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S-HYPERCYCLIC, MIXING AND CHAOTIC STRONGLY CONTINUOUS SEMIGROUPS

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Abstract

We introduce and extensively study S-hypercyclic semigroups whose index set is an appropriate sector of the complex plane. Concerning supercyclicity, chaoticity and weakly mixing properties of strongly continuous semigroups, we supplement many structural results proved by other authors. Plenty of various examples of translation S-hypercyclic semigroups and S-hypercyclic semigroups induced by semiflows illustrates our theoretical approach.

1 Introduction

Throughout this paper, we assume that X is a separable infinite-dimensional Fréchet space and that E is a sequentially complete barreled locally convex space over the field $\mathbb{K} \in \{\mathbb{R},\ \mathbb{C}\}$. The Montel property of the space X plays a crucial role in some statements concerning S-hypercyclicity of operator semigroups and every employment of this property will be explicitly quoted. We assume that the topology of X is induced by the fundamental system $(p_n)_{n\in\mathbb{N}}$ of increasing seminorms. Then the translation invariant metric d, defined by:

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)},\ x,\ y \in X, \tag{1}$$

satisfies, among many other properties, the following: $d(x+u, y+v) \leq d(x,y) + d(u,v)$ and $d(cx, cy) \leq (|c|+1)d(x,y),\ c \in \mathbb{K},\ x,\ y,\ u,\ v \in X$. In what follows, we designate by $L(X)$ the space of all bounded, linear operators on X and assume that S is a non-empty closed subset of $\mathbb{K}$ satisfying $\mathrm{S} \setminus \{0\} \neq \emptyset$. In order to simplify the notation, we will simply set $\inf \mathrm{S} := \inf\{|s| : s \in \mathrm{S}\}$ and $\sup \mathrm{S} := \sup\{|s| : s \in \mathrm{S}\}$.
A continuous mapping $T : X \to X$ is said to be *hypercyclic* if there exists an element $x \in X$ whose orbit $\mathrm{Orb}(x, T) := \{T^n x \mid n \in \mathbb{N}_0\}$ is dense in X while

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T is said to be *topologically transitive* iff for for any pair of open non-empty sets $U,\ V$ of X there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. In our framework, T is hypercyclic iff T is topologically transitive. More generally, a sequence (T_n) in $L(X)$ is called *hypercyclic* if there exists $x \in X$ so that its *orbit under* (T_n), defined by $\{T_n x : n \in \mathbb{N}_0\}$, is dense in X while (T_n) is said to be *hereditarily hypercyclic* [10] if every subsequence of (T_n) is hypercyclic. A continuous mapping $T : X \rightarrow X$ is said to be *chaotic* if T is hypercyclic and the set of periodic points of T, defined by $\{x \in X :$ there exists $n \in \mathbb{N}$ such that $T^n x = x\}$, is dense in X.
Let $\alpha \in (0, \frac{\pi}{2}]$, $\delta > 0$ and $s \in \mathbb{R}$. Define $\Delta(\alpha) := \{re^{i\theta} : r \geq 0,\ \theta \in [-\alpha, \alpha]\}$ and suppose $\Delta \in \{[0, \infty),\ \mathbb{R},\ \mathbb{C}\}$ or $\Delta = \Delta(\alpha)$ for an appropriate $\alpha \in (0, \frac{\pi}{2}]$. Further on, put $\Delta_\delta := \{z \in \Delta : |z| \leq \delta\}$, $\lfloor s \rfloor := \sup\{k \in \mathbb{Z} : k \leq s\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : k \geq s\}$. For a closed, linear operator A acting on E, we denote by $\mathrm{R}(A)$, $\rho(A)$ and $\sigma(A)$ its range, resolvent set and spectrum, respectively; by $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ are denoted the point, continuous and residual spectrum of A, respectively.
An operator family $(T(t))_{t \in \Delta}$ $(T(t) \in L(E),\ t \in \Delta)$ is a strongly continuous semigroup if:

(i) $T(0) = I$,

(ii) $T(t + s) = T(t)T(s),\ t,\ s \in \Delta$ and

(iii) the mapping $t \mapsto T(t)x,\ t \in \Delta$ is continuous for every fixed $x \in E$.

We refer the reader to [2], [11], [26], [41], [43], [46], [55], [64]-[65] and [68] for the basic theory of semigroups of operators in locally convex spaces. It is said that the strongly continuous semigroup $(T(t))_{t \in \Delta}$ is:

(i) *hypercyclic*, if there exists $x \in E$ whose orbit $\mathrm{Orb}(x, T) := \{T(t)x : t \in \Delta\}$ is dense in E. Such an element is called a *hypercyclic vector* for $(T(t))_{t \in \Delta}$ and $\mathrm{HC}(T)$ denotes the set of all hypercyclic vectors for $(T(t))_{t \in \Delta}$,

(ii) *chaotic*, if $(T(t))_{t \in \Delta}$ is hypercyclic and the set of *periodic points* of $(T(t))_{t \in \Delta}$, defined by $\{x \in E : T(t_0)x = x$ for some $t_0 \in \Delta \setminus \{0\}\}$, is dense in E,

(iii) *topologically transitive*, if for every pair of open non-empty sets $U,\ V$ of E, there exists $t \in \Delta$ such that $T(t)U \cap V \neq \emptyset$,

(iv) *topologically mixing*, if for every pair of open non-empty sets $U,\ V$ of E, there exists $t_0 \in \Delta$ such that $T(t)U \cap V \neq \emptyset$ for every $t \in \Delta$ with $|t| \geq |t_0|$,

(v) *weakly mixing*, if the semigroup $(T \oplus T(t))_{t \in \Delta}$ is topologically transitive in $E \oplus E$, where $T \oplus T(t)(x, y) := (T(t)x, T(t)y),\ x,\ y \in E,\ t \in \Delta$,

(vi) *supercyclic*, if there exists $x \in E$ such that the *projective orbit* $\{cT(t)x : c \in \mathbb{K},\ t \in \Delta\}$ is dense in E; $\mathrm{SHC}(T)$ denotes the set of all $x \in E$ whose projective orbit is dense in X,

