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ON SEMIGROUP INVERSES

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Abstract

Schmoeger's characterization of "group invertible" algebra elements is deconstructed, and the structure of their "semigroup inverses" explored.

Suppose A is a semigroup, with identity 1 and invertible group A^{-1} (more generally, a category): then $a \in A$ is said to be *regular* if

0.1 $$a \in aAa\ ,$$

so that it has *generalized inverses*, $b \in A$ for which [2],[3],[6],[7]

0.2 $$a = aba\ .$$

As is familiar, if (0.2) holds then ab and ba are idempotents; also if (0.2) holds with $b = c$ then it also holds with $b = cac$, the passage from c to cac does not alter the idempotents ac and ca, and in addition if $b = cac$ then also

0.3 $$b = bab\ .$$

Under this normalising process the relationship between a and b therefore becomes symmetric; alternatively it is possible that

0.4 $$a \in aA^{-1}a$$

is *decomposably regular*, in the sense of having an invertible generalized inverse $b \in A^{-1}$. When the semigroup A comes from a ring, then one condition which guarantees (0.4) is that $a \in A$ have a commuting generalized inverse, $b \in A$ satisfying (0.2) together with

0.5 $$ba = ab\ .$$

It turns out that $b \in A$ satisfying (0.2), (0.3) and (0.5) is uniquely determined, and double commutes with $a \in A$:

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1. Lemma *If A is a ring, and $a, b, c, d \in A$ satisfy*

1.1 $$a = aba = aca \text{ with } ab = ca$$

then

1.2 $$da = ad \Longrightarrow cad = cadab = dab \text{ and } (cab)d = d(cab)\ .$$

There is implication

1.3 $$a = ab'a = ac'a\ ,\ ab' = c'a \implies c'ab' = cab\ .$$

Proof. Observe, for the first part of (1.2),

$$cad - cadab = cad(1 - ab) = cda(1 - ab) = 0\ ,$$

and similarly $cadab = dab$. Also for the second part

$$cabd = c^2ad = cdab = cadb = dab^2 = dcab\ .$$

Finally, for (1.3),

$$c'a = ab = ca\ ,\ ab' = ca = ab \bullet$$

The unique $b = a^\times \in A$ satisfying (0.2), (0.3) and (0.5) is known as the *group inverse* of $a \in A$, since $\{a, b\} \subseteq A$ evidently generates a group. In this note we discuss what we shall call *semigroup inverses*:

2. Definition *If A is a ring, with identity 1 and invertible group A^{-1}, then a semigroup inverse for $a \in A$ is an element $b \in A$ for which*

2.1 $$a = aba \text{ and } ab + ba - 1 \in A^{-1}\ .$$

Elements of the Calkin algebra with semigroup inverses were christened "generalized Fredholm" by Caradus [2],[3], and the idea extended to more general algebras by Schmoeger [6],[7]. Various authors [1],[4],[9] have looked at conditions under which a difference $e - f$ of projections $e = e^2$ and $f = f^2$ is invertible, and Kato [8] has noticed that anyway $(e - f)^2$ commutes with both e and f. When $e - f \in A^{-1}$ is invertible then $g = e(e - f)^{-2}f$ is known as *Kovarik's formula* [10] for the "poor man's path" from e to f, a projection $g = g^2$ for which

2.2 $$eg = g = gf\ ;\ ge = e\ ;\ fg = f\ ,$$

and Chen, Du and Feng [4] have shown that also $g = e(e - f)^{-1}f$.

It is rather easy to see that commuting generalized inverses must be semigroup inverses: for if $b \in A$ satisfies (0.2) and (0.5) then

2.3 $$(ab + ba - 1)^2 = 1\ .$$

Conversely it has been shown by Schmoeger that if $a \in A$ has semigroup inverses then it also has a group inverse. Schmoeger establishes this as part of a more complex result ([12], I Theorem 3.3) involving "ascent" and "descent"; it is the purpose of this note to deconstruct the argument, and look for some structure among semigroup inverses. We shall write

2.4 $$S(a) \equiv S_A(A) = \{b \in A : a = aba\ ,\ ab + ba - 1 \in A^{-1}\} \subseteq A$$

for the (possibly empty) set of semigroup inverses of $a \in A$, and

2.5 $$S(A) = \{a \in A : S(a) \neq \emptyset\} = \bigcup_{a \in A} S(a)$$

for the set of those elements which have semigroup inverses. We begin by introducing two operations on the set $S(a)$:

3. Definition *If $a \in A$ and $b, c \in S(a)$ write*

3.1 $$a_b = (ab + ba - 1)^{-1} \in A^{-1}$$

and

3.2 $$b *_a c = a_b baca_c \in A\ .$$

The set $S(a)$ is, rather dramatically, closed under multiplication; we begin by noticing how a_b "intertwines" ab and ba:

4. Theorem *If $a \in A$ and $b \in S(a)$ then*

4.1 $$a^2 ba_b = a = a_b ba^2$$

and

4.2 $$a_b ab = baa_b\ ;\ a_b ba = aba_b\ ,$$

and hence also

4.3 $$aba_b a = a = aa_b ba\ .$$

If also $c \in S(a)$ then

4.4 $$(ac)a_c = a_b(ba)\ .$$

Proof. Towards (4.1) and (4.2), observe

$$a^2b = a(ab + ba - 1); (ab + ba - 1)a = ba^2$$

and

$$(ab+ba-1)ab = ba^2b = ba(ab+ba-1)\ ;\ ab(ab+ba-1) = ab^2a = (ab+ba-1)ba\ ;$$

now (4.3) follows. Finally, for (4.4),

$$(ab + ba - 1)ac = ba^2c = ba(ac + ca - 1) \bullet$$

Part of the argument for Theorem 4 is contained in Lemma 1. We can now see where Schmoeger's group inverse comes from:

5. Theorem *If* $\{b, c\} \subseteq S(a)$ *then*

5.1 $$a(b *_a c)a = a \ ,$$

5.2 $$(b *_a c)a(b *_a c) = b *_a c$$

and

5.3 $$(b *_a c)a = a(b *_a c) \ .$$

Thus if $S(a) \neq \emptyset$

5.4 $$S(a) *_a S(a) = \{a^\times\} \subseteq S(a) \ .$$

Proof. Towards (5.1), if $b, c \in S(a)$ then by (4.3)

$$a(b *_a c)a = aa_b baca_c a = aca_c a = a \ .$$

For (5.2) we argue

$$(b *_a c)a(b *_a c) = a_b b(aca_c a)a_b baca_c = a_b b(aa_b ba)ca_c = a_b baca_c = b *_a c \ .$$

Finally, for (5.3),

5.5 $$a(b *_a c) = (aa_b ba)ca_c = (ac)a_c$$

and

5.6 $$(b *_a c)a = a_b b(aca_c) = a_b(ab) \ ;$$

Now recall (4.4) •

Theorem 5 says among other things that $S(A) = A^g$, where (cf Schmoeger) A^g is the set of group invertible elements of A. Elements with group inverses are always decomposably invertible, and decomposably invertible elements which have either left or right inverses are always invertible:

6. Theorem *If* A *is a ring then*

6.1 $$S(A) \cap \left(A^{-1}_{left} \cup A^{-1}_{right}\right) = A^{-1} \ .$$

If $a \in A$ *and* $b \in S(a)$ *then*

6.2 $$a + 1 - ab \in A^{-1}_{left} \ ; \ a + 1 - ba \in A^{-1}_{right} \ .$$

Proof. If $a'a = 1$ and $b \in S(a)$ then

$$1 = a'a = a'aba = ba$$

and hence

$$ab = ab + ba - 1 \in A^{-1} \ ,$$

so that a is also right invertible. This, with the corresponding argument for right invertibles, gives (6.1). For the first part of (6.2) note

$$(-bab+1-ab)(a+1-ab) = -baba+(1-ab)a-bab(1-ab)+(1-ab)^2 = -ba+1-ab \in A^{-1} \ ;$$

the second part is similar •

Theorem 6 is in some sense dual to Schmoeger's result ([13] Proposition 2) that $ab \in A^{-1} \Longrightarrow ba \in S(A)$:

6.3 $$ab = c^{-1} \in A^{-1} \Longrightarrow ba(bc^2a) = (bc^2a)ba \ , \ ba(bc^2a)ba = bcaba = ba \ .$$

When $ab = ba$ then of course (6.2) says that $a + (1 - ab)$ is invertible. From (6.1) it also follows that if $a \in S(A)$ then it is "left-right consistent" in the sense [5] that for arbitrary $c \in A$ there is implication

6.4 $$ac \in A^{-1} \Longleftrightarrow ca \in A^{-1} \ .$$

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