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ON DIVERGENCE A.E. OF FOURIER EXPANSIONS WITH RESPECT TO NON-DISCRETE LAGUERRE-SOBOLEV INNER PRODUCT

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Abstract

Let us introduce the Sobolev inner product

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)d\mu(x) + \lambda \int_0^\infty f'(x)g'(x)d\mu(x)$$

where $d\mu(x) = x^\alpha e^{-x}dx$ with $\alpha > -1$, and $\lambda > 0$. In this paper we prove the failure of a.e. convergence of Fourier expansion in terms of the orthogonal polynomials with respect to the above Sobolev inner product.

1 Introduction

First we will introduce some notation. We will consider two sets of weighted Lebesgue spaces (see, for instance, [16]): the spaces

$$L^p_{\omega(\alpha)} = \begin{cases} f: \ \|f\|_{L^p_{\omega(\alpha)}} = \left(\int_0^\infty |f(x)e^{-x/2}|^p x^\alpha dx\right)^{\frac{1}{p}} < \infty, & \text{if } 1 \le p < \infty, \\ f: \ \|f\|_{L^p_{\omega(\alpha)}} = \underset{0<x<\infty}{ess\ sup}\ |f(x)e^{-x/2}| < \infty, & \text{if } p = \infty, \end{cases}$$

for $\alpha > -1$, as well as the classical spaces

$$L^p_{u(\alpha)} = \begin{cases} f: \ \|f\|_{L^p_{u(\alpha)}} = \left(\int_0^\infty |f(x)e^{-x/2}x^{\alpha/2}|^p dx\right)^{\frac{1}{p}} < \infty, & \text{if } 1 \le p < \infty, \\ f: \ \|f\|_{L^p_{u(\alpha)}} = \underset{0<x<\infty}{ess\ sup}\ |f(x)e^{-x/2}x^{\alpha/2}| < \infty, & \text{if } p = \infty, \end{cases}$$

for $\alpha > -2/p$ if $1 \le p < \infty$ and $\alpha \ge 0$ if $p = \infty$.

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Now, in the linear space of polynomials $\mathbf{P}$ we define the norms $||\cdot||_{W^{1,p}_{\omega(\alpha)}}$ and $||\cdot||_{W^{1,p}_{u(\alpha)}}$, as follows:

$$||f||^p_{W^{1,p}_{\omega(\alpha)}} = \|f\|^p_{L^p_{\omega(\alpha)}} + \lambda\|f'\|^p_{L^p_{\omega(\alpha)}}, \qquad 1 \le p < \infty,$$

$$||f||_{W^{1,\infty}_{\omega(\alpha)}} = \max\{||f||_{L^\infty_{\omega(\alpha)}}, \lambda||f'||_{L^\infty_{\omega(\alpha)}}\},$$

for $\alpha > -1$ and $\lambda \ge 0$, and

$$||f||^p_{W^{1,p}_{u(\alpha)}} = \|f\|^p_{L^p_{u(\alpha)}} + \lambda\|f'\|^p_{L^p_{u(\alpha)}}, \qquad 1 \le p < \infty,$$

$$||f||_{W^{1,\infty}_{u(\alpha)}} = \max\{||f||_{L^\infty_{u(\alpha)}}, \lambda||f'||_{L^\infty_{u(\alpha)}}\},$$

for $\alpha > -2/p$ if $1 \le p < \infty$ and $\alpha \ge 0$ if $p = \infty$ and $\lambda \ge 0$. Clearly, the spaces $\left(\mathbf{P},\ ||\cdot||_{W^{1,p}_{\omega(\alpha)}}\right)$ and $\left(\mathbf{P},\ ||\cdot||_{W^{1,p}_{u(\alpha)}}\right)$ are normed spaces. Taking into account that the completion of every normed space exists, we define the Sobolev-type normed spaces $\left(W^{1,p}_{\omega(\alpha)},\ ||\cdot||_{W^{1,p}_{\omega(\alpha)}}\right)$ and $\left(W^{1,p}_{u(\alpha)},\ ||\cdot||_{W^{1,p}_{u(\alpha)}}\right)$, as the closure of the spaces $\left(\mathbf{P},\ ||\cdot||_{W^{1,p}_{\omega(\alpha)}}\right)$ and $\left(\mathbf{P},\ ||\cdot||_{W^{1,p}_{u(\alpha)}}\right)$, respectively (see, for instance, [1], [11], [21], [22]). Notice that for $p = 2$ the norms are the same norm, and as a consequence, the topology of the space $\mathbf{P}$ is the same.

