Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/faac

Functional Analysis, Approximation and Computation **1:1** (2009), 31–48

APPROXIMATING FIXED POINTS OF NOOR ITERATION WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

G. S. Saluja

Abstract

In this paper, we give the sufficient condition for convergence of fixed point of Noor iteration with errors for asymptotically quasi nonexpansive mapping in real Banach space. Also we have proved that (i) strong convergence of fixed point of Noor iteration with errors for completely continuous uniformly L-Lipschitzian asymptotically quasi nonexpansive mapping on a nonempty closed convex subset of a real uniformly convex Banach space and (ii) convergence of fixed point of Noor iteration with errors for (L, α) uniform Lipschitz asymptotically quasi nonexpansive mapping on a nonempty compact convex subset of a real uniformly convex Banach space. The results presented in this paper extend and improve the corresponding results of Xu and Noor [19], Qihou [10, 11], Rhoades [13] and many others.

1 Introduction and preliminaries

Let C be a nonempty subset of a real normed linear space E. Let T be a self mapping of C. Then T is said to be asymptotically nonexpansive with sequence $\{r_n\} \subset [0, \infty)$ if $\lim_{n\to\infty} r_n = 0$ and

$$\|T^n x - T^n y\| \leq (1 + r_n)\,\|x - y\|$$

for all $x, y \in C$ and $n \geq 1$; and is said to be asymptotically quasi-nonexpansive with sequence $\{r_n\} \subset [0, \infty)$ if $F(T) = \{x \in C : Tx = x\} \neq \emptyset$, $\lim_{n\to\infty} r_n = 0$ and

$$\|T^n x - x^*\| \leq (1 + r_n)\,\|x - x^*\|$$

2000 *Mathematics Subject Classifications.* 47H09, 47H10.
Key words and Phrases. Asymptotically quasi nonexpansive mapping, fixed point, (L, α) uniform Lipschitz mapping, Noor iteration with errors, strong convergence, uniformly convex Banach space.
Received: November 10, 2008
Communicated by Dragan S. Djordjević

for all $x \in C$, $x^* \in F(T)$ and $n \geq 1$. It is clear that an asymptotically nonexpansive mapping with a nonempty fixed point set is asymptotically quasi-nonexpansive. The converse do not hold in general.

The mapping T is called uniformly (L, α) Lipschitzian if there exist constants $L > 0$ and $\alpha > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|^{\alpha}$$

for all $x, y \in C$ and $n \geq 1$.

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [2] as an important generalization of the class of nonexpansive maps. They established that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is an asymptotically nonexpansive self mapping of K, then T has a fixed point. In [3], they extended this result to the broader class of uniformly L-Lipschitzian mappings with $L < \lambda$, where λ is sufficiently near 1.

Iterative techniques for approximating fixed points of nonexpansive mappings and their generalizations (asymptotically nonexpansive mappings etc.) have been studied by a number of authors (see e.g. Chidume [1], Rhoades [13], Schu [15], Tan and Xu [18]), using the Mann iteration process [7] or the Ishikawa iteration process [6].

In 2001, Noor [8, 9] have introduced the three-step iterative sequences and he studied the approximate solutions of variational inequalities in Hilbert space. The three-step iterative approximation problem were studied extensively by Noor [8, 9], Glowinski and Le Tallec [4], Haubruge et al [5].

In 2002, Xu and Noor [19] introduced the three-step iterative for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach space:

Theorem XN([19], Theorem 2.1): Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, and

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For a given $x_0 \in C$, define

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1-\gamma_n)x_n \\ y_n &= \beta_n T^n z_n + (1-\beta_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n. \end{aligned} \tag{1.0.1}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T.

Motivated and inspired by Xu and Noor [19] and many others, we study the following iteration scheme which we call it Noor iteration with errors as follows:

Noor Iteration With Errors:

Let C be a nonempty subset of normed space X and let $T\colon C \to C$ be a mapping. For a given $x_0 \in C$, define the sequence $\{x_n\}$ as follows:

$$\begin{aligned} z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n \\ y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n. \end{aligned} \tag{1.0.2}$$

where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C.