(vii) *positively supercyclic*, if there exists $x \in E$ such that its *positive projective orbit* $\{cT(t)x : c \in [0, \infty),\ t \in \Delta\}$ is dense in E; $\mathrm{SHC}_{\mathrm{pos}}(T)$ denotes the set of all $x \in E$ whose positive projective orbit is dense in X,

(viii) S-*hypercyclic*, if there exists $x \in E$ such that that its S-*projective orbit* $\{cT(t)x : c \in \mathrm{S},\ t \in \Delta\}$ is dense in E; $\mathrm{HC}_{\mathrm{S}}(T)$ denotes the set of all $x \in E$ whose S-projective orbit is dense in X,

(ix) S-*topologically transitive*, if for every pair of open non-empty sets $U,\ V$ of X, there exist $c \in \mathrm{S}$ and $t \in \Delta$ such that $cT(t)U \cap V \neq \emptyset$,

(x) *uniformly continuous*, if the mapping $t \mapsto T(t) \in L(E),\ t \in \Delta$ is continuous, where we assume that $L(E)$ is endowed with the strong operator topology,

(xi) *analytic semigroup of angle* $\alpha \in (0, \frac{\pi}{2}]$, if the mapping $t \mapsto T(t),\ t \in (\Delta(\alpha))^\circ$ is analytic ($\Delta = \Delta(\alpha)$),

(xii) *locally equicontinuous*, if for any $r > 0$, the family $\{T(t) : t \in \Delta_r\}$ is equicontinuous.

Chronologically, the first examples of hypercyclic operators were given on the space $\mathrm{H}(\mathbb{C})$ of entire functions equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. In 1929, G. D. Birkhoff proved that the translation operator $f \mapsto f(\cdot + a),\ f \in \mathrm{H}(\mathbb{C}),\ a \in \mathbb{C} \setminus \{0\}$ is hypercyclic in $\mathrm{H}(\mathbb{C})$ while the hypercyclicity of the derivative operator $f \mapsto f',\ f \in \mathrm{H}(\mathbb{C})$ was proved by G. R. MacLane in 1952. In 1969, S. Rolewicz [60] presented the first example of a hypercyclic operator on the Banach space $l^2(\mathbb{N})$. The first examples of chaotic semigroups were given by C. R. MacCluer [50] and V. Protopopescu, Y. Azmy [59] in 1992.
Let us recollect some well-known assertions concerning the existence of such classes of semigroups. The recent result of J. A. Conejero [13] says that every separable infinite-dimensional Fréchet space, except the space $\omega := \prod_{n \in \mathbb{N}} \mathbb{K}$, admits a hypercyclic semigroup and an interesting result of T. Bermúdez, A. Bonilla, J. A. Conejero and A. Peris [4] shows that every separable infinite-dimensional Banach space admits a topologically mixing, analytic semigroup of angle $\frac{\pi}{2}$. Further on, L. Bernal-Gonzáles and K.-G. Grosse Erdmann [8] have recently proved that every separable infinite-dimensional Banach space admits a uniformly continuous, weakly mixing semigroup. The existence of chaotic and supercyclic semigroups on locally convex spaces is more delicate: T. Bermúdez, A. Bonilla and A. Martinón [7] proved that there exists a separable infinite-dimensional Banach space which does not admit a chaotic semigroup and, due to L. Bernal-Gonzáles and K.-G. Grosse Erdmann [8], we know that the space $\varphi := \bigoplus_{n \in \mathbb{N}} \mathbb{K}$ does not admit a supercyclic semigroup. A slight modification of the arguments employed in [4] and [8] shows that, for every complex infinite-dimensional separable Banach space E, there exists a bounded linear operator $A \in L(E \oplus \mathbb{C})$ which generates an analytic supercyclic semigroup $(T(z))_{z \in \mathbb{C}}$ in $E \oplus \mathbb{C}$ such that the operator $T(1)$ is not supercyclic.

