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Functional Analysis, Approximation and Computation **1:1** (2009), 19–29

SPECTRAL PROPERTIES OF LINEAR OPERATOR THROUGH INVARIANT SUBSPACES

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Abstract

In this note we will give conditions for invertibility of mapping T between two normed spaces using properties of its restrictions to invariant subspaces and mappings induced by T over quotient subspaces.

1 Introduction

Given normed spaces X and Y, let $\mathcal{L}(X,Y)$ denote the algebra of all linear transformations from X into Y, and if X is a Banach space, then let $\mathcal{B}(X)$ denote the space of all bounded linear transformations (equivalently, operators) from X to X. For $T \in \mathcal{L}(X,Y)$, let $N(T)$ and $R(T)$ denote, respectively, the null space and the range of the mapping T. Let $n(T)$ and $d(T)$ denote, respectively, the dimension of $N(T)$ and the the codimension of $R(T)$. If the range $R(T)$ of $T \in \mathcal{B}(X)$ is closed and $n(T) < \infty$ (resp. $d(T) < \infty$), then T is said to be an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm* operator, and then the *index* of T is defined by $ind(T) = n(T) - d(T)$. If both $n(T)$ and $d(T)$ are finite, then T is a *Fredholm* operator. The essential (Fredholm) spectrum $\sigma_e(T)$ is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}.$$

We say that $T \in \mathcal{B}(X)$ has *the single valued extension property*, (SVEP), at $\lambda \in \mathbb{C}$ if for every open neighborhood U of λ, the only solution of the equation $(T-\mu)f(\mu) = 0$ that is analytic on U is the constant function $f \equiv 0$. Let $S(T)$ be the set of all λ on which T does not have SVEP.

In this paper, we start by considering the invertibility of a linear mapping, respectively operator, T by considering the restriction $T_{|E}$ of T to an invariant subspace E and the mapping $T_{|X/E}$ determined by T on the quotient space X/E of this invariant subspace. (We refer the reader to the recent publications by B.A.

2000 *Mathematics Subject Classifications.* Primary 47A10, 47A15. Secondary 47A05, 15A29.
Key words and Phrases. Invariant subspace, spectrum of operator.
Received: April 15, 2009
Communicated by Dragan S. Djordjević

Barnes, [2] and [3], for some pioneering work in this direction.) It is seen that for an operator $T \in \mathcal{B}(X)$: (a) if $T_{|E}$ has closed range,then T is invertible if and only if $T_{|E}$ is bounded below, $T_{|X/E}$ is onto and $N(T_{|X/E})$ is isomorphic to $E/R(T_{|E})$; (b) $\sigma(T_{|E}) \cup \sigma(T_{|X/E}) = \sigma(T) \cup \{S(T_{|E})^* \cap S(T_{|X/E})\} = \sigma(T) \cup \{\sigma(T_{|E}) \cap \sigma(T_{|X/E})\}$. It is known that for an operator $T \in \mathcal{B}(X)$, if any two of T, $T_{|E}$ and $T_{|X/E}$ are Fredholm, then so is the third one: we prove that the Fredholm spectrum $\sigma_e(T)$ of T, $T_{|E}$ and $T_{|X/E}$ satisfy the equality $\sigma_e(T_{|E}) \cup \sigma_e(T_{|X/E}) \cup \{S(T_{|E})^* \cap S(T_{|X/E})\} = \sigma_e(T) \cup \{S(T_{|E})^* \cap S(T_{|X/E})\}$. The Browder spectrum $\sigma_b(.)$ satisfies a more satisfactory property: we prove that $\sigma_b(T_{|E}) \cup \sigma_b(T_{|X/E}) = \sigma_b(T) \cup \{S(T_{|E})^* \cap S(T_{|X/E})\}$. The relationship between the Weyl spectra of T, $T_{|E}$ and $T_{|X/E}$ is a bit more delicate: it is proved that $\sigma_w(T_{|E}) \cup \sigma_w(T_{|X/E}) \subseteq \sigma_b(T) \cup \{S(T^*_{|E}) \cap S(T_{|X/E})\} \subseteq \sigma_w(T) \cup \{S_e(P) \cup S(Q)\}$, where either $P = T_{|E}$ and $Q = T_{|X/E}$ or $P = T^*_{|X/E}$ and $Q = T^*_{|E}$, and $S_e(P) = \{\lambda \notin \sigma_e(T) : P$ does not have SVEP at $\lambda\}$. This implies that if $S_e(P) \cup S(Q) = \emptyset$, then T satisfies Browder's theorem.