It is clear that the Mann and Ishikawa iterations processes are all special case of the Noor iteration with errors.

In this paper, we will extend the iteration process (1.0.1) to Noor iteration with errors (1.0.2) for asymptotically quasi-nonexpansive mappings. The purpose of this paper is to study convergence of fixed point for three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in real uniformly convex Banach space. The results presented in this paper extend and improve the corresponding results of Xu and Noor [19], Qihou [10, 11], Rhoades [13] and many other known results.

We need the following result and lemmas to prove our main result:

Theorem LQ [11, Theorem 3]: Let E be a nonempty closed convex subset of a Banach space, T is an asymptotically quasi-nonexpansive mapping on E, and $F(T)$ nonempty. Given $\sum_{n=1}^{\infty} u_n < +\infty$, $\forall x_1 \in E$, defined $\{x_n\}_{n=1}^{\infty}$ as

$$x_{n+1} = a_n x_n + b_n T^n y_n + c_n m_n,$$

$$y_n = \bar{a}_n x_n + \bar{b}_n T^n x_n + \bar{c}_n l_n, \ \forall n \in N,$$

where $m_n, l_n \in E$, and $\{\|m_n\|\}_{n=1}^{\infty}$, $\{\|l_n\|\}_{n=1}^{\infty}$ are bounded, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n$, $0 \le a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \le 1$. Then $\{x_n\}_{n=1}^{\infty}$ converges to some fixed point p of T if and only if there exists some infinite subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges to p.

Lemma 1.1[[**14**] **J.Schu′s Lemma**]: Let X be a real uniformly convex Banach space, $0 < \alpha \le t_n \le \beta < 1$, $x_n, y_n \in X$, $\limsup_{n\to\infty} \|x_n\| \le a$, $\limsup_{n\to\infty} \|y_n\| \le a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = a$, $a \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.2[**11**, **Lemma 2**]: Let nonnegative series $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$ satisfy $\alpha_n \le (1+\beta_n)\alpha_n + r_n$, $\forall n \in N$, and $\sum_{n=1}^{\infty} \beta_n < +\infty$, $\sum_{n=1}^{\infty} r_n < +\infty$; then $\lim_{n\to\infty} \alpha_n$ exists. Moreover if $\liminf_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.3: Let X be a real Banach space, C a nonempty closed convex subset of X. Let $T\colon C \to C$ be an asymptotically quasi nonexpansive mapping with sequence $\{r_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$. For a given $x_0 \in C$, let $\{x_n\}$ be the sequence defined by (1.0.2), where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C with the following restrictions

(i)$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

(**a**) Then for each $p \in F(T)$, $\lim_{n\to\infty} \|x_n - p\|$ exists.

(**b**) There exists a constant $M > 0$ such that

$$\|x_{n+m} - p\| \le M \|x_n - p\| + M \sum_{k=n}^{n+m-1} B_k$$

for all $n, m \ge 1$ and $p \in F(T)$, where $M = e^{3\sum_{k=n}^{n+m-1} r_k}$.

Proof (**a**): By the Schauder's fixed point theorem, we obtain that $F(T) \neq \emptyset$. Let $p \in F(T)$, since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C, so we put

$$K = \sup_{n\ge1} \|u_n - p\| \vee \sup_{n\ge1} \|v_n - p\| \vee \sup_{n\ge1} \|w_n - p\|.$$

For each $n \geq 1$, we note that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\leq \alpha_n (1 + r_n) \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n K \end{aligned} \tag{1.0.3}$$

and

$$\begin{aligned} \|y_n - p\| &= \|\alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &\leq \alpha'_n \|T^n z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leq \alpha'_n (1 + r_n) \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n K \end{aligned} \tag{1.0.4}$$

and

$$\begin{aligned} \|z_n - p\| &= \|\alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n - p\| \\ &\leq \alpha''_n \|T^n x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\| \\ &\leq \alpha''_n (1 + r_n) \|x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\| \\ &\leq (\alpha''_n + \beta''_n)(1 + r_n) \|x_n - p\| + \gamma''_n \|u_n - p\| \\ &= (1 - \gamma''_n)(1 + r_n) \|x_n - p\| + \gamma''_n \|u_n - p\| \\ &\leq (1 + r_n) \|x_n - p\| + \gamma''_n K \end{aligned} \tag{1.0.5}$$