This paper is organized as follows. In Section 2, we introduce the notion of S-hypercyclicity of strongly continuous semigroups and characterize basic structural properties of S-hypercyclic semigroups whose index set is an appropriate sector of the complex plane. The concept of S-hypercyclicity of strongly continuous semigroups is meaningful and does not coincide with hypercyclicity, resp. positive supercyclicity, if $\sup \mathrm{S} < \infty$, resp. $\sup \mathrm{S} = \infty$. The important relationship between S-topological transitivity and S-hypercyclicity of a strongly continuous semigroup $(T(t))_{t\in\Delta}$ is presented in Theorem 1 and Theorem 2. In Theorem 4 and Theorem 6, we transfer several assertions proved by W. Desch, W. Schappacher and G. F. Webb in their systematic exposition [28] to S-hypercyclic semigroups in Fréchet spaces. The spectral mapping theorem for strongly continuous semigroups in locally convex spaces (cf. Theorem 5) is multi-functionally used in the paper and its proof follows by making use of the arguments given in the monographs of K. J. Engel, R. Nagel [32] and A. Pazy [56]. In order to better explain the importance of Theorem 5 in our research, we begin with recalling of a profound result of J. A. Conejero, V. Müler and A. Peris [17] which states that a strongly continuous semigroup $(T(t))_{t\geq 0}$ in a separable Fréchet space is hypercyclic iff every its single operator $T(t),\ t > 0$ is hypercyclic. On the other hand, F. Bayart and A. Bonilla have recently proved that every single operator $T(t),\ t > 0$ of a chaotic strongly continuous semigroup $(T(t))_{t\geq 0}$ need not be chaotic itself. In the case when all suppositions quoted in the formulation of [28, Theorem 3.1] hold, T. Kalmes proved that $T(t)$ must be chaotic for all $t > 0$ (cf. [39, Theorem 4.9, Corollary 4.10]). We transfer the above assertions to chaotic semigroups in complex separable Fréchet spaces by means of Theorem 5. Furthermore, Theorem 5 is essentially applied in proving Proposition 8 (cf. also Proposition 9) which states that the infinitesimal generator A as well as every single operator of a hypercyclic semigroup $(T(t))_{t\in\Delta}$ (Δ is either $[0,\infty)$ or $\mathbb{R}$) possesses the empty residual spectrum. The third section is devoted to the extensive study of weakly mixing semigroups. It turns out that K.-G. Grosse Erdmann's collapse/blow-up version of the Hypercyclicity Criterion for single operators and operator semigroups (cf. [8, Definition 2.1] and [9]) presents a natural framework for investigation of weakly mixing semigroups whose index set is an appropriate sector of the complex plane. In Theorem 12, we recollect several results obtained by J. A. Conejero, A. Peris [14], L. Bernal-Gonzáles, K.-G. Grosse Erdmann [9] and T. Kalmes [39] concerning weakly mixing properties of strongly continuous semigroups whose index set $\Delta = [0,\infty)$. The main result of Section 3 is Theorem 13 which almost completely describes weakly mixing semigroups $(T(t))_{t\in\Delta}$ in the case $\Delta \neq [0,\infty)$. Concerning hypercyclicity of products of strongly continuous semigroups whose index set is $[0,\infty)$, it is worthwhile to point out that W. Desch and W. Schappacher [27] have recently introduced a strengthened version of the Hypercyclicity Criterion (cf. [27, Definition 2.1, Proposition 2.2]) called by authors the Recurrent Hypercyclicity Criterion. Although the analysis of strongly continuous semigroups $(T(t))_{t\in\Delta},\ \Delta \neq [0,\infty)$ which satisfy the Recurrent Hypercyclicity Criterion falls out from the framework of this paper, we present a slight modification of [28, Example 4.11] which shows that the Recurrent Hypercyclicity Criterion is strictly stronger than the Hypercyclicity Criterion. The inheritance law for the Hypercyclicity Criterion (cf. [10] and [27, Proposition 3.1]) is clarified in Proposition 16. The main objective in Section 4 is to profile S-hypercyclic translation semigroups as well as S-hypercyclic strongly continuous semigroups induced by semiflows on various kinds of weighted function spaces. In this section, we also continue the researches of M. Matsui, M. Yamada, F. Takeo [51], [53], F. Takeo [62]-[63], T. Kalmes [38, Section 4] and transfer the assertion of the Positive Supercyclicity Theorem (cf. [48, Theorem 1]) proved by F. León-Saavedra, V. Müler to operator semigroups in complex Fréchet spaces. Chaoticity and mixing properties of the translation semigroup $(T(t))_{t\in\Delta}$ on the Fréchet space $C^m(\Delta,\mathbb{K})$, $m\in\mathbb{N}_0\cup\{\infty\}$ are proved in Example 27 with the help of extension type theorems for continuously differentiable functions and the Whitney extension theorem.

2 S-hypercyclic, S-topologically transitive and chaotic semigroups

We start this section with the following fundamental assertion whose proof is omitted.

Theorem 1. *Let $(T(t))_{t\in\Delta}$ be a strongly continuous semigroup in X. Then the following assertions are equivalent:*

(i) $(T(t))_{t\in\Delta}$ *is S-topologically transitive.*

(ii) $(T(t))_{t\in\Delta}$ *is S-hypercyclic and $HC_S(T)$ is a dense subset of X.*

(iii) For every $y,\ z\in X$ and $\varepsilon>0$, there exist $c\in S$, $t\in\Delta$ and $v\in X$ so that $d(y,v)<\varepsilon$ and $d(z,cT(t)v)<\varepsilon$.

(iv) For every $\varepsilon>0$, there exists a dense subset D of X such that for every $z\in D$ there exists a dense subset D' of X such that for every $y\in D'$ there exist $c\in S$, $t\in\Delta$ and $v\in X$ so that $d(y,v)<\varepsilon$ and $d(z,cT(t)v)<\varepsilon$.

If any of conditions (i)-(iv) holds, then $HC_S(T)$ is a dense G_δ-subset of X.

The next theorem is inspired by [8, Theorem 5.1], [16, Remark 1], [51, Lemma 1, Theorem 1] and the analysis given on [14, p. 771]. Notice only that the local equicontinuity of $(T(t))_{t\in\Delta}$ automatically holds provided that $\Delta\in\{[0,\infty),\ \mathbb{R}\}$ ([43], [64]).

Theorem 2. *(i) Suppose S is bounded, $\alpha\in(0,\frac{\pi}{2})$, $\Delta\in\{[0,\infty),\ \Delta(\alpha)\}$, $(T(t))_{t\in\Delta}$ is an S-hypercyclic strongly continuous semigroup in X and $x\in HC_S(T)$. Then the set $\overline{\{cT(t)x : c\in S,\ t\in\Delta\setminus\Delta_s\}}$ is dense in X for all $s>0$. If $\Delta=\Delta(\alpha)$ and $z\in(\partial\Delta)\setminus\{0\}$, then $\overline{\{cT(t)x : c\in S,\ t\in z+\Delta\}}=X$ or $\overline{\{cT(t)x : c\in S,\ t\in\overline{z}+\Delta\}}=X$.*

(ii) Suppose $\Delta\in\{\mathbb{R},\ \mathbb{C}\}$. Then the S-hypercyclicity of a strongly continuous $(T(t))_{t\in\Delta}$ is equivalent to its S-topological transitivity. The previous statement remains true if $\Delta=[0,\infty)$ and S is bounded.

(iii) Suppose S is bounded, $\Delta = \Delta(\alpha)$ for some $\alpha \in (0, \frac{\pi}{2}]$, $(T(t))_{t\in\Delta}$ is a strongly continuous semigroup in X and the set

$$\{cT(t)x : c \in S,\ t \in \Delta(\beta)\} \tag{2}$$

is dense in X for an appropriate $\beta \in (0, \alpha)$ and an $x \in X$. Then the semigroup $(T(t))_{t\in\Delta}$ is S-topologically transitive.

(iv) Suppose $\Delta = [0, \infty)$ and X is not a Montel space. Then the S-hypercyclicity of a strongly continuous semigroup $(T(t))_{t\in\Delta}$ is equivalent to its S-topological transitivity.