2 Spectrum of an operator using invariant subspaces

An operator $T \in \mathcal{B}(X)$, X a Banach space, is said to be invertible if there exists a bounded linear operator $S \in \mathcal{B}(X)$ such that $TS = ST = I$. In this case, it is clear that T is one-one and onto. In pure set theory, the converse is also true: for every linear operator T such that T is one-one and onto, the inverse exists in the sense that there is a linear transformation S such that $ST = TS = I$. Here, it is not obvious that such an S must be bounded. For this, we need something more: T be bounded below (i.e., there exists $\epsilon > 0$ such that $\|Tx\| \geq \epsilon\|x\|$ for every $x \in X$). Indeed, for invertibility of bounded linear operators, a necessary and sufficient condition is that T is bounded below and has dense range (see, for example, [7, pg. 57] or [6, pg. 26]).

In the following, we shall use $Inv(T)$ to denote the set of closed (in X) invariant subspaces of T . For $T \in \mathcal{B}(X)$ and $E \in Inv(T)$, we shall denote by $A : E \to E$ the restriction of T on E, and by B the operator $B(\pi(y)) = \pi(T(y))$ on the quotient space X/E, where π is the natural homoeomorphism between X and X/E.,

Theorem 2.1. *If $T \in B(X)$ is a bounded operator and $E \in Inv(T)$, then the following holds.*

(i) $\sigma(T) \subset \sigma(A) \cup \sigma(B)$;

(ii) $\sigma(A) \subset \sigma(T) \cup \sigma(B)$;

(iii) $\sigma(B) \subset \sigma(T) \cup \sigma(A)$.

Proof. The proofs of (i) and (iii) are quite straightforward and well known (see [3, Proposition 3 (i)] and [5, Proposition 1.2.4]). For the sake of completeness, we give a proof of part (ii).

Suppose that $0 \in \rho(T) \cap \rho(B)$ and $Ax = 0$. Then since $0 = Ax = Tx$, $x = 0$. Now let $y \in E$ be an arbitrary vector. Then there exists $x \in X$ such that $y = Tx$.

Clearly, $B(x+E) = y+E = E$, i.e. $x \in E$. Hence, $y = Ax$. It is easy to see that A is bounded below since T is bounded below. □

Corollary 2.2. *Let $T \in B(X)$ be a bounded operator and let $E \in Inv(T)$. Then the following properties hold:*

(i) if $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(T)$, then $\lambda \in \sigma(A) \cap \sigma(B)$;
(ii) if $\lambda \in (\sigma(T) \cup \sigma(B)) \setminus \sigma(A)$, then $\lambda \in \sigma(T) \cap \sigma(B)$;
(iii) if $\lambda \in (\sigma(T) \cup \sigma(A)) \setminus \sigma(B)$, then $\lambda \in \sigma(T) \cap \sigma(A)$.

If we introduce the mapping $\xi : T^{-1}(E)/E \to E/R(A)$ with

$$\xi(x+E) = Tx + R(A), \quad x \in T^{-1}(E),$$

then we can get more information about the invertibility of the operator T using A, B and ξ. It is easily seen that $N(B) = T^{-1}(E)/E$

Theorem 2.3. *If $T \in B(X)$ and $E \in Inv(T)$, then T is invertible if and only if the following conditions hold:*

(i) A is bounded below;
(ii) B is onto;
(iii) ξ is one-one and onto.

Proof. if T is invertible operator, then A is bounded below and B is onto. Let $x+E \in T^{-1}(E)/E$, $x \in T^{-1}(E)$, be such that $\xi(x+E) = \mathbf{0} = R(A)$. Then $\xi(x+E) = Tx + R(A) = R(A)$, i.e. $Tx \in R(A)$. Let $e \in E$ be a vector such that $Tx = Ae$. Then $0 = T(x-e)$. Since T is one-one, it follows that $x = e \in E$. Hence $x+E = E$, i.e., ξ is one-one.

Now let $z + R(A)$ be an arbitrary vector in $E/R(A)$. Then there exists $x \in X$ such that $z = Tx$. Evidently, $x \in T^{-1}(E)$ and $\xi(x+E) = Tx + R(A) = z + R(A)$, i.e., ξ is onto.

To prove the converse, assume that A is bounded below, B is onto and ξ is one-one and onto. Suppose that $Tx = 0$ for some $x \in X$. Then $x + E \in N(B) = T^{-1}(E)/E$ and $\xi(x+E) = Tx + R(A) = R(A)$. Since ξ is one-one, it follows that $x + E = E$, or $x \in E$. But then $0 = Tx = Ax$, which, since A is one-one, implies that $x = 0$. Hence T is one-one.