Let f and g in $W^{1,2}_{\omega(\alpha)} \equiv W^{1,2}_{u(\alpha)}$. We can introduce the non-discrete Sobolev inner product

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)d\mu(x) + \lambda \int_0^\infty f'(x)g'(x)d\mu(x) \qquad (1)$$

where $d\mu(x) = x^\alpha e^{-x}dx$ with $\alpha > -1$, and $\lambda \ge 0$. If $\lambda = 0$ then we get the classical Laguerre inner product. We denote by $\{S^{(\alpha)}_n(x)\}^\infty_{n=0}$ the sequence of orthogonal polynomials with respect to the inner product (1), normalized by the condition that $S^{(\alpha)}_n$ has the same leading coefficient as the classical Laguerre polynomials $L^{(\alpha)}_n(x) = \frac{(-1)^n}{n!}x^n + \dots$. We call these polynomials the Laguerre-Sobolev polynomials. The Laguerre-Sobolev orthogonal polynomials and their asymptotics have been studied by several authors (see, for instance, [2], [3], [12], [13], [14], [20]). These polynomials constitute a particular case of the so-called coherent pairs of measures, studied in [10] (see also [19]).

For $f \in W^{1,1}_{\omega(\alpha)}$ $\left(f \in W^{1,1}_{u(\alpha)}\right)$, the Fourier expansion in terms of Laguerre-Sobolev polynomials is

$$\sum_{k=0}^\infty \hat{f}(k)S^{(\alpha)}_k(x), \qquad (2)$$

with Fourier coefficients

$$\hat{f}(k) = \left(\|S^{(\alpha)}_k\|_{W^{1,2}_{\omega(\alpha)}}\right)^{-2}\langle f, S^{(\alpha)}_k\rangle = \left(\|S^{(\alpha)}_k\|_{W^{1,2}_{u(\alpha)}}\right)^{-2}\langle f, S^{(\alpha)}_k\rangle, \quad k = 0,\ 1,\ \dots\ . \qquad (3)$$

For $\lambda = 0$, Stempak [23] shows that for each $4 < p \leq \infty$ there exists a function $f \in L^p_{u(\alpha)}$ such that its Fourier expansion (2) diverges a.e. on $(0, \infty)$. Later on, Meaney [18] extends the result for Cesàro means. The results about divergence a.e. of the Fourier expansions associated to systems of orthogonal polynomials on $[-1, 1]$ and Bessel systems have been studied in [17], [9], [24], [4]. Recently, the analysis of the divergence a.e. of the Fourier expansions with respect to discrete Sobolev inner products was done by the author and F. Marcellán (see [5], [6], [7], [8]). Thus it seems to be natural to analyze the divergence a.e. of the Fourier expansions with respect to non-discrete Sobolev inner product as well as to compare it with the Fourier expansions in terms of classical Laguerre orthogonal polynomials.

In such a direction, the main goal of this paper is to prove that there are functions $f \in W^{1,p}_{\omega(\alpha)}$ $(f \in W^{1,p}_{u(\alpha)})$, $\frac{4\alpha+4}{2\alpha+1} < p \leq \infty$ $(4 < p \leq \infty)$, whose Fourier expansions (2) are divergent almost everywhere on $(0, \infty)$ in the norm of $W^{1,\infty}_{\omega(\alpha)}$ $(W^{1,\infty}_{u(\alpha)})$. In order to prove these results, the asymptotic behavior of $S_n^{(\alpha)}$, the topological dual of $W^{1,p}_{\beta}$, and Cantor-Lebesgue theorem are used.

2 The classical Laguerre polynomials

In this section we list some properties of the Laguerre polynomials which we will use in the sequel. Throughout this paper positive constants are denoted by c, $c_1, \ldots$ and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $c_1 u_n \leq v_n \leq c_2 u_n$ for sufficiently large n.