Proof. Let $s > 0$ be fixed. We will prove $\overline{\{cT(t)x : c \in \mathrm{S},\ t \in \Delta \setminus \Delta_s\}} = X$ only in the case $\Delta = \Delta(\alpha)$ since the consideration is similar if $\Delta = [0, \infty)$. Put $\Delta_{s,1} := \{t \in \Delta : \mathrm{Re}(t) \geq s\}$. Then the strong continuity of $(T(t))_{t\in\Delta}$ implies:

$$\overline{\{T(t)X : t \in \Delta_{s,1}\}} \subset \overline{\{cT(t)x : c \in \mathrm{S},\ t \in \Delta_{s,1}\}}. \tag{3}$$

Using again the strong continuity of $(T(t))_{t\in\Delta}$, we obtain that the set $\{cT(t)x : c \in \mathrm{S},\ t \in (\Delta_{s,1})^c\}$ is bounded. Let p be a continuous seminorm on X and let M be a positive real number so that $p(x) \neq 0$ and that $p(cT(t)x) < M,\ c \in \mathrm{S},\ t \in (\Delta_{s,1})^c$. Certainly, there exists $k \in (0, \infty)$ and a sequence (c_n) in S such that $p(kx) > M,\ \lim_{n\to\infty} c_n T(t_n)x = kx$ and $\lim_{n\to\infty} p(T(c_n t_n)x) = p(kx)$. Hence, there exists a subsequence (t_{n_m}) of (t_n) and a subsequence (c_{n_m}) of (c_n) satisfying $t_{n_m} \in \Delta_{s,1},\ m \in \mathbb{N},\ \lim_{m\to\infty} c_{n_m} T(t_{n_m})x = kx$ and $\lim_{m\to\infty} T(t_{n_m})(c_{n_m} \frac{x}{k}) = x$. Owing to (3), one concludes that $x \in \overline{\{cT(t)x : c \in \mathrm{S},\ t \in \Delta_{s,1}\}}$. Since $x \in \mathrm{HC}_{\mathrm{S}}(T)$, the previous inclusion yields that any open ball contains an element of the set $\{T(t)X : t \in \Delta_{s,1}\}$, and thanks to (3), we obtain that $\overline{\{cT(t)x : c \in \mathrm{S},\ t \in \Delta_{s,1}\}} = X$. Further on, let $z \in (\partial\Delta) \setminus \{0\}$ be fixed. Suppose

$$x_1 \notin \overline{\{cT(t)x : c \in \mathrm{S},\ t \in z + \Delta\}} = \overline{\{T(t)X : t \in z + \Delta\}}$$

and

$$x_2 \notin \overline{\{cT(t)x : c \in \mathrm{S},\ t \in \overline{z} + \Delta\}} = \overline{\{T(t)X : t \in \overline{z} + \Delta\}}.$$

The first part of proof shows that there exist a sequence (t_n) in $\Delta_{2|z|,1}$ and a sequence (c_n) in S such that $\lim_{n\to\infty} c_n T(t_n)x = x_1 + x_2$. On the other hand, there exist a sequence (t'_k) in $z + \Delta$ and a sequence (t''_k) in $\overline{z} + \Delta$ as well as two sequences (c'_k) and (c''_k) in S such that $\lim_{k\to\infty} c'_k T(t'_k)x = x_2$ and that $\lim_{k\to\infty} c''_k T(t''_k)x = x_1$. Without loss of generality, we may assume that there exists a subsequence (t_{n_k}) of (t_n) satisfying $t_{n_k} \in z + \Delta,\ k \in \mathbb{N}$. Hence, $x_1 = \lim_{k\to\infty}(c_{n_k} T(t_{n_k})x - c''_k T(t''_k)x) \in \overline{\{T(t)X : t \in z + \Delta\}}$ which is a contradiction. To prove (ii), suppose $(T(t))_{t\in\Delta}$ is S-hypercyclic, where either $\Delta \in \{\mathbb{R},\ \mathbb{C}\}$ or $\Delta = [0, \infty)$ and S is bounded. Taking into account (i), one obtains that the range of $T(t),\ t \in \Delta$ is dense in X and an application of [37, Theorem 1, Proposition 1], with $I = \{(c, t) : c \in \mathrm{S},\ t \in \Delta\}$, gives that $(T(t))_{t\in\Delta}$ is S-topologically transitive, as required. Let us prove (iii). The prescribed assumptions imply that $(T(t))_{t\in\Delta(\beta)}$ is an S-hypercyclic strongly continuous semigroup in X and an employment of (i) gives that the set $\{cT(t)x : c \in \mathrm{S}\backslash\{0\},\ t \in \Delta(\beta),\ |t| \geq r\}$ is dense in X for every $r > 0$. Let $t \in \Delta$ and $c \in \mathrm{S}\backslash\{0\}$ be fixed. Then there exists $R \in (0,\infty)$ such that $\{z \in \Delta(\beta) : \mathrm{Re}(z) \geq R\} \subset t + \Delta$. In conclusion, $\{c'T(s)x : c' \in \mathrm{S},\ s \in t+\Delta\}$ is dense in X, which implies that $\mathrm{R}(T(t))$ and $\mathrm{R}(cT(t))$ are also dense in X. Denote $I = \{(c,t) : c \in \mathrm{S}\setminus\{0\},\ t \in \Delta\}$ and put, for every $\tau = (c,t) \in I$, $T_\tau := cT(t)$. Apply [37, Theorem 1, Proposition 1] again to end the proof of (iii). In order to prove (iv), suppose that $(T(t))_{t\geq 0}$ is S-hypercyclic and that the set $\{cT(t)x : c \in \mathrm{S},\ t \geq 0\}$ is dense in X for some $x \neq 0$. We will slightly alter the arguments given in the proof [51, Lemma 1] in order to obtain that $T(t)x \neq 0,\ t \geq 0$ and that the set $\{cT(t)x : c \in \mathrm{S},\ t \geq s\}$ is dense in X for every $s \geq 0$. Suppose $t_0 = \min\{t \geq 0 : T(t)x = 0\}$; obviously, $t_0 > 0$. We will prove that for every $y \in X$ there exist $c \in \mathrm{S}$ and $t \in [0, t_0]$ such that $y = cT(t)x$. We consider only the non-trivial case $y \neq 0$. It is evident that there exist a sequence (t_n) in $[0, t_0]$ converging to some $t \in [0, t_0]$, and a sequence (c_n) in S so that $\lim_{n\to\infty} c_nT(t_n)x = y$. Assume first that $t = t_0$. In this case, we get