Now let $y \in X$ be an arbitrary vector. Since B is onto, there exists $x \in X$ such that $y+E = B(x+E) = Tx+E$. Then $Tx-y \in E$. Fix $(Tx-y)+R(A) \in E/R(A)$. Since ξ is onto, there exists $z \in T^{-1}(E)$ such that

$$(Tx-y) + R(A) = \xi(z + R(A)) = Tz + R(A).$$

But there exists $e \in E$ such that $Tx - y - Tz = Ae$, or $y = T(x-z-e)$. Hence T is onto.

To complete the proof of the invertibility of T, we have to show that T is bounded below too.

Suppose to the contrary T is not bounded below. Then there exists a sequence of vectors $\{x_n\} \subset X$ with norm one such that $\|Tx_n\| \to 0$. Assume without loss of generality that for every positive integer n, $x_n \in X \setminus E$, or equivalently that $\|x_n + E\| \neq 0$. (Indeed, if $\{x_n\} \subset E$, then

$$\|Ax_n\| = \|Tx_n\| \to 0,$$

which is in contradiction with A is bounded below.) Also, we may assume that there exists a $x_0 \in X \setminus \{0\}$ such that $x_n \to x_0$, and consequently that $Tx_0 = 0$. Then $B(x_0 + E) = E$, i.e., $x_0 + E \in N(B) = T^{-1}(E)/E$. Now,

$$\xi(x_0 + E) = Tx_0 + R(A) = R(A),$$

and, since ξ is one-one, it follows that $x_0 + E = E$, i.e., $x_0 \in E$. But then $Ax_0 = Tx_0 = 0$. Since A is bounded below (hence one-one) it follows that $x_0 = 0$: this contradicts the fact that $x_n \to x_0$. □

It is of interest to find conditions, using isomorphism between some subspaces, for invertibility of the operator T which mirror the results for the spectrum of upper triangular operator matrices. Here, we say that two normed spaces X and Y are *isomorphic*, $X \cong Y$, if there is a one-one correspondence in both directions that preserves linear algebra and topology (see [7, pg. 12]). The following proposition will be needed.

Proposition 2.4. *If $T \in B(X)$ is invertible and A has closed range, then ξ is an isomorphism between $N(B)$ and $E/R(A)$.*

Proof. By the proof of Theorem 2.3, ξ is a one-one correspondence in both directions. Also, this is easily seen, ξ preserves linear algebra. Hence, we need only to show that ξ preserves topologies in both spaces, or that ξ is bounded and bounded below.

Suppose that ξ is not bounded below. Then there exists a sequence $\{x_n + E\}$ of norm one elements in $T^{-1}(E)/E$ such that $\|\xi(x_n + E)\| = \|Tx_n + R(A)\| \to 0$, or equivalently that there exists a $z \in R(A) = \overline{R(A)}$ such that $Tx_n \to z = Ae_0$, $e_0 \in E$. Evidently, $T(x_n - e_0) \to 0$. Since T is bounded below, it follows that $x_n \to e_0$. Consequently,

$$1 = \|x_n + E\| \to \|e_0 + E\| = 0,$$

which is a contradiction. Hence, ξ is bounded below.

Let $x + E$ be an arbitrary norm one vector in $T^{-1}(E)/E$. Since E is closed, there exists $e_0 \in E$ such that $\|x + E\| = \|x - e_0\|$. Then

$$\begin{aligned} \|\xi(x + E)\| &= \|Tx + R(A)\| \leq \|Tx - Ae_0\| = \|T(x - e_0)\| \\ &\leq \|T\|\,\|x - e_0\| = \|T\|\,\|x + E\|, \end{aligned}$$

i.e., ξ is bounded. □

The following theorem is an easy consequence of Theorem 2.3 and Proposition 2.4.

Theorem 2.5. *If $T \in B(X)$ and $E \in Inv(T)$ is such that the restriction A of T on E has closed range, then T is invertible if and only if the following conditions hold:*
(i) A is bounded below;
(ii) B is onto;
(iii) $N(B) \cong E/R(A)$.