substituting (1.0.5) into (1.0.4), we have

$$\begin{aligned} \|y_n - p\| &\leq \alpha'_n (1 + r_n)[(1 + r_n) \|x_n - p\| + \gamma''_n K] \\ &\quad + \beta'_n \|x_n - p\| + \gamma'_n K \\ &\leq (1 + r_n)^2 (\alpha'_n + \beta'_n) \|x_n - p\| + \alpha'_n (1 + r_n) \gamma''_n K \\ &\quad + \gamma'_n K \\ &\leq (1 + r_n)^2 (\alpha'_n + \beta'_n) \|x_n - p\| + (1 + r_n) \gamma''_n K \\ &\quad + \gamma'_n K \\ &= (1 + r_n)^2 (1 - \gamma'_n) \|x_n - p\| + (1 + r_n) K (\gamma''_n + \gamma'_n) \\ &\leq (1 + r_n)^2 \|x_n - p\| + A_n \end{aligned} \tag{1.0.6}$$

where $A_n = (1 + r_n) K (\gamma''_n + \gamma'_n)$. Note that $\sum_{n=1}^{\infty} A_n < \infty$, since $\sum_{n=1}^{\infty} r_n < \infty$ and by condition (ii).

Substituting (1.0.6) into (1.0.3), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \alpha_n(1+r_n)[(1+r_n)^2\|x_n-p\|+A_n] \\
&\quad +\beta_n\|x_n-p\|+\gamma_n K \\
&\leq (1+r_n)^3(\alpha_n+\beta_n)\|x_n-p\|+\alpha_n(1+r_n)A_n \\
&\quad +\gamma_n K \\
&\leq (1+r_n)^3(\alpha_n+\beta_n)\|x_n-p\|+(1+r_n)A_n \\
&\quad +\gamma_n K \\
&= (1+r_n)^3(1-\gamma_n)\|x_n-p\|+(1+r_n)A_n+\gamma_n K \\
&\leq (1+r_n)^3\|x_n-p\|+(1+r_n)(A_n+\gamma_n K) \\
&\leq (1+r_n)^3\|x_n-p\|+(1+r_n)^2K(\gamma_n+\gamma'_n+\gamma''_n) \\
&= (1+r_n)^3\|x_n-p\|+B_n \qquad (1.0.7)
\end{aligned}$$

where $B_n = (1+r_n)^2K(\gamma_n+\gamma'_n+\gamma''_n)$. Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$ and $\sum_{n=1}^{\infty}\gamma''_n < \infty$, it follows that $\sum_{n=1}^{\infty} B_n < \infty$, thus by Lemma 1.2, we have $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof.

(**b**) Since $1+x \leq e^x$ for all $x > 0$. Then from (a) it can be obtained that

$$\begin{aligned}
\|x_{n+m}-p\| &\leq (1+r_{n+m-1})^3\|x_{n+m-1}-p\|+B_{n+m-1} \\
&\leq e^{3r_{n+m-1}}\|x_{n+m-1}-p\|+B_{n+m-1} \\
&\leq e^{3r_{n+m-1}}[e^{3r_{n+m-2}}\|x_{n+m-2}-p\|+B_{n+m-2}]+B_{n+m-1} \\
&\leq e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2}-p\|+e^{3r_{n+m-1}}B_{n+m-2}+B_{n+m-1} \\
&\leq e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2}-p\|+e^{3r_{n+m-1}}[B_{n+m-2}+B_{n+m-1}] \\
&\leq \ldots\ldots\ldots \\
&\leq \ldots\ldots\ldots \\
&\leq e^{3\sum_{k=n}^{n+m-1} r_k}.\|x_n-p\|+e^{3\sum_{k=n}^{n+m-1} r_k}.\sum_{k=n}^{n+m-1} B_k \\
&\leq M\|x_n-p\|+M\sum_{k=n}^{n+m-1} B_k, \text{ where } M = e^{3\sum_{k=n}^{n+m-1} r_k}.
\end{aligned}$$

This completes the proof.