The following estimate for the standard norm of Laguerre polynomials holds (see [25, formula (5.1.1)])

$$\int_0^\infty [L_n^{(\alpha)}(x)]^2 d\mu(x) \cong n^\alpha. \tag{4}$$

They satisfy a structure relation (see [25, formula (5.1.13])

$$L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x), \qquad \alpha > 0, \tag{5}$$

as well as the following relation for the derivatives (see [25, formula (5.1.14]):

$$\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x), \qquad \alpha > -1. \tag{6}$$

The following result was obtained by Markett ([15, Lemma 1])

$$\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+\gamma+1)}}||L_n^{(\alpha+\gamma)}||_{L^p_{u(\alpha)}} \sim \begin{cases} n^{1/p-1/2-\gamma/2} & \text{if } \gamma < 2/p-1/2, \ \ 1 \le p \le 4, \\ n^{\gamma/2-1/3+1/3p} & \text{if } \gamma \le 4/3p-1/3, \ \ 4 < p \le \infty, \\ n^{1/p-1/2-\gamma/2}(\log n)^{1/p} & \text{if } \gamma = 2/p-1/2, \ \ 1 \le p \le 4, \\ n^{\gamma/2-1/p} & \text{if } \gamma > 2/p-1/2, \ \ 1 \le p \le 4, \\ n^{\gamma/2-1/p} & \text{if } \gamma > 4/3p-1/3, \ \ 4 < p \le \infty. \end{cases}$$

In particular, we get (see also [16, Lemma 1]):

Lemma 1. *For $\alpha \ge 0$*

$$||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \sim \begin{cases} n^{(\alpha+1)/p-1/2} & \text{if } 1 \le p < \frac{4\alpha+4}{2\alpha+1}, \\ n^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ n^{\alpha-(\alpha+1)/p} & \text{if } \frac{4\alpha+4}{2\alpha+1} < p < \infty, \end{cases}$$

and

$$||L_n^{(\alpha)}||_{L^p_{u(\alpha)}} \sim \begin{cases} n^{\alpha/2+1/p-1/2} & \text{if } 1 \le p < 4, \\ n^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p = 4, \\ n^{\alpha/2-1/p} & \text{if } 4 < p < \infty. \end{cases}$$

3 The Laguerre-Sobolev orthogonal polynomials

Now let us summarize some properties of Laguerre-Sobolev polynomials we need, cf. [2], [20]. We have the following relation between Laguerre polynomials and Laguerre-Sobolev polynomials

$$L_n^{(\alpha-1)}(x) = S_n^{(\alpha)}(x) - a_{n-1}S_{n-1}^{(\alpha)}(x), \qquad n \ge 1, \tag{7}$$

where

$$a_n = \frac{\|L_n^{(\alpha)}\|^2_{L^2_{\omega(\alpha)}}}{\|S_n^{(\alpha)}\|^2_{W^{1,2}_{\omega(\alpha)}}} = \frac{\|L_n^{(\alpha)}\|^2_{L^2_{u(\alpha)}}}{\|S_n^{(\alpha)}\|^2_{W^{1,2}_{u(\alpha)}}}. \tag{8}$$

Moreover

$$\lim_{n\to\infty} a_n = \frac{1}{\phi((\lambda+2)/2)} = a \in (0,1),$$

where ϕ is the conformal maping of $\mathbf{C}\backslash[-1,1]$ onto the exterior of the unit circle given by

$$\phi(x) = x + \sqrt{x^2-1}, \quad x \in \mathbf{C}\backslash[-1,1],$$

with $\sqrt{x^2-1}>0$ when $x>1$. Using (7) in a recursive way and taking into account (5) we get

$$S_n^{(\alpha)}(x)=\sum_{i=0}^{n} b_i^{(n)} L_{n-i}^{(\alpha-1)}(x)=\sum_{i=0}^{n} b_i^{(n)}\left(L_{n-i}^{(\alpha)}(x)-L_{n-i-1}^{(\alpha)}(x)\right), \qquad n\geq 0, \tag{9}$$

where $b_i^{(n)}=\prod_{j=1}^{i} a_{n-j}$ and $b_0^{(n)}=1$.