Proposition 2.6. *Let $T \in B(X)$ and $E \in Inv(T)$. Then*

$$\sigma(A) \cup \sigma(B) = \sigma(T) \cup (S(A^*) \cap S(B)) = \sigma(T) \cup \{\sigma(A) \cap \sigma(B)\}.$$

Proof. If $\lambda \notin \sigma(A) \cup \sigma(B)$, then we have the following implications:

$$\begin{aligned} & \lambda \notin \sigma(A) \cup \sigma(B) \\ \Longleftrightarrow\ & A-\lambda, B-\lambda \text{ are invertible}, \lambda \notin \{S(A) \cap S(A^*) \cap S(B) \cap S(B^*)\} \\ \Longrightarrow\ & T-\lambda \text{ is invertible and } \lambda \notin \{S(A) \cap S(A^*) \cap S(B) \cap S(B^*)\} \\ \Longrightarrow\ & \lambda \notin \sigma(T) \cup \{S(A) \cup S(A^*) \cup S(B) \cup S(B^*)\}. \end{aligned}$$

Since $\sigma(T) \cup \{S(A^*) \cap S(B)\} \subseteq \sigma(T) \cup \{S(A) \cup S(A^*) \cup S(B) \cup S(B^*)\}$, it follows that $\sigma(A) \cup \sigma(B) \supseteq \sigma(T) \cup \{S(A^*) \cap S(B)\}$. For the reverse inclusion, we observe (from Theorem 2.5) that the following implications hold:

$$\begin{aligned} & \lambda \notin \sigma(T) \cup \{S(A^*) \cap S(B)\} \\ \Longleftrightarrow\ & T-\lambda \text{ is invertible and } \lambda \notin \{S(A^*) \cap S(B)\} \\ \Longrightarrow\ & A-\lambda \text{ is bounded below}, B-\lambda \text{ is onto}, \\ & N(B-\lambda) \cong E/R(A-\lambda) \text{ and } \lambda \notin \{S(A^*) \cap S(B)\}. \end{aligned}$$

Recall, [1, Corollary 2.24], that if an operator $S \in \mathcal{B}(X)$ is surjective, then S has SVEP at a point λ if and only if it is injective. Since $A-\lambda$ bounded below implies that $A^* - \lambda I^*$ is surjective, it follows that if A^* has SVEP at λ then $A-\lambda$ is invertible. But then $N(B-\lambda) = \{0\}$, which implies that $B-\lambda$ is invertible. Again, if $\lambda \notin S(B)$, then $B-\lambda$ is invertible, which implies that $R(A-\lambda) = E$, and hence that $A-\lambda$ is invertible. In either case, we have that $\lambda \notin \sigma(A) \cup \sigma(B)$. Hence $\sigma(A) \cup \sigma(B) \subseteq \sigma(T) \cup \{S(A^*) \cap S(B)\}$.

Observe that if any two of $T-\lambda$, $A-\lambda$ and $B-\lambda$ are invertible, then so also is the third one (see Theorems 2.3 and 2.5). Since

$$\begin{aligned} \lambda \notin \sigma(A) \cup \sigma(B) \quad & \Longleftrightarrow \quad A-\lambda, B-\lambda \text{ are invertible} \\ & \Longleftrightarrow \quad T-\lambda \text{ and } A-\lambda \text{ or } B-\lambda \text{ are invertible} \\ & \Longleftrightarrow \quad \lambda \notin \sigma(T) \cup \{\sigma(A) \cap \sigma(B)\}, \end{aligned}$$

it follows that $\sigma(A) \cup \sigma(B) = \sigma(T) \cup \{\sigma(A) \cap \sigma(B)\}$. □

It is meaningful to describe the spectrum of T in terms of the spectra of the operators A and B. Even more meaningful is the finding of necessary and (or) sufficient conditions under which the spectrum of T coincides with the union of the spectra of A and B. The following proposition gives some sufficient conditions for this.

Proposition 2.7. *Let $T \in B(X)$ and $E \in Inv(T)$. If one of following conditions holds:*

(i) E is T-hyperinvariant;

(ii) there exists $F \in Inv(T)$ such that $X = E \oplus F$;

(iii) $\sigma(A) \cap \sigma(B) = \emptyset$;

(iv) $\sigma(A) \subset \sigma(T)$ or $\sigma(B) \subset \sigma(T)$;

(v) A^* or B has SVEP,

then $\sigma(T) = \sigma(A) \cup \sigma(B)$.

Proof. (i) [3, Proposition 3(3)].

(ii) Let $T = A \oplus B_1$ on $X = E \oplus F$. Then B_1 and B are similar operators, and $\sigma(T) = \sigma(A) \cup \sigma(B_1) = \sigma(A) \cup \sigma(B)$.

(iii) and (iv) are direct consequences of Corollary 2.2.