$$\begin{aligned} d(0, T(t_0 - t_n)y) &= d(0 + 0, T(t_0 - t_n)(y - c_nT(t_n)x) + T(t_0 - t_n)(c_nT(t_n)x)) \\ &\leq d(0, T(t_0 - t_n)(y - c_nT(t_n)x)) + d(0, T(t_0 - t_n)(c_nT(t_n)x)) \\ &= d(0, T(t_0 - t_n)(y - c_nT(t_n)x)), \end{aligned} \tag{4}$$

and moreover, the strong continuity of $(T(t))_{t\geq 0}$ implies $\lim_{n\to\infty} d(0, T(t_0 - t_n)y) = d(0, y)$. Let us show that $\lim_{n\to\infty} T(t_0 - t_n)(y - c_nT(t_n)x) = 0$. So let p be an arbitrary continuous seminorm on X. Using the equicontinuity of the family $\{T(t) : t \in [0, t_0]\}$, one obtains the existence of a continuous seminorm q on X such that:

$$p(T(t)x) \leq q(x),\ t \in [0, t_0],\ x \in X. \tag{5}$$

Due to (5), one yields $0 \leq p(T(t_0 - t_n)(y - c_nT(t_n)x)) \leq q(y - c_nT(t_n)x) \to 0$ as $n \to \infty$. This implies $\lim_{n\to\infty} T(t_0 - t_n)(y - c_nT(t_n)x) = 0$ and one can employ (4) to conclude that $d(0, y) = 0$, i.e., $y = 0$, which is a contradiction. Suppose now $t < t_0$. Then $T(t)x \neq 0$ and there exists a continuous seminorm p on X so that $p(T(t)x) \neq 0$. Since $p(T(t_n)x) \to p(T(t)x) \neq 0$ and $|c_n|p(T(t_n)x) \to p(T(t)y)$ as $n \to \infty$, we have the existence of an integer $n_0 \in \mathbb{N}$ and a positive real number m satisfying $p(T(t_n)x) \geq m$ and $|c_n|p(T(t_n)x) \leq p(T(t)y) + 1,\ n \geq n_0$. Therefore, $|c_n| \leq \frac{p(T(t)y)+1}{m},\ n \geq n_0$ and the closedness of S yields that there exist a subsequence (c_{n_k}) of (c_n) and a number $c \in \mathrm{S}$ satisfying $\lim_{k\to\infty} c_{n_k} = c$ and $y = cT(t)x$. Proceeding as in the proof of [51, Lemma 1], one obtains $T(t)x \neq 0,\ t \geq 0$, as required. Assume now that the set $\{cT(t)x : c \in \mathrm{S},\ t \geq s\}$ is not dense in X for some $s \geq 0$. Then there exists an open, bounded subset $U \subset X$ which fulfills:

$$U \cap \overline{\{cT(t)x : c \in \mathrm{S},\ t \geq s\}} = \emptyset \text{ and } U \subset \overline{\{cT(t)x : c \in \mathrm{S},\ t \in [0, s]\}}. \tag{6}$$

Let $t \in [0,s]$ be fixed. Since $T(t)x \neq 0$, there exists a continuous seminorm p_t on X such that $p_t(T(t)x) \neq 0$ and the strong continuity of $(T(t))_{t\geq 0}$ implies the existence of a number $\varepsilon_t \in (0,\infty)$ satisfying:

$$p_t(T(t')x) \neq 0,\ t' \in (t-\varepsilon_t, t+\varepsilon_t) \cap [0,s]. \tag{7}$$

Therefore, there exists a finite subset $\{t_1,\cdots,t_k\}$ of $[0,s]$ such that $[0,s] \subset \bigcup_{i=1}^{k} (t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i})$. Choose a continuous seminorm p on X so that $p \geq \max(p_{t_1},\cdots,p_{t_k})$. This implies $p(T(t)x) > 0,\ t \in [0,s]$ and the continuity of $t \mapsto p(T(t)x),\ t \in [0,s]$ allows one to deduce that there are positive real numbers m_1 and m_2 with:

$$m_1 < p(T(t)x) < m_2,\ t \in [0,s]. \tag{8}$$

The boundedness of U gives the existence of a positive real number M so that $p(u) < M,\ u \in U$. Let $u \in U$ be an arbitrary vector; clearly, there exist a sequence (t_n) in $[0,s]$ and a sequence (c_n) in S with $|c_n|p(T(t_n)x) \to p(u)$ as $n \to \infty$. Hence, there is an appropriate $n_0 \in \mathbb{N}$ which satisfies $|c_n| \leq \frac{M}{m_1},\ n \geq n_0$. This inequality enables one to see that:

$$U \subset \overline{\{cT(t)x : c \in \mathrm{S},\ |c| \leq \frac{M}{m_1},\ 0 \leq t \leq s\}}. \tag{9}$$

So $\overline{U}$ is a compact subset of X. Since X is not a Montel space, there exists a bounded set W such that $\overline{W}$ is not compact. In the meantime, there exist $u \in U$ and $\alpha > 0$ so that $\overline{W} \subset \alpha(-u+\overline{U})$, which is a compact set. Hence, $\overline{W}$ is compact and this is a contradiction. We have proved that $\mathrm{R}(cT(s)),\ c \in \mathrm{S}\setminus\{0\},\ s \geq 0$ is dense in X and the proof of (iv) follows from an application [37, Theorem 1, Proposition 1].

Herein the following questions arise immediately:

1. Suppose S is bounded and $\inf \mathrm{S} > 0$. Does the S-hypercyclicity of a strongly continuous semigroup $(T(t))_{t\in\Delta}$ reduce to its hypercyclicity?