Lemma 2. [2, *Lemma* 3.2]. *There exist constants C and r with $C>1$ and $0<r<1$ such that the coefficients $b_i^{(n)}$ in* (9) *satisfy $0<b_i^{(n)}<Cr^i$ for all $n\geq 0$ and $0\leq i\leq n$.*

Now we give the inner strong asymptotics of $S_n^{(\alpha)}$ and its derivative.

Lemma 3. *Let $\alpha>-1$. Uniformly on compact subsets of $(0,\infty)$*

$$\frac{S_n^{(\alpha)}(x)}{n^{(\alpha-1)/2}}=e^{x/2}x^{-(\alpha-1)/2}J_{\alpha-1}(2\sqrt{nx})+O(n^{-1/4}) \tag{10}$$

and

$$\frac{{S'}_n^{(\alpha)}(x)}{(n-1)^{\alpha/2}}=-e^{x/2}x^{-\alpha/2}J_{\alpha}(2\sqrt{(n-1)x})+O(n^{-1/4}), \tag{11}$$

where J_α is the Bessel function of the first kind of order α.

Proof. The relation (10) has been proved in [2, Theorem 3.1]. On the other hand, taking into account (6) and (7) we get

$${S'}_n^{(\alpha)}(x)=L_{n-1}^{(\alpha)}(x)+a_{n-1}{S'}_{n-1}^{(\alpha)}(x), \qquad n\geq 1.$$

Now, the proof of the (11) can be done in a similar way as in [2, Theorem 3.1]. □

The proof of our main result is based on the following:

Lemma 4. *For $\alpha>-1/2$*

$$||{S'}_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}\geq\begin{cases} cn^{(\alpha+1)/p-1/2} & \text{if } 1\leq p<\frac{4\alpha+4}{2\alpha+1},\\ cn^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p=\frac{4\alpha+4}{2\alpha+1},\\ cn^{\alpha-(\alpha+1)/p} & \text{if } \frac{4\alpha+4}{2\alpha+1}<p<\infty,\end{cases} \tag{12}$$

and for $\alpha>-2/p$

$$||{S'}_n^{(\alpha)}||_{L^p_{u(\alpha)}}\geq\begin{cases} cn^{\alpha/2+1/p-1/2} & \text{if } 1\leq p<4,\\ cn^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p=4,\\ cn^{\alpha/2-1/p} & \text{if } 4<p<\infty.\end{cases} \tag{13}$$

Proof. According to (11)

$$\int_{\frac{\pi^2}{4(n-1)}}^{\frac{(n-1)\pi^2}{4}} |{S'}_n^{(\alpha)}(x)e^{-x/2}|^p x^\beta dx$$
$$\geq cn^{p\alpha/2}\int_{\frac{\pi^2}{4(n-1)}}^{\frac{(n-1)\pi^2}{4}} |J_{\alpha+1}(2\sqrt{(n-1)x})|^p x^{\beta-p\alpha/2}dx$$
$$\geq cn^{p\alpha-\beta-1}\int_{\pi}^{(n-1)\pi} t^{2\beta-p\alpha+1}|J_{\alpha+1}(t)|^p dt.$$

On the other hand, from [24, Lemma 2.1], for $\nu > -1$, $\gamma > -1-p\nu$, and $1 \leq p < \infty$ we have

$$\int_{\pi}^{n\pi} t^\gamma |J_\nu(t)|^p dt \sim \begin{cases} c & \text{if } \gamma < p/2-1, \\ c\log n & \text{if } \gamma = p/2-1, \\ cn^{\gamma-p/2+1} & \text{if } \gamma < p/2-1. \end{cases}$$

Therefore, for $\beta = \alpha$ relation (12) follows and for $\beta = p\alpha/2$ we get (13). □