(v) See Proposition 2.6. □

3 Relating Browder, Weyl and Fredholm essential spectra of T, A and B

For the Banach space X, with $\Phi_+(X)$, respectively $\Phi_-(X)$, we denote the set of upper, respectively lower, semi-Fredholm operators, i.e.

$$\Phi_+(X) = \{T \in B(X) : R(T) \text{ is closed and } n(T) < \infty\},$$
$$\Phi_-(X) = \{T \in B(X) : d(T) < \infty\}$$

and then $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ is set of all Fredholm operators.

Let $E \in Inv(T)$, A and B be like in the previous section. Barnes in [3] showed that if T is Fredholm operator, then A is upper semi-Fredholm and B is lower semi-Fredholm. Moreover, if $n(A) < \infty$ and $n(B) < \infty$, then

$$n(A) \leq n(T) \leq n(A) + n(B),$$

and if $d(A) < \infty$ and $d(B) < \infty$, then

$$d(B) \leq d(T) \leq d(A) + d(B).$$

(see [3, Proposition 7]). Also, by [3, Theorem 8] , if two of the operators A, B and T are Fredholm, then the third one is Fredholm too. Hence, we have the following theorem:

Theorem 3.1. *Let $T \in B(X)$, be a bounded operator and $E \in Inv(T)$. Then the following properties hold.*

(i) $\sigma_e(T) \subset \sigma_e(A) \cup \sigma_e(B)$;

(ii) $\sigma_e(A) \subset \sigma_e(T) \cup \sigma_e(B)$;

(iii) $\sigma_e(B) \subset \sigma_e(T) \cup \sigma_e(A)$.

Moreover,

(iv) $\sigma_e(A) \cup \sigma_e(B) = \sigma_e(T) \cup \{\sigma_e(A) \cap \sigma_e(B)\}$;

(v) $\sigma_e(T) \cup \sigma_e(B) = \sigma_e(A) \cup \{\sigma_e(T) \cap \sigma_e(B)\}$;

(vi) $\sigma_e(T) \cup \sigma_e(A) = \sigma_e(B) \cup \{\sigma_e(T) \cap \sigma_e(A)\}$.

Proof. The proof of (i), (ii) and (iii) is straightforward, and (iv), (v) and (vi) follow from an argument of type:

$$\begin{aligned}\lambda \notin (\sigma_e(A) \cup \sigma_e(B)) \quad &\Longleftrightarrow \quad A-\lambda \text{ and } B-\lambda \text{ are Fredholm} \\ &\Longleftrightarrow \quad T-\lambda \text{ , and } A-\lambda \text{ or } B-\lambda \text{ are Fredholm} \\ &\Longleftrightarrow \quad \lambda \notin \sigma_e(T) \cup \{\sigma_e(A) \cap \sigma_e(B)\}.\end{aligned}$$

□

Remark 3.2. If $E \in Inv(T)$ is complemented in X by a T-invariant subspace, say $X = E \oplus F$, then $T = \begin{pmatrix} A & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} E \\ F \end{pmatrix}$, where B_2 is similar to B. An argument of Djordjević [4, Theorem 3.2] proves that T is Fredholm if and only if A is upper semi–Fredholm, B_2 is lower semi–Fredholm and $N(B_2)$ is isomorphic to $E/R(A)$ upto a finite dimensional subspace. Evidently, B is lower semi–Fredholm if and only if B_2 is lower semi–Fredholm and $N(B_2)$ is isomorphic to $N(B)$. Hence if $E \in Inv(T)$ is complemented in X, then T is Fredholm if and only if A is upper semi–Fredholm, B is lower semi–Fredholm and $N(B)$ is isomorphic to $E/R(A)$ upto a finite dimensional subspace. An obvious question here is the following: Suppose that A has closed range. Then, is T is Fredholm if and only if A is upper semi–Fredholm, B is lower semi–Fredholm and $N(B)$ is isomorphic to $E/R(A)$ upto a finite dimensional subspace?

The following Theorem relates $\sigma_e(A)$, $\sigma_e(B)$, $\sigma_e(T)$ and $S(A^*) \cap S(B)$.

Theorem 3.3. *Let $T \in B(X)$ and $E \in Inv(T)$. Then*

$$\sigma_e(A) \cup \sigma_e(B) \cup (S(A^*) \cap S(B)) = \sigma_e(T) \cup (S(A^*) \cap S(B)).$$

Proof. By Theorem 3.1 (i), the inclusion $\sigma_e(A) \cup \sigma_e(B) \cup (S(A^*) \cap S(B)) \supseteq \sigma_e(T) \cup (S(A^*) \cap S(B))$ is obvious.