2 Main results

Theorem 2.1: Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be an asymptotically quasi nonexpansive mapping with sequence $\{r_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$. From an arbitrary $x_0 \in C$, let $\{x_n\}$ be the sequence defined by (1.0.2), where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ with the following restrictions:

(i)$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty}$ $d(x_n, F(T)) = 0$.

Proof: By the Schauder's fixed point theorem, we obtain that $F(T) \neq \emptyset$. It is suffices that we only prove the sufficiency. By equation (1.0.7) of Lemma 1.3, we have

$$\|x_{n+1} - p\| \leq (1 + r_n)^3 \|x_n - p\| + B_n, \ \forall n \in N \ \mathit{and} \ p \in F(T) \quad (2.0.8)$$

where $B_n = (1 + r_n)^2[\gamma_n + \gamma'_n + \gamma''_n]K$. Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma''_n < \infty$, thus we know $\sum_{n=1}^{\infty} B_n < \infty$. So from equation (2.0.8), we obtain

$$d(x_{n+1}, F(T)) \leq (1 + r_n)^3 d(x_n, F(T)) + B_n \quad (2.0.9)$$

Since $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ and from Lemma 1.2, we have $\lim_{n\to\infty} d(x_n, F(T))$ $= 0$.

Next we will show that $\{x_n\}$ is a Cauchy sequence. For all $\varepsilon_1 > 0$, from Lemma 1.3, it can be known there must exists a constant $M > 0$ such that

$$\begin{aligned}\|x_{n+m} - p\| &\leq M \|x_n - p\| \\ &\quad + M \sum_{k=n}^{n+m-1} B_k, \ \forall n, m \in N, \ \forall p \in F(T).\end{aligned} \quad (2.0.10)$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{k=n}^{\infty} B_k < \infty$, then there must exists a constant N_1, such that when $n \geq N_1$

$$d(x_n, F(T)) < \frac{\varepsilon_1}{3M}, \ \mathit{and} \ \sum_{k=n}^{\infty} B_k < \frac{\varepsilon_1}{6M}.$$

So there must exists $p^* \in F(T)$, such that

$$d(x_{N_1}, F(T)) = \|x_{N_1} - p^*\| < \frac{\varepsilon_1}{3M}.$$

From (2.0.10), it can be obtained that when $n \geq N_1$

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\
&\leq M\,\|x_{N_1} - p^*\| + M\,\|x_{N_1} - p^*\| + 2M \sum_{k=N_1}^{\infty} B_k \\
&< M.\frac{\varepsilon_1}{3M} + M.\frac{\varepsilon_1}{3M} + 2M.\frac{\varepsilon_1}{6M} \\
&< \varepsilon_1
\end{aligned}$$

that is

$$\|x_{n+m} - x_n\| < \varepsilon_1.$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = y^*$. Then $y^* \in C$. It remains to show that $y^* \in F(T)$. Let $\varepsilon_2 > 0$ be given. Then there exists a natural number N_2 such that

$$\|x_n - y^*\| < \frac{\varepsilon_2}{2(2 + r_1)}, \;\; \forall n \geq N_2.$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there must exists a natural number $N_3 \geq N_2$ such that for all $n \geq N_3$, we have

$$d(x_n, F(T)) < \frac{\varepsilon_2}{3(2 + r_1)},$$

and in particular, we have

$$d(x_{N_3}, F(T)) < \frac{\varepsilon_2}{3(2 + r_1)}.$$

Therefore, there exists $z^* \in F(T)$ such that

$$\|x_{N_3} - z^*\| < \frac{\varepsilon_2}{2(2 + r_1)}.$$

Consequently we have

$$\begin{aligned}
\|Ty^* - y^*\| &= \|Ty^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\
&\leq \|Ty^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\leq (1 + r_1)\,\|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\leq (1 + r_1)\,\|y^* - x_{N_3} + x_{N_3} - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\leq (1 + r_1)[\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\
&\leq (2 + r_1)\,\|y^* - x_{N_3}\| + (2 + r_1)\,\|z^* - x_{N_3}\| \\
&< (2 + r_1).\frac{\varepsilon_2}{2(2 + r_1)} + (2 + r_1).\frac{\varepsilon_2}{2(2 + r_1)} \\
&< \varepsilon_2.
\end{aligned}$$

This shows that $y^* \in F(T)$. Thus $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof.