Corollary 1. *For $\alpha > -1/2$*

$$||S_n^{(\alpha)}||_{W^{1,p}_{\omega(\alpha)}} \geq \begin{cases} cn^{(\alpha+1)/p-1/2} & \text{if } 1 \leq p < \frac{4\alpha+4}{2\alpha+1}, \\ cn^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ cn^{\alpha-(\alpha+1)/p} & \text{if } \frac{4\alpha+4}{2\alpha+1} < p < \infty, \end{cases}$$

and for $\alpha > -2/p$

$$||S_n^{(\alpha)}||_{W^{1,p}_{u(\alpha)}} \geq \begin{cases} cn^{\alpha/2+1/p-1/2} & \text{if } 1 \leq p < 4, \\ cn^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p = 4, \\ cn^{\alpha/2-1/p} & \text{if } 4 < p < \infty. \end{cases}$$

Lemma 5. *For $\alpha \geq 0$ and n large enough*

$$||S_n^{(\alpha)}||_{W^{1,p}_{\omega(\alpha)}} \sim \begin{cases} n^{(\alpha+1)/p-1/2} & \text{if } 1 \leq p < \frac{4\alpha+4}{2\alpha+1}, \\ n^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ n^{\alpha-(\alpha+1)/p} & \text{if } \frac{4\alpha+4}{2\alpha+1} < p < \infty, \end{cases}$$

and

$$||S_n^{(\alpha)}||_{W^{1,p}_{u(\alpha)}} \sim \begin{cases} n^{\alpha/2+1/p-1/2} & \text{if } 1 \leq p < 4, \\ n^{\alpha/2-1/4}(\log n)^{1/p} & \text{if } p = 4, \\ n^{\alpha/2-1/p} & \text{if } 4 < p < \infty. \end{cases}$$

Proof. Here we will only analyze the upper estimate for $||S_n^{(\alpha)}||_{W^{1,p}_{\omega(\alpha)}}$. The proof of the upper estimate for $||S_n^{(\alpha)}||_{W^{1,p}_{u(\alpha)}}$ can be done in a similar way. Using (9) and Minkowski's inequality we have

$$||S_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \leq \sum_{i=0}^{n} b_{n-i}^{(n)} ||L_i^{(\alpha-1)}||_{L^p_{\omega(\alpha)}} \leq \sum_{i=0}^{n} b_{n-i}^{(n)} ||L_i^{(\alpha)}||_{L^p_{\omega(\alpha)}} + \sum_{i=0}^{n} b_{n-i}^{(n)} ||L_{i-1}^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

It is easy to prove that, for $\alpha \geq 1/2$, $1 \leq p < \infty$ and $i = 0, 1, ..., n$, by Lemma 1

$$||L_i^{(\alpha)}||_{L^p_{\omega(\alpha)}} \leq ||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

On the other hand, from Lemma 3

$$\sum_{i=0}^{n} b_{n-i}^{(n)} = \sum_{i=0}^{n} b_i^{(n)} < C \sum_{i=0}^{n} r^i = C \frac{1 - r^{n+1}}{1 - r} \to \frac{C}{1 - r}.$$

Thus, for $\alpha \geq 1/2$ and $1 \leq p < \infty$

$$||S_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \leq c ||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

In a similar way we can prove the case when either $0 < \alpha < 1/2$ and $p \geq \frac{\alpha+1}{\alpha}$ or $p \leq 2(\alpha+1)$. Now let $0 \leq \alpha < 1/2$ and $2(\alpha+1) < p < \frac{\alpha+1}{\alpha}$. Since $(1-\delta)n - n/2 \geq 1$, for $0 < \delta < 1/2 - 1/n$ and n large enough, then there exists $n_o \in \mathbf{N}$ such that $n/2 \leq n_0 \leq (1-\delta)n$. Again, from Lemma 1 and Lemma 2

$$\sum_{i=0}^{n} b_{n-i}^{(n)} ||L_i^{(\alpha)}||_{L^p_{\omega(\alpha)}} = \sum_{i=0}^{n_0} b_{n-i}^{(n)} ||L_i^{(\alpha)}||_{L^p_{\omega(\alpha)}} + \sum_{n=n_0+1}^{n} b_{n-i}^{(n)} ||L_i^{(\alpha)}||_{L^p_{\omega(\alpha)}} \leq c ||L_0^{(\alpha)}||_{L^p_{\omega(\alpha)}} \sum_{i=0}^{n_0} b_{n-i}^{(n)} + c ||L_{n_0+1}^{(\alpha)}||_{L^p_{\omega(\alpha)}} \sum_{i=n_0+1}^{n} b_{n-i}^{(n)} \leq c \sum_{i=0}^{n_0} r^{n-i} + ||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \sum_{i=n_0+1}^{n} r^{n-i}.$$