Now, let $\lambda \in (\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(T)$. Then Theorem 3.1 implies that $\lambda \in \sigma_e(A) \cap \sigma_e(B)$, and [3, Proposition 5] implies that $A-\lambda$ is upper and $B-\lambda$ is lower semi-Fredholm. Observe that if A^* has SVEP at λ, then (ind$(A-\lambda) \geq 0$, which implies that) $A-\lambda$ is Fredholm, and consequently also that $B-\lambda$ is Fredholm. Again, if B has SVEP at λ, then (ind$(B-\lambda) \leq 0$, which implies that) $B-\lambda$ is Fredholm, and consequently also that $A-\lambda$ is Fredholm. Hence $\lambda \in (\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(T)$ implies $\lambda \in S(A^*) \cap S(B)$. □

The following proposition gives some sufficient conditions for $\sigma_e(T)$ to coincide with the union of the spectra $\sigma_e(A)$ and $\sigma_e(B)$.

Proposition 3.4. *Let $T \in B(X)$ and $E \in Inv(T)$. If one of following conditions holds:*

(i) E is T-hyperinvariant;

(ii) exists $F \in Inv(T)$ such that $X = E \oplus F$;

(iii) $\sigma_e(A) \cap \sigma_e(B) = \emptyset$;

(iv) $\sigma_e(A) \subset \sigma_e(T)$ *or* $\sigma_e(B) \subset \sigma_e(T)$*;*
(v) A^* *or* B *has SVEP,*
then $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$.

Proof. (i) If E is T-hyperinvariant, then it is easy to see that $T^{-1}(E) = E$, and by [3, Corollary 9] follows that $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$.

(ii) Let $T = A \oplus B_1$ on $X = E \oplus F$. Then B_1 and B are similar operators, and $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B_1) = \sigma_e(A) \cup \sigma_e(B)$.

(iii) and (iv) are direct consequences of Theorem 3.1.

(v) By Theorem 3.3. □

The *ascent*, denoted asc(T), and the *descent*, denoted dsc(T), of T are given by

$$\mathrm{asc}(T) = \inf\{n : N(T^n) = N(T^{n+1})\}, \quad \mathrm{dsc}(T) = \inf\{n : R(T^n) = R(T^{n+1})\};$$

if no such n exists, then asc$(T) = \infty$, respectively dsc$(T) = \infty$. The following inequalities relating the ascent, and the descent, of T, A and B are known to hold for all linear operators $T \in \mathcal{L}(X)$ [8, Exercise 7, p 293]:

$$\mathrm{asc}(A) \le \mathrm{asc}(T) \le \mathrm{asc}(A) + \mathrm{asc}(B);$$
$$\mathrm{dsc}(B) \le \mathrm{dsc}(T) \le \mathrm{dsc}(A) + \mathrm{dsc}(B),$$

where the inequalities are best possible.

A bounded linear operator T is *Browder* (resp., Weyl) if it is Fredholm of finite ascent and descent (resp., it is Fredholm and has index 0). The Browder spectrum $\sigma_b(T)$, and the Weyl spectrum $\sigma_w(T)$ of T are the set

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.$$

Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{ acc } \sigma(T),$$

where for a subset $K \subseteq \mathbb{C}$, we write acc K (resp. iso K) for accumulation (resp. isolated) points of K.

The following theorem is the Browder spectrum version of Theorem 3.1: the proof of the lemma is straightforward, hence left to the reader.

Theorem 3.5. *Let* $T \in B(X)$*, be a bounded operator and* $E \in Inv(T)$*. Then*
(i) $\sigma_b(T) \subset \sigma_b(A) \cup \sigma_b(B)$*;*
(ii) $\sigma_b(A) \subset \sigma_b(T) \cup \sigma_b(B)$*;*
(iii) $\sigma_b(B) \subset \sigma_b(T) \cup \sigma_b(A)$*.*
Furthermore
(iv) $\sigma_b(A) \cup \sigma_b(B) = \sigma_b(T) \cup \{\sigma_b(A) \cap \sigma_b(B)\}$*;*
(v) $\sigma_b(T) \cup \sigma_b(B) = \sigma_b(A) \cup \{\sigma_b(T) \cap \sigma_b(B)\}$*;*
(vi) $\sigma_b(T) \cup \sigma_b(A) = \sigma_b(B) \cup \{\sigma_b(T) \cap \sigma_b(A)\}$*.*

Recall from [1, Corollary 3.19] that if an operator $B-\lambda$ is semi–Fredholm, then B (resp., B^*) has SVEP at λ implies that $\mathrm{ind}(B-\lambda)\leq 0$ (resp., $\mathrm{ind}(B-\lambda)\geq 0$). In particular, if $B-\lambda$ is semi–Fredholm, and both B and B^* have SVEP at λ, then $B-\lambda$ is Weyl. The following theorem is the Browder spectrum analogue of 3.3.