Remark 2.2: Theorem 2.1 extends Theorem 1 of [11] to the case of three step iteration scheme considered here and also it extends Theorem 1 of [10] to the case of three step iteration scheme with errors considered here.

Theorem 2.3: Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be uniformly L-Lipschitzian asymptotically quasi nonexpansive mapping with sequence $\{r_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$. From an arbitrary $x_0 \in C$, let $\{x_n\}$ be the sequence defined by (1.0.2), where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0, 1]$ with the following restrictions:

(i)$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof: By Schauder's fixed point theorem, we have $F(T) \neq \emptyset$. By Lemma 1.3(a), we have $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geq 0$. From (1.0.6), we have

$$\|y_n - p\| \leq (1 + r_n)^2\,\|x_n - p\| + A_n,\ \ \forall n \geq 1.$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = a.$$

Note that

$$\limsup_{n\to\infty} \|T^n y_n - p\| \leq \limsup_{n\to\infty} (1 + r_n) \|y_n - p\| = \limsup_{n\to\infty} \|y_n - p\| \leq a.$$

Next, consider

$$\|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq \|T^n y_n - p\| + \gamma_n \|w_n - x_n\| .$$

Thus,

$$\limsup_{n\to\infty} \|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq a.$$

Also,

$$\|x_n - p + \gamma_n(w_n - x_n)\| \leq \|x_n - p\| + \gamma_n \|w_n - x_n\| ,$$

gives that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(w_n - x_n)\| \leq a,$$

and

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \lim_{n\to\infty} \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n T^n y_n + (1 - \alpha_n) x_n - \gamma_n x_n + \gamma_n w_n - (1 - \alpha_n) p - \alpha_n p\| \\ &= \lim_{n\to\infty} \|\alpha_n [T^n y_n - p + \gamma_n(w_n - x_n)] + (1 - \alpha_n)[x_n - p + \gamma_n(w_n - x_n)]\| . \end{aligned}$$

By J.Schu's Lemma [14], we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0.$$

Again, note that for each $n \geq 1$

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - T^n y_n\| + \|T^n y_n - p\| \\ &\leq \|x_n - T^n y_n\| + (1 + r_n) \|y_n - p\| . \end{aligned}$$

Since $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$, we obtain that

$$a = \lim_{n\to\infty} \|x_n - p\| \leq \liminf_{n\to\infty} \|y_n - p\| .$$

It follows that

$$a \leq \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq a.$$

This implies that

$$\lim_{n\to\infty} \|y_n - p\| = a.$$

On the other hand, we note that

$$\begin{aligned}
\|z_n - p\| &= \|\alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n - p\| \\
&\leq \alpha_n'' \|T^n x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|u_n - p\| \\
&\leq \alpha_n''(1 + r_n) \|x_n - p\| + \beta_n'' \|x_n - p\| + \gamma_n'' \|u_n - p\| \\
&\leq (\alpha_n'' + \beta_n'')(1 + r_n) \|x_n - p\| + \gamma_n'' \|u_n - p\| \\
&\leq (1 - \gamma_n'')(1 + r_n) \|x_n - p\| + \gamma_n'' \|u_n - p\| \\
&\leq (1 + r_n) \|x_n - p\| + \gamma_n'' \|u_n - p\| .
\end{aligned}$$

Since $\sum_{n=1}^{\infty} r_n < \infty$ and $\lim_{n\to\infty} \gamma_n'' = 0$, we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = a,$$

and

$$\limsup_{n\to\infty} \|T^n z_n - p\| \leq \limsup_{n\to\infty} (1 + r_n) \|z_n - p\| = \lim_{n\to\infty} \|z_n - p\| = a.$$