On the other hand

$$\sum_{i=0}^{n_0} r^{n-i} = r^{n-n_0} \frac{1 - r^{n_0+1}}{1 - r} \leq c r^{\delta n} \leq c ||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}, \quad as \quad n \to \infty,$$

and

$$\sum_{i=n_0+1}^{n} r^{n-i} = \frac{1-r^{n-n_0}}{1-r} \to \frac{1}{1-r}.$$

Thus

$$\sum_{i=0}^{n} b_{n-i}^{(n)} ||L_i^{(\alpha)}||_{L^p_{\omega(\alpha)}} \le c||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

In a similar way

$$\sum_{i=0}^{n} b_{n-i}^{(n)} ||L_{i-1}^{(\alpha)}||_{L^p_{\omega(\alpha)}} \le c||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

Thus, for $0 \le \alpha < 1/2$ and $2(\alpha-1) < p < \frac{\alpha+1}{\alpha}$

$$||S_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \le c||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

As a conclusion, for $\alpha \ge 0$ and $1 \le p < \infty$

$$||S_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \le c||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

On the other hand, from (6), (9) and Minkowski's inequality

$$||{S'}_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \le \sum_{i=0}^{n} b_{n-i}^{(n)} ||L_{i-1}^{(\alpha)}||_{L^p_{\omega(\alpha)}}.$$

As above we can show that

$$||{S'}_n^{(\alpha)}||_{L^p_{\omega(\alpha)}} \le c||L_n^{(\alpha)}||_{L^p_{\omega(\alpha)}},$$

where $\alpha \ge 0$ and $1 \le p < \infty$.

The proof of lemma is completed. □

4 Divergence almost everywhere

Let $S_n f$ be the n-th partial sum of the expansion (2). If $||S_n f||_{W^{1,\infty}_{\omega(\alpha)}}$ $\left(||S_n f||_{W^{1,\infty}_{u(\alpha)}}\right)$ is bounded then

$$||\hat{f}(n) S_n^{(\alpha)}||_{W^{1,\infty}_{\omega(\alpha)}} < c \quad \left(||\hat{f}(n) S_n^{(\alpha)}||_{W^{1,\infty}_{u(\alpha)}} < c\right), \quad n \in \mathbf{N}.$$

Therefore

$$|\hat{f}(n) {S'}_n^{(\alpha)}(x) e^{-x/2}| < c \quad \left(|\hat{f}(n) {S'}_n^{(\alpha)}(x) e^{-x/2} x^{\alpha/2}| < c\right), \quad n \in \mathbf{N},$$

a.e on $(0,\infty)$. From Egorov's Theorem it follows that there is a subset $E \subset (0,\infty)$ of positive measure such that

$$|\hat{f}(n) {S'}_n^{(\alpha)}(x) e^{-x/2}| < c \quad \left(|\hat{f}(n) {S'}_n^{(\alpha)}(x) e^{-x/2} x^{\alpha/2}| < c\right)$$

uniformly for $x \in E$.

On the other hand, from (11) and using a strong asymptotics form of Bessel function $J_\alpha(x)$ (see, for instance, [24, formula (2.1)]) we obtain

$$S'^{(\alpha)}_n(x) = \frac{e^{x/2}(n-1)^{\alpha/2-1/4}}{\pi^{1/2}x^{\alpha/2+1/4}}\cos(2\sqrt{(n-1)x}-\alpha\pi/2-\pi/4)+O(n^{\alpha/2-3/4}),$$

for $x \in [\epsilon, \omega]$.