Theorem 3.6. *Let $T\in B(X)$, be a bounded operator and $E\in Inv(T)$. Then*

$$\sigma_b(T)\cup(S(A^*)\cap S(B))=\sigma_b(A)\cup\sigma_b(B).$$

Proof. The proof of the inclusion $\sigma_b(A)\cup\sigma_b(B)\subseteq\sigma_b(T)\cup(S(A^*)\cap S(B))$ follows from the implications

$$\begin{aligned}
& \lambda\notin\sigma_b(T)\cup(S(A^*)\cap S(B)\\
\Longleftrightarrow\quad & T-\lambda\in\Phi(X),\mathrm{asc}(A-\lambda)<\infty,\mathrm{dsc}(B-\lambda)<\infty,\\
& A^* \text{ and } B \text{ have SVEP at } \lambda\\
\Longrightarrow\quad & A-\lambda\ \text{ is lower semi–Fredholm, }\ B-\lambda\ \text{ is upper semi–Fredholm },\\
& \mathrm{asc}(A-\lambda)<\infty,\mathrm{dsc}(B-\lambda)<\infty,A^*\ \text{ and }\ B\ \text{ have SVEP at }\ \lambda\\
\Longrightarrow\quad & A-\lambda\ \text{ and }\ B-\lambda\ \text{ are Browder}\\
\Longrightarrow\quad & \lambda\notin\sigma_b(A)\cup\sigma_b(B)
\end{aligned}$$

and the reverse inclusion $\sigma_b(A)\cup\sigma_b(B)\subseteq\sigma_b(T)\cup(S(A^*)\cap S(B))$ follows from the implications

$$\begin{aligned}
& \lambda\notin\sigma_b(A)\cup\sigma_b(B)\\
\Longleftrightarrow\quad & A-\lambda\ \text{ and }\ B-\lambda\ \text{ are Fredholm},\\
& \mathrm{asc}(A-\lambda)=\mathrm{dsc}(A-\lambda)<\infty,\mathrm{asc}(B-\lambda)=\mathrm{dsc}(B-\lambda)<\infty\\
\Longrightarrow\quad & T-\lambda\ \text{ is Fredholm},\mathrm{asc}(T-\lambda)=\mathrm{dsc}(T-\lambda)<\infty,\\
& A^*\ \text{ and }\ B\ \text{ have SVEP at }\ \lambda\\
\Longrightarrow\quad & \lambda\notin\sigma_b(T)\cup(S(A^*)\cap S(B)).
\end{aligned}$$

□

The following corollary is immediate from the above.

Corollary 3.7. *Let $T\in B(X)$ and $E\in Inv(T)$. If one of following conditions holds*

(i) E is T-hyperinvariant;

(ii) exists $F\in Inv(T)$ such that $X=E\oplus F$;

(iii) $\sigma_b(A)\cap\sigma_b(B)=\emptyset$;

(iv) $\sigma_b(A)\subset\sigma_b(T)$ or $\sigma_b(B)\subset\sigma_b(T)$;

(v) A^* or B has SVEP,

then $\sigma_b(T)=\sigma_b(A)\cup\sigma_b(B)$.

The relationship between the Weyl spectra of A, B and T is a bit more delicate, and an equality of the type of Theorem 3.6 is not possible for the Weyl spectrum.

Thus, let $U \in \mathcal{B}(\ell^2)$ denote the forward unilateral shift $U(x_1, x_2, ...) = (0, x_1, x_2, ...)$, $A = U^*$, $B = U$ and $T = A \oplus B$. Then $\sigma_w(T) = \partial\mathbf{D}$ is the boundary of the unit disc $\mathbf{D}$ in $\mathbf{C}$, $\sigma_w(A) = \sigma_w(B) = \mathbf{D}$ and $S(A^*) = S(B) = \emptyset$.