2. Suppose S is bounded, $\Delta = \Delta(\alpha)$ for some $\alpha \in (0, \frac{\pi}{2}]$ and $(T(t))_{t\in\Delta}$ is an S-hypercyclic strongly continuous semigroup in X. Is $(T(t))_{t\in\Delta}$ S-topologically transitive?

As we will see in Section 4, the answers are affirmative in the case of translation semigroups and strongly continuous semigroups induced by semiflows.

We continue by introducing the following subsets of X which play an important role in the analysis of hypercyclic and chaotic semigroups in Banach spaces [28]:

X_0 : is the set of all $x \in X$ so that $\lim_{t\to\infty,\ t\in\Delta} T(t)x = 0$ and

X_∞ : is the set of all $x \in X$ such that for every $\varepsilon > 0$ there exist $\omega \in X$ and $t \in \Delta \setminus \{0\}$ satisfying $d(\omega,0) < \varepsilon$ and $d(T(t)\omega, x) < \varepsilon$.

Lemma 3. *Suppose $(T(t))_{t\in\Delta}$ is locally equicontinuous and $x \in X_\infty$. Then, for every $s > 0$ and $\varepsilon > 0$, there exist $\omega \in X$ and $t \in \Delta \setminus \Delta_s$ so that $d(\omega,0) < \varepsilon$ and $d(T(t)\omega, x) < \varepsilon$.*

Proof. The assertion is trivial if $x=0$. Suppose $x \in X_\infty \setminus \{0\}$; then one gets the existence of a sequence (t_n) in $\Delta \setminus \{0\}$ and a sequence (ω_n) in X such that $d(\omega_n, 0) < \frac{1}{n}$ and that $d(T(t_n)\omega_n, x) < \frac{1}{n}$. Hence, $\lim_{n\to\infty} \omega_n = 0$ and $\lim_{n\to\infty} T(t_n)\omega_n = x$. Let p be an arbitrary continuous seminorm on X. The assumption $|t_n| \leq s,\ n \in \mathbb{N}$ for some $s \in (0,\infty)$ and the local equicontinuity of $(T(t))_{t\in\Delta}$ imply the existence of a continuous seminorm q on X which fulfills $p(T(t)x) \leq q(x),\ t \in \Delta_s,\ x \in X$. In particular, $0 \leq p(T(t_n)\omega_n) \leq q(\omega_n),\ n \in \mathbb{N}$. Letting $n \to \infty$, one deduces $p(x) = 0$ and the arbitrariness of p yields $x = 0$. Therefore, given $\varepsilon > 0$, there exists a subsequence (t_{n_k}) of (t_n) satisfying $d(\omega_{n_k}, 0) < \frac{1}{n_k}$, $d(T(t_{n_k})\omega_{n_k}, x) < \frac{1}{n_k}$, $|t_{n_k}| > s$ and $n_k > \frac{1}{\varepsilon},\ k \in \mathbb{N}$. This completes the proof.

Having in mind Lemma 3, Theorem 1 and Theorem 2, one can repeat literally the argumentation given in the proofs of [28, Theorems 2.2, 2.3, 2.5; Remark 2.4] to verify the validity of the next theorem.

Theorem 4. *(i) Suppose that $(T(t))_{t\in\Delta}$ is locally equicontinuous. If X_0 and X_∞ are dense subsets of X, then $(T(t))_{t\in\Delta}$ is topologically transitive and $X_\infty = X$.*

(ii) Suppose $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group in X and $\{\frac{1}{y} : y \in S\setminus\{0\}\} \subset S$. Then $(T(t))_{t\geq 0}$ is S-topologically transitive iff $(T(-t))_{t\geq 0}$ is S-topologically transitive. If X is not a Montel space or S is bounded, then the preceding assertions are also equivalent to the existence of an element $x \in X$ such that both S-projective orbits $\{cT(t)x : c \in S,\ t \geq 0\}$ and $\{cT(-t)x : c \in S,\ t \geq 0\}$ are dense in X.

Suppose $\Delta \in \{[0,\infty),\ \mathbb{R}\}$. The infinitesimal generator A of a strongly continuous semigroup $(T(t))_{t\in\Delta}$ in E is defined by

$$D(A) := \{x \in E \mid \lim_{t\to 0,\ t\in\Delta} \frac{T(t)x - x}{t} \text{ exists}\} \text{ and}$$

$$Ax := \lim_{t\to 0,\ t\in\Delta} \frac{T(t)x - x}{t},\ x \in D(A).$$

We need the following useful extensions of [56, Theorems 2.4-2.6, p. 46-48]; see also [32, Chapter IV] and [45].

Theorem 5. *Suppose that a closed linear operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ in a complex sequentially complete barreled locally convex space E. Then:*

(i)
$$e^{t\sigma_p(A)} \subset \sigma_p(T(t)) \subset e^{t\sigma_p(A)} \cup \{0\},\ t \geq 0. \tag{10}$$

(ii) If $\lambda \in \sigma_r(A)$, $t > 0$ and $(\lambda + \frac{2\pi i\mathbb{Z}}{t}) \cap \sigma_p(A) = \emptyset$, then $e^{\lambda t} \in \sigma_r(T(t))$.

(iii) If $\lambda \in \mathbb{C},\ t > 0$ and $e^{\lambda t} \in \sigma_r(T(t))$, then $(\lambda + \frac{2\pi i\mathbb{Z}}{t}) \cap \sigma_p(A) = \emptyset$ and there exists $k \in \mathbb{Z}$ such that $\lambda_k := \lambda + \frac{2\pi ik}{t} \in \sigma_r(A)$.

(iv) If $\lambda \in \sigma_c(A)$, $t > 0$ and $(\lambda + \frac{2\pi i\mathbb{Z}}{t}) \cap (\sigma_p(A) \cup \sigma_r(A)) = \emptyset$, then $e^{\lambda t} \in \sigma_c(A)$.