Next, consider

$$\|T^n z_n - p + \gamma_n'(v_n - x_n)\| \leq \|T^n z_n - p\| + \gamma_n' \|v_n - x_n\| .$$

Thus

$$\limsup_{n\to\infty} \|T^n z_n - p + \gamma_n'(v_n - x_n)\| \leq a.$$

Also,

$$\|x_n - p + \gamma_n'(v_n - x_n)\| \leq \|x_n - p\| + \gamma_n' \|v_n - x_n\|$$

gives that

$$\limsup_{n\to\infty} \|x_n - p + \gamma'_n(v_n - x_n)\| \leq a,$$

and

$$\begin{aligned} a &= \lim_{n\to\infty} \|y_n - p\| \\ &= \lim_{n\to\infty} \|\alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &= \lim_{n\to\infty} \|\alpha'_n T^n z_n + (1-\alpha'_n)x_n - \gamma'_n x_n + \gamma'_n v_n - (1-\alpha'_n)p - \alpha'_n p\| \\ &= \lim_{n\to\infty} \|\alpha'_n[T^n z_n - p + \gamma'_n(v_n - x_n)] + (1-\alpha'_n)[x_n - p + \gamma'_n(v_n - x_n)]\| \end{aligned}$$

By J.Schu's Lemma [14], we have

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0.$$

Again, note that

$$\|x_{n+1} - x_n\| \leq \alpha_n \|T^n y_n - x_n\| + \gamma_n \|w_n - x_n\| \to 0, \text{ as } n \to \infty.$$

Thus,

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ &\leq (1+r_n)\|x_n - y_n\| + \|T^n y_n - x_n\| \\ &\leq (1+r_n)[\alpha'_n \|x_n - T^n z_n\| + \gamma'_n \|v_n - x_n\|] + \|T^n y_n - x_n\| \\ &\quad \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now, we have

$$\begin{aligned} \|x_n - Tx_n\| \leq\; & \|x_{n+1} - x_n\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| \\ & + \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| + \left\|T^{n+1}x_n - Tx_n\right\|. \end{aligned}$$

Since T is uniformly L-Lipschitzian, we obtain that

$$\begin{aligned} \|x_n - Tx_n\| \leq\; & \|x_{n+1} - x_n\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| \\ & + L\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\| \end{aligned}$$

using the above inequality, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof.

Theorem 2.4: Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be completely continuous uniformly L-Lipschitzian asymptotically quasi nonexpansive mapping with sequence $\{r_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$. From an arbitrary $x_0 \in C$, let $\{x_n\}$ be the sequence defined by (1.0.2), where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ with the following restrictions:

(i)$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T.

Proof: By Lemma 1.3(a), $\{x_n\}$ is bounded. It follows by our assumption that T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to p \in C$ as $k \to \infty$. Moreover, by Theorem 2.3, we have $\|Tx_{n_k} - x_{n_k}\| \to 0$ as $k \to \infty$ which implies that $x_{n_k} \to p$ as $k \to \infty$. Again by Theorem 2.3, we have

$$\|p - Tp\| = \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

It follows that $p \in F(T)$. Furthermore, since $\lim_{n\to\infty} \|x_n - p\|$ exists. Therefore $\lim_{n\to\infty} \|x_n - p\| = 0$, that is $\{x_n\}$ converges to some fixed point of T.

Now, we have

$$\|y_n - x_n\| \le \alpha'_n \|T^n z_n - x_n\| + \gamma'_n \|v_n - x_n\| \to 0 \text{ as } n \to \infty,$$

and

$$\|z_n - x_n\| \le \alpha''_n \|T^n x_n - x_n\| + \gamma''_n \|u_n - x_n\| \to 0 \text{ as } n \to \infty.$$

Therefore $\lim_{n\to\infty} y_n = p = \lim_{n\to\infty} z_n$. Thus $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to some fixed point of T. This completes the proof.

Remark 2.5: Theorem 2.4 extend Theorem 2 and 3 of Rhoades [13] and Theorem 1.5 of Schu [15] to the case of more general class of asymptotically nonexpansive mapping and three step iteration scheme with errors considered here and no boundedness condition imposed on C.