Thus

$$|n^{\alpha/2-1/4}\hat{f}(n)\left(c\cos(2\sqrt{(n-1)x}-\alpha\pi/2-\pi/4)+O(n^{-1/2})\right)| < c$$

uniformly for $x \in E$. Using the Cantor-Lebesgue Theorem, as described in [18, Subsection 1.5] (see also [26, p.316]), we obtain

$$n^{\alpha/2-1/4}|\hat{f}(n)| < c. \tag{14}$$

Theorem 1. *For $\alpha > -1/2$ and $\frac{4\alpha+4}{2\alpha+1} < q \le \infty$ there is an $f \in W^{1,q}_{\omega(\alpha)}$ whose Fourier expansion* (2) *diverges a.e. on* $(0,\infty)$ *in the norm of* $W^{1,\infty}_{\omega(\alpha)}$.

Proof. From (3), (4) and (8)

$$n^{\alpha/2-1/4}|\hat{f}(n)| \ge cn^{-\alpha/2-1/4}|\langle f, S^{(\alpha)}_n\rangle|.$$

Consider the linear functionals

$$L_n(f) = n^{-\alpha/2-1/4}\langle f, S^{(\alpha)}_n\rangle$$

on $W^{1,q}_{\omega(\alpha)}$, $\frac{4\alpha+4}{2\alpha+1} < q \le \infty$. By using [1, Theorem 3.8] we have

$$||L_n|| = n^{-\alpha/2-1/4}||S^{(\alpha)}_n||_{W^{1,p}_{\omega(\alpha)}}, \quad 1 \le p < \frac{4\alpha+4}{2\alpha+3}.$$

Thus, from Corollary 1

$$\sup_n ||L_n|| = \infty.$$

Then, the Banach-Steinhaus theorem shows that there exists $f \in W^{1,q}_{\omega(\alpha)}$, $\frac{4\alpha+4}{2\alpha+1} < q \le \infty$, such that

$$\sup_n |L_n(f)| = \infty,$$

hence, also

$$\sup_n n^{\alpha/2-1/4}|\hat{f}(n)| = \infty.$$

Since this result contradicts (14) then for this f the Fourier expansion (2) diverges almost everywhere on $(0,\infty)$ in the norm of $W^{1,\infty}_{\omega(\alpha)}$. □

In a similar way we can prove the following theorem.

Theorem 2. *For $\alpha > -1/2$ and $4 < q \leq \infty$ there is an $f \in W^{1,q}_{u(\alpha)}$ whose Fourier expansion* (2) *diverges a.e. on* $(0, \infty)$ *in the norm of* $W^{1,\infty}_{u(\alpha)}$.

Proof. From (3), (4) and (8)

$$n^{\alpha/2-1/4}|\hat{f}(n)| \geq cn^{-\alpha/2-1/4}|\langle f, S_n^{(\alpha)}\rangle|.$$

Consider the linear functionals

$$T_n(f) = n^{-\alpha/2-1/4}\langle f, S_n^{(\alpha)}\rangle$$

on $W^{1,q}_{u(\alpha)}$, $4 < q \leq \infty$. Again, by using [1, Theorem 3.8] we have

$$||T_n|| = n^{-\alpha/2-1/4}||S_n^{(\alpha)}||_{W^{1,p}_{u(\alpha)}}, \quad 1 \leq p < \frac{4}{3}.$$

Thus, from Corollary 1

$$\sup_n ||T_n|| = \infty.$$

Then, the Banach-Steinhaus theorem shows that there exists $f \in W^{1,q}_{u(\alpha)}$, $4 < q \leq \infty$, such that

$$\sup_n |T_n(f)| = \infty,$$

hence, also

$$\sup_n n^{\alpha/2-1/4}|\hat{f}(n)| = \infty.$$

Since this result contradicts (14) then for this f the Fourier expansion (2) diverges almost everywhere on $(0, \infty)$ in the norm of $W^{1,\infty}_{u(\alpha)}$. □

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, (1975).

[2] M. Alfaro, J. J. Moreno-Balcázar, and M. L. Rezola, *Laguerre-Sobolev orthogonal polynomials: asymptotics for coherent pairs of type II*, J. Approx. Theory **122** (2003), 79-96.

[3] J. Brenner, *Über eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen*, In G. Alexits and S. B. Stechkin, editors, "Constructive Theory of Functions", Akadémiai Kiadó Budapest, (1972), 77-83.

[4] B. Xh. Fejzullahu, *Divergent Cesàro means of Fourier expansions with respect to polynomials associated with the measure* $(1-x)^\alpha(1+x)^\beta + M\Delta_{-1}$, Filomat **21:2** (2007), 153-160.