Proof. We will prove only (i). Owing to [64, Theorem P.2], we know that A is a closed, densely defined operator which satisfies $T(t)A \subset AT(t),\ t \geq 0$. Furthermore, the following equality can be simply justified:

$$(A-\lambda)\int\limits_0^t e^{-\lambda s}T(s)xds = e^{-\lambda t}T(t)x - x,\ x \in E,\ \lambda \in \mathbb{C},\ t \geq 0. \tag{11}$$

Therefore, the assumption $Ax = \lambda x$, for some $\lambda \in \mathbb{C}$ and $x \in E \setminus \{0\}$, implies $T(t)x = e^{\lambda t}x,\ t \geq 0$; in other words, $e^{t\sigma_p(A)} \subset \sigma_p(T(t)),\ t \geq 0$. In order to prove the second spectral inclusion, we must adapt the arguments given in the proof of [56, Theorem 2.4, p. 46] since $\rho(A)$ can be the empty set (cf. [2, p. 164], [43] and [68, Definition 3.1, p. 12]). We consider only the non-trivial case $t > 0$. Suppose $T(t)x = e^{\lambda t}x$ for some $x \in E \setminus \{0\}$ and $\lambda \in \mathbb{C}$. It is clear that there exists $x^* \in X^*$ such that $x^*(x) \neq 0$. Further on, the function $f : [0,\infty) \to \mathbb{C}$ defined by $f(s) := x^*(e^{-\lambda s}T(s)x),\ s \geq 0$ is continuous and periodic with period t. Since the function $f(\cdot)$ does not vanish identically on $[0,\infty)$, one gets the existence of an integer $k \in \mathbb{Z}$ such that $\frac{1}{t}\int\limits_0^t e^{-\frac{2\pi iks}{t}}x^*(e^{-\lambda s}T(s)x)ds \neq 0$. This clearly implies:

$$x_k := \frac{1}{t}\int\limits_0^t e^{-\frac{2\pi iks}{t}}(e^{-\lambda s}T(s)x)ds \neq 0. \tag{12}$$

Define $\Omega := \mathbb{C} \setminus \{\lambda + \frac{2\pi ni}{t} : n \in \mathbb{Z}\}$ and the function $g : \Omega \to E$ by:

$$g(\eta) := (1 - e^{(\lambda-\eta)t})^{-1}\int\limits_0^t e^{-\eta s}T(s)xds,\ \eta \in \Omega. \tag{13}$$

As a matter of routine, one obtains $\int\limits_0^t e^{-\eta s}T(s)xds \in D(A)$ and:

$$A\int\limits_0^t e^{-\eta s}T(s)xds = e^{-\eta t}T(t)x - x + \eta\int\limits_0^t e^{-\eta s}T(s)xds,\ \eta \in \mathbb{C}. \tag{14}$$

Therefore, $g(\eta) \in D(A),\ \eta \in \Omega$ and

$$(\eta - A)g(\eta) = (1-e^{(\lambda-\eta)t})^{-1}[\eta\int\limits_0^t e^{-\eta s}T(s)xds - e^{-\eta t}T(t)x + x - \eta\int\limits_0^t e^{-\eta s}T(s)xds]$$

$$= (1 - e^{(\lambda-\eta)t})^{-1}(1 - e^{(\lambda-\eta)t})x = x,\ \eta \in \Omega.$$

By the definition of $g(\cdot)$, one directly sees that:

$$\lim_{\eta \to \lambda + \frac{2\pi k i}{t}} (\eta - \lambda - \frac{2\pi k i}{t}) g(\eta) = x_k \text{ and} \tag{15}$$

$$\lim_{\eta \to \lambda + \frac{2\pi k i}{t}} (\eta - \lambda - \frac{2\pi k i}{t})^2 g(\eta) = 0. \tag{16}$$

On the other hand, $(\lambda + \frac{2\pi ki}{t} - A)(\eta - \lambda - \frac{2\pi ki}{t}) g(\eta) = (\eta - \lambda - \frac{2\pi ki}{t})[(\lambda + \frac{2\pi ki}{t}) g(\eta) - Ag(\eta)] = (\eta - \lambda - \frac{2\pi ki}{t})[(\lambda + \frac{2\pi ki}{t}) g(\eta) - \eta g(\eta) + x] = (\eta - \lambda - \frac{2\pi ki}{t}) x - (\eta - \lambda - \frac{2\pi ki}{t})^2 g(\eta),\ \eta \in \Omega$. Hence, (16) implies:

$$\lim_{\eta \to \lambda + \frac{2\pi k i}{t}} (\lambda + \frac{2\pi k i}{t} - A)[(\eta - \lambda - \frac{2\pi k i}{t}) g(\eta)] = 0. \tag{17}$$

The closedness of A, (15) and (17) imply $x_k \in D(A)$, $(\lambda + \frac{2\pi ki}{t} - A) x_k = 0$, $Ax_k = (\lambda + \frac{2\pi ki}{t}) x_k$ and $\lambda + \frac{2\pi ki}{t} \in \sigma_p(A)$.

With Theorem 5 in view, one can simply prove the next generalizations of [28, Theorem 3.1] and [39, Theorem 4.9, Corollary 4.10].

Theorem 6. *Suppose X is a complex space, $(T(t))_{t \geq 0}$ is a strongly continuous semigroup in X generated by A and U is an open, non-empty, connected subset of $\mathbb{C}$ which intersects the imaginary axis. Suppose, further, that there exists a family $\{x_\lambda : \lambda \in U\}$ which satisfies the following properties:*

(i) $x_\lambda \in D(A) \setminus \{0\}$ *and* $Ax_\lambda = \lambda x_\lambda,\ \lambda \in U$.

(ii) For every $\phi \in X^*$, the function $F_\phi : U \to \mathbb{C}$, defined by $F_\phi(\lambda) := \phi(x_\lambda),\ \lambda \in U$ is analytic.

(iii) The function F_ϕ does not vanish identically on U unless $\phi = 0$.

Then $(T(t))_{t \geq 0}$ is chaotic and, for every $t_0 > 0$, the operator $T(t_0)$ is chaotic.