Theorem 2.6: Let E be a real uniformly convex Banach space and C be a nonempty compact subset of E. Let $T\colon C \to C$ be (L,α) uniformly Lipschitz asymptotically quasi nonexpansive mapping with sequence $\{r_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_n < \infty$. From an arbitrary $x_0 \in C$, let $\{x_n\}$ be the sequence defined by (1.0.2), where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are real sequences in $[0,1]$ with the following restrictions:

(i)$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$ and $\lim_{n\to\infty} \alpha'_n = 0$.

Then $\{x_n\}$ converges strongly to some fixed point of T.

Proof: By Schauder's fixed point theorem, we obtain $F(T) \neq \emptyset$ and by Lemma 1.3(a), we have $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geq 0$. From (1.0.6), we have

$$\|y_n - p\| \leq (1 + r_n)^2 \|x_n - p\| + A_n, \ \forall n \geq 1.$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = a.$$

Note that

$$\limsup_{n\to\infty} \|T^n y_n - p\| \leq \limsup_{n\to\infty} (1 + r_n) \|y_n - p\| = \limsup_{n\to\infty} \|y_n - p\| \leq a.$$

Next, consider

$$\|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq \|T^n y_n - p\| + \gamma_n \|w_n - x_n\|.$$

Thus,

$$\limsup_{n\to\infty} \|T^n y_n - p + \gamma_n(w_n - x_n)\| \leq a.$$

Also,

$$\|x_n - p + \gamma_n(w_n - x_n)\| \leq \|x_n - p\| + \gamma_n \|w_n - x_n\|,$$

gives that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(w_n - x_n)\| \leq a,$$

and

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \lim_{n\to\infty} \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n T^n y_n + (1-\alpha_n)x_n - \gamma_n x_n + \gamma_n w_n - (1-\alpha_n)p - \alpha_n p\| \\ &= \lim_{n\to\infty} \|\alpha_n [T^n y_n - p + \gamma_n(w_n - x_n)] + (1-\alpha_n)[x_n - p + \gamma_n(w_n - x_n)]\|. \end{aligned}$$

By J.Schu's Lemma [14], we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0. \tag{2.0.11}$$

Since E is compact, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.
Let

$$\lim_{k\to\infty} x_{n_k} = p. \tag{2.0.12}$$

Then from equation (2.0.11) and $\lim_{n\to\infty} \gamma_n = 0$, we have

$$\begin{aligned} \|x_{n_k+1} - x_{n_k}\| &\leq \alpha_{n_k} \|T^{n_k} y_{n_k} - x_{n_k}\| + \gamma_{n_k} \|w_{n_k} - x_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{2.0.13}$$

Note that $\lim_{n\to\infty} \alpha'_n = 0$, $\lim_{n\to\infty} \gamma'_n = 0$, therefore we have

$$\begin{aligned} \|y_n - x_n\| &\leq \alpha'_n \|T^n z_n - x_n\| + \gamma'_n \|v_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.0.14}$$

Thus from (2.0.11) and (2.0.12), we have

$$\lim_{k\to\infty} T^{n_k} y_{n_k} = p. \tag{2.0.15}$$

Thus $\lim_{k\to\infty} x_{n_k+1} = p$. Similarly, $\lim_{k\to\infty} x_{n_k+2} = p$, and