[5] B. Xh. Fejzullahu, *Divergent Cesàro means of Jacobi-Sobolev expansions*, Rev. Mat. Complut. **21** (2008), 427-433.

[6] B. Xh. Fejzullahu, *Divergent Legendre-Sobolev polynomial series*, Novi Sad J. Math. **38** (2008), 35-41.

[7] B. Xh. Fejzullahu and F. Marcellán, *On Convergence and Divergence of Fourier expansions with respect to some Gegenbauer-Sobolev type inner product*, submitted.

[8] B. Xh. Fejzullahu, *On Divergence of Fourier expansions with respect to some Laguerre-Sobolev type inner product*, in "International Workshop on Orthogonal Polynomials and Approximation Theory," Conference in honor of Guillermo López Lagomasino's 60th birthday, Madrid, (2008); http://turan.uc3m.es/uc3m/dpto/MATEM/iwopa08/abstracts_iwopa.pdf

[9] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal system*, Proc. Amer. Math. Soc. **116** (1992), 457-464.

[10] A. Iserles, P. E. Koch, S. P. Nϕrsett, and J. M. Sanz-Serna, *On polynomials orthogonal with respect to certain Sobolev inner products*, J. Approx. Theory **65** (1991), 151-175.

[11] A. Kufner, *Weighted Sobolev Spaces*, John Wiley and Sons, Wiley-Interscience, New York, (1985).

[12] F. Marcellán, T. E. Pérez, and M. A. Pinar, *Laguerre-Sobolev orthogonal polynomials*, J. Comput. Appl. Math. **71** (1996), 245-265.

[13] F. Marcellán, H. G. Meijer, T. E. Pérez, and M. A. Pinar, *An asymptotic result for Laguerre-Sobolev orthogonal polynomials*, J. Comput. Appl. Math. **87** (1997), 87-94.

[14] F. Marcellán and J. J. Moreno-Balcázar, *Strong and Plancherel-Rotach asymptotics of nondiagonal Laguerre-Sobolev orthogonal polynomials*, J. Approx. Theory **110** (2001), 54-73.

[15] C. Markett, *Mean Cesàro summability of Laguerre expansions and norm estimates with shifted parameter*, Anal. Math. **8** (1982), 19-37.

[16] C. Markett, *Cohen type inequalities for Jacobi, Laguerre and Hermite expansions*, SIAM J. Math. Anal. **14** (1983), 819-833.

[17] Ch. Meaney, *Divergent Jacobi polynomial series*, Proc. Amer. Math. Soc. **87** (1983), 459-462.

[18] Ch. Meaney, *Divergent Cesàro and Riesz means of Jacobi and Laguerre expansions*, Proc. Amer. Math. Soc. **131** (2003), 3123-3128.

[19] H. G. Meijer, *Determination of all coherent pairs*, J. Approx. Theory **89** (1997), 321-343.

[20] H. G. Meijer, T. E. Pérez, and M. A. Pinar, *Asymptotics of Sobolev Orthogonal Polynomials for Coherent Pairs of Laguerre Type*, J. Math. Anal. Appl. **245** (2000), 528-546.

[21] J. M. Rodrígues, V. Alvarez, E. Romero, and D. Pestana, *Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials I*, Acta Applicandae Mathematicae **80** (2004), 273-308.

[22] J. M. Rodrígues, V. Alvarez, E. Romero, and D. Pestana, *Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials II*, Approximation Theory and its Applications **18** (2002), 1-32.

[23] K. Stempak, *Divergent Laguerre series*, Proc. Amer. Math. Soc. **129** (2000), 1123-1126.

[24] K. Stempak, *On convergence and divergence of Fourier-Bessel series*, Electron. Trans. Numer. Anal. **14** (2002), 223-235.

[25] G. Szegő, *Orthogonal polynomials,* Amer. Math. Soc. Colloq. Pub. 23, Amer. Math. Soc., Providence, RI, Fourth Edition, (1975).

[26] A. Zygmund, *Trigonometric series: Vols. I, II*, Cambridge University Press, London (1968).

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