^\dagger aa^\dagger = a^*a^\dagger.$$

Hence, a satisfies the condition (iii) of Theorem 3.1.

(x) If $a^*a^\# a^\# = a^\#$, then

$$aa^\# = a^\# a = a^*a^\# a^\# a = a^*a^\#.$$

The equality (ii) is satisfied.

(xi) Let $a^\# a^*a^\# = a^\dagger$. Now, we get

$$a^\# a = a^\# aa^\dagger a = a^\# aa^\# a^*a^\# a = (a^\# a^*a^\#)a = a^\dagger a.$$

Since $a^\# a$ is symmetric, then a is EP, by Theorem 1.3. So, $a^\# = a^\dagger$, by definition of EP element. From this equality and (xi), we have

$$a^\dagger a = a^\# a^*a^\# a = a^\# a^*aa^\# = a^\# a^*aa^\dagger = a^\# a^*.$$

Hence condition (iv) is satisfied.

(xii) Using $aa^*a^\dagger = a$, we obtain

$$a = aa^*a^\dagger = (aa^*a^\dagger)aa^\dagger = aaa^\dagger.$$

Then

$$a^\# a = a^\# aaa^\dagger = aa^\dagger,$$

and we deduce that $a^{\#}a$ is symmetric. Thus, a is EP, by Theorem 1.3 and $aa^\dagger = a^\dagger a$. By the last equality and (xii), we get

$$aa^\dagger = a^\dagger a = a^\dagger a a^* a^\dagger = a^* a^\dagger.$$

Thus condition (iii) of Theorem 3.1 holds. □

It should be mentioned that Theorems 3.1 and 3.2 generalize the observation by Baksalary and Trenkler, following their Theorem 6 in [1].

4 Conclusions

In this paper we considered Moore-Penrose invertible elements in rings with involution. Precisely, we characterized MP-invertible normal and Hermitian elements in terms of equations involving their adjoints, Moore-Penorse and group inverse. All of these results are already known for complex matrices, and for closed range linear bounded operators on Hilbert spaces. However, we demonstrated the new technique in proving the results. In the theory of complex matrices various authors used the matrix rank to characterize normal and Hermitian matrices. In the case of linear bounded operators on Hilbert spaces, it seems that the method of operator matrices is very useful. In this paper we applied a purely algebraic technique, involving different characterizations of the Moore-Penrose inverse.

Acknowledgement. We are grateful to the referee for helpful comments concerning the paper.

References

[1] O.M. Baksalary, G. Trenkler, *Characterizations of EP, normal and Hermitian matrices*, Linear and Multilinear Algebra 56 (2008), 299–304.

[2] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.

[3] S. Cheng, Y. Tian, *Two sets of new characterizations for normal and EP matrices*, Linear Algebra Appl. 375 (2003), 181–195.

[4] D. Cvetković, D.S. Djordjević, J.J. Koliha, *Moore–Penrose inverse in rings with involution*, Linear Algebra Appl. 426 (2007), 371–381.

[5] D.S. Djordjević, *Characterization of normal, hyponormal and EP operators*, J. Math. Anal. Appl. 329 (2) (2007), 1181-1190.

[6] D S. Djordjević, J.J. Koliha, *Characterizing hermitian, normal and EP operators*, Filomat 21:1 (2007), 39–54.

[7] D.S. Djordjević, V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, 2008.

[8] M.P. Drazin, *Pseudoinverse in associative rings and semigroups*, Amer. Math. Monthly 65 (1958), 506-514.

[9] L. Elsner, K.D. Ikramov, *Normal matrices: an update*, Linear Algebra Appl. 285 (1998), 291-303.

[10] R. Grone, C.R. Johnson, E.M. Sa, H. Wolkowicz, *Normal matrices*, Linear Algebra Appl. 87 (1987), 213-225.

[11] R.E. Harte, M. Mbekhta, *On generalized inverses in C*-algebras*, Studia Math. 103 (1992), 71–77.

[12] R.E. Hartwig, K. Spindelböck, *Matrices for which A^* and $A^\dagger$ commute*, Linear and Multilinear Algebra 14 (1984), 241–256.

[13] J.J. Koliha, *The Drazin and Moore–Penrose inverse in C*-algebras*, Math. Proc. Royal Irish Acad. 99A (1999), 17–27.

[14] J.J. Koliha, P. Patrício *Elements of rings with equal spectral idempotents*, J. Australian Math. Soc. 72 (2002), 137–152.

[15] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

[16] F. Zhang, *Matrix Theory. Basic results and techniques*, Springer-Verlag, New York, 1999.

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