$$\lim_{k\to\infty} T^{n_k+1} y_{n_k+1} = p. \tag{2.0.16}$$

From (2.0.11) - (2.0.16), we have

$$\begin{aligned}
0 &\leq \|p - Tp\| \\
&= \|p - T^{n_k+1} y_{n_k+1} + T^{n_k+1} y_{n_k+1} - T^{n_k+1} x_{n_k+1} + T^{n_k+1} x_{n_k+1} \\
&\quad - T^{n_k+1} x_{n_k} + T^{n_k+1} x_{n_k} - T^{n_k+1} y_{n_k} + T^{n_k+1} y_{n_k} - Tp\| \\
&\leq \|p - T^{n_k+1} y_{n_k+1}\| + \|T^{n_k+1} y_{n_k+1} - T^{n_k+1} x_{n_k+1}\| + \|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\| \\
&\quad + \|T^{n_k+1} x_{n_k} - T^{n_k+1} y_{n_k}\| + \|T^{n_k+1} y_{n_k} - Tp\| \\
&\leq \|p - T^{n_k+1} y_{n_k+1}\| + L\|y_{n_k+1} - x_{n_k+1}\|^\alpha + L\|x_{n_k+1} - x_{n_k}\|^\alpha \\
&\quad + L\|x_{n_k} - y_{n_k}\|^\alpha + L\|T^{n_k} y_{n_k} - p\|^\alpha \\
&\quad \to 0.
\end{aligned}$$

This shows that p is a fixed point of T. Since the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to p from (2.0.12), we have $\lim_{n\to\infty} x_n = p$ from Theorem LQ [11]. Thus $\{x_n\}$ converges strongly to some fixed point p of T. This completes the proof.

Remark 2.7: (i) Theorem 2.6 extends the corresponding result of Xu and Noor [19] to the case of more general class of asymptotically nonexpansive mapping and three step iteration scheme with errors considered here.

(ii) Theorem 2.6 also extends the corresponding result of Qihou [12] to the case of three step iteration scheme considered here.

(iii) Theorem 2.6 also extends corollary 3.6 of Shahzad and Udomene [17] to the case of more general class of continuous asymptotically quasi nonexpansive mapping and three step iteration scheme with errors considered here.

References

[1] Chidume C.E., *Nonexpansive mappings, generalizations and iterative algorithms*. In: Agarwal R.P., O'Reagan D.eds. Nonlinear Analysis and Application. To V. Lakshmikantam on his 80th Birthday (Research Monograph), Dordrecht: Kluwer Academic Publishers, pp. 383-430.

[2] Goebel K. and Kirk W.A., *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35(1972), no.1, 171-174.

[3] Goebel K. and Kirk W.A., *A fixed point theorem for transformations whose iterates have uniform Lipschitz constant*, Studia Mathematica 47(1973), 135-140.

[4] Glowinski R. and Le Tallec P., *"Augemented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics"* Siam, Philadelphia, (1989).

[5] Haubruge S., Nguyen V.H. and Strodiot J.J., *Convergence analysis and applications of the Glowinski Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl. 97(1998), 645-673.

[6] Ishikawa S., *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. 44(1974), 147-150.

[7] Mann W.R., *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4(1953), 506-510.

[8] Noor M.A., *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. 251(2000), 217-229.

[9] Noor M.A., *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl. 255(2001).

[10] Qihou L., *Iterative sequences for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. 259(2001), 1-7.

[11] Qihou L., *Iterative sequences for asymptotically quasi-nonexpansive mapping with error member*, J. Math. Anal. Appl. 259(2001), 18-24.

[12] Qihou L., *Iterative sequences for asymptotically quasi-nonexpansive mapping with an error member of uniformly convex Banach space*, J. Math. Anal. Appl. 266(2002), 468-471.

[13] Rhoades B.E., *Fixed point iteration for certain nonlinear mappings*, J. Math. Anal. Appl. 183(1994), 118-120.

[14] Schu J., *Iterative construction of fixed points of strictly quasicontractive mappings*, Appl. Anal. 40(1991), 67-72.

[15] Schu J., *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 158(1991), 407-413.

[16] Senter H.F. and Dotson W.G., *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. 44(1974), 375-380.

[17] Shahzad N. and Udomene A., *Approximating common fixed points of two asymptotically quasi nonexpansive mappings in Banach spaces*, Fixed point Theory and Applications 2006(2006), Article ID 18909, Pages 1-10.

[18] Tan K.K. and Xu H.K., *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178(1993), 301-308.

[19] Xu B.L. and Noor M.A., *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 267(2002), No.2, 444-453.

Address

Department of Mathematics & Information Technology, Govt. Nagarjun P.G. College of Science , Raipur (C.G.), India
E-mail: saluja_1963@rediffmail.com