However, for some kind of relationship of Weyl spectrums we need some extra conditions. For example, if $\text{ind}(T-\lambda) = \text{ind}(A-\lambda)+\text{ind}(B-\lambda)$, whenever either the left hand side or the right hand side in the equality is finite, then $\sigma_w(A) \cup \sigma_w(B) = \sigma_w(T) \cup \{\sigma_w(A) \cap \sigma_w(B)\}$: this follows from the following implications.

$$\begin{aligned} & \lambda \notin \sigma_w(A) \cup \sigma_w(B) \\ \iff & A - \lambda \text{ and } B - \lambda \text{ are Weyl} \\ \iff & T - \lambda \text{ and } A - \lambda \text{ , or, } T - \lambda \text{ and } B - \lambda \text{ are Weyl} \\ \iff & \lambda \notin \sigma_w(T) \cup \{\sigma_w(A) \cap \sigma_w(B)\}. \end{aligned}$$

For an operator $C \in \mathcal{B}(X)$, let $S_e(C) = \{\lambda \notin \sigma_e(C) : C$ dose not have SVEP at $\lambda\}$. Then:

Theorem 3.8. $\sigma_w(A) \cup \sigma_w(B) \subseteq \sigma_b(T) \cup \{S(A^*) \cap S(B)\} \subseteq \sigma_w(T) \cup \{S_e(P) \cup S(Q)\}$, *where either* $P = A$ *and* $Q = B$ *or* $P = B^*$ *and* $Q = A^*$.

Proof. Recall from Theorem 3.6 that $\sigma_b(T) \cup (S(A^*) \cap S(B)) = \sigma_b(A) \cup \sigma_b(B)$. Hence, since $\sigma_w(C) \subseteq \sigma_b(C)$ for every $C \in \mathcal{B}(X)$, it would suffice to prove that $\sigma_b(A) \cup \sigma_b(B) \subseteq \sigma_w(T) \cup \{S_e(P) \cup S(Q)\}$. We consider the case in which $P = A$ and $Q = B$; the proof for the other case is similar.

If $\lambda \notin \sigma_w(T) \cup \{S_e(A) \cup S(B)\}$, then $T - \lambda$ is Weyl, B has SVEP at λ and A has SVEP at λ whenever $\lambda \notin \sigma_e(A)$. The conclusion $T - \lambda$ is Weyl implies that $A - \lambda$ is upper semi–Fredholm, $B - \lambda$ is lower semi–Fredholm and $\text{ind}(T - \lambda) = 0$. This, since B has SVEP at λ, implies that ($\text{asc}(B - \lambda) < \infty \Longrightarrow$) $\text{ind}(B - \lambda) \leq 0$, which in turn implies that $B - \lambda$ is Fredholm, and hence also that $A - \lambda$ is Fredholm. But then A has SVEP at λ; hence $\text{asc}(A - \lambda) < \infty$. Recall that $\text{asc}(T - \lambda) \leq \text{asc}(A-\lambda)+\text{asc}(B-\lambda)$. Hence $\text{asc}(T-\lambda) < \infty$, which (since $T-\lambda$ is Weyl) implies that $\text{asc}(T - \lambda) = \text{dsc}(T - \lambda) < \infty$ [1, Theorem 3.77]. Consequently, $\lambda \in \text{iso}\sigma(T)$. Since $\text{dsc}(B - \lambda) \leq \text{dsc}(T - \lambda)$, $\text{asc}(B - \lambda) = \text{dsc}(B - \lambda) < \infty$, $\lambda \notin \sigma_b(B)$, and $\lambda \in \text{iso}\sigma(B)$. The hypothesis that B has SVEP implies by Proposition 2.7 that $\sigma(T) = \sigma(A) \cup \sigma(B)$. Hence $\lambda \in \text{iso}\sigma(A)$. (Observe that if $\lambda \notin \sigma(A)$, then $\lambda \notin \sigma_b(A)$, trivially.) Since $A - \lambda$ is Fredholm, it follows that $\text{asc}(A - \lambda) = \text{dsc}(A - \lambda) < \infty$, i.e., $\lambda \notin \sigma_b(A)$. Hence $\lambda \notin \sigma_b(A) \cup \sigma_b(B)$. □

In accordance with current terminology, [1, page 156], we say that T satisfies Browder's theorem if $\sigma_w(T) = \sigma_b(T)$ and it is known that, if T or T^* has SVEP, then Browder's theorem holds for T (see [1, Theorem 3.52]). It is easy to see that if $S_e(P) \cup S(Q) = \emptyset$, where either $P = A$ and $Q = B$ or $P = B^*$ and $Q = A^*$, then $S(A) \cap S(B) = \emptyset$, and by Theorem 3.8 we have $\sigma_w(T) = \sigma_b(T)$. Hence, the next corollary is proved:

Corollary 3.9. *If* $S_e(P) \cup S(Q) = \emptyset$, *where either* $P = A$ *and* $Q = B$ *or* $P = B^*$ *and* $Q = A^*$, *then* T *satisfies Browder's theorem.*

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