Proposition 7. *Suppose X is a complex space and $(T(t))_{t \geq 0}$ is a strongly continuous semigroup in X generated by A. If $\sigma_p(A)$ is an open, non-empty, connected subset of $\mathbb{C}$ such that there exists a family $\{x_\lambda : \lambda \in \sigma_p(A)\}$ which satisfies the properties (i)-(iii) given in the formulation of Theorem 6, then the following assertions are equivalent:*

(i) $T(t)$ is chaotic for every $t > 0$.

(ii) $(T(t))_{t \geq 0}$ is chaotic.

(iii) $(T(t))_{t \geq 0}$ possesses a non-trivial periodic point.

(iv) $\sigma_p(A)$ intersects the imaginary axis.

It could be of importance to state the following assertion which extends [17, Corollary 2.2], [28, Theorem 3.3] and [39, Corollary 4.11].

Proposition 8. *Suppose X is a complex space, $\Delta \in \{[0,\infty),\ \mathbb{R}\}$ and $(T(t))_{t\in\Delta}$ is a hypercyclic strongly continuous semigroup in X generated by A. Then:*

(i) $\sigma_r(A) = \emptyset$ *and* $\sigma_r(T(t)) = \emptyset,\ t \in \Delta \setminus \{0\}$.

(ii) *Suppose* $\Delta = \mathbb{R}$. *Then:*

(ii.1) $R(T(t) - \lambda I)$ *is dense in* X *for every* $t \in \mathbb{R} \setminus \{0\}$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$.

(ii.2) $R(T(t) - e^{\lambda t} I)$ *is dense in* X *for every* $t \in \mathbb{R} \setminus \{0\}$ *and* $\lambda \in \mathbb{C}$ *with*

$$(\lambda + \frac{2\pi i\mathbb{Z}}{t}) \cap \sigma_p(A) = \emptyset. \tag{18}$$

(iii.1) *If* $\Delta = [0,\infty)$, $(\alpha_1, \alpha_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, $t_1 \geq 0$, $t_2 \geq 0$ *and* $0 \leq t_1 < t_2$, *then* $R(\alpha_1 T(t_1) + \alpha_2 T(t_2))$ *is dense in* X.

(iii.2) *If* $\Delta = \mathbb{R}$, $(\alpha_1, \alpha_2) \in \mathbb{C}^2$, $|\alpha_1| = |\alpha_2| > 0$, $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$ *and* $t_1 \neq t_2$, *then* $R(\alpha_1 T(t_1) + \alpha_2 T(t_2))$ *is dense in* X.

(iv.1) *Let* Δ, (α_1, α_2) *and* (t_1, t_2) *be as in the formulation of (iii.1) and let* $x \in HC(T)$. *Then* $\alpha_1 T(t_1)x + \alpha_2 T(t_2)x \in HC(T)$.

(iv.2) *Let* Δ, (α_1, α_2) *and* (t_1, t_2) *be as in the formulation of (iii.2) and let* $x \in HC(T)$. *Then* $\alpha_1 T(t_1)x + \alpha_2 T(t_2)x \in HC(T)$.

Proof. We will prove (i) by making use of two different ideas. Suppose first $\Delta = [0,\infty)$ and $\lambda \in \sigma_r(A)$. Since $\mathrm{R}(\lambda I - A)$ is not dense in X, there exists a functional $x^* \in X^* \setminus \{0\}$ such that $x^*(Ax - \lambda x) = 0$ for all $x \in D(A)$. Due to (11), $x^*(e^{-\lambda t} T(t)x - x) = 0,\ t \geq 0,\ x \in D(A)$ and the denseness of A implies:

$$x^*(T(t)x - e^{\lambda t}x) = 0,\ t \geq 0,\ x \in X. \tag{19}$$

In conclusion, one yields that $\mathrm{R}(T(t) - e^{\lambda t} I)$ is not dense in X and this contradicts [17, Lemma 2.1]. Using again [17, Lemma 2.1], we obtain $\sigma_r(A) = \emptyset$ and (i) in the case $\Delta = [0,\infty)$. Suppose now $\Delta = \mathbb{R}$ and $\lambda \in \sigma_r(A)$. Then $\pm A$ generate strongly continuous semigroups $(T(\pm t))_{t\geq 0}$, and by the foregoing, we have the existence of a functional $x^* \in X^* \setminus \{0\}$ such that:

$$x^*(T(t)x - e^{\lambda t}x) = 0,\ t \in \mathbb{R},\ x \in X. \tag{20}$$

Suppose now $x \in \mathrm{HC}(T)$. Since $x^* \neq 0$, it must be surjective, and therefore, $\mathbb{C} = x^*(X) = \overline{\{x^*(T(t)x) : t \in \mathbb{R}\}} = \overline{\{e^{\lambda t} x^*(x) : t \in \mathbb{R}\}}$. If $x^*(x) = 0$, the contradiction is obvious; otherwise, $\overline{\{e^{\lambda t} : t \in \mathbb{R}\}} = \mathbb{C}$. This is again a contradiction since the mapping $t \mapsto e^{\lambda t},\ t \in \mathbb{R}$ is continuous and, for every $R \in (1,\infty)$, $\mathrm{card}(\{e^{\lambda t} : t \in \mathbb{R}\} \cap \{z \in \mathbb{C} : |z| = R\}) \leq 1$. Hence, $\sigma_r(A) = \emptyset$, and moreover, $0 \notin \sigma_r(T(t))$ since $T(t)$ is bijective for all $t \in \mathbb{R}$. Now the proof of (i) finishes an application of Theorem 5(iii). To prove (ii.1), one can repeat literally the argumentation used in the proof of [17, Lemma 2.1] while (ii.2) is a simple consequence of (i) as well as

Theorem 5(i). The proofs of (iii.1) and (iii.2) follow by using [17, Lemma 2.1], (ii) and decompositions

$$\alpha_1 T(t_1) + \alpha_2 T(t_2) = \alpha_2 T(t_1)[\frac{\alpha_1}{\alpha_2} I + T(t_2 - t_1)] \text{ if } \alpha_2 \neq 0 \text{ and:}$$

$$\alpha_1 T(t_1) + \alpha_2 T(t_2) = \alpha_1 T(t_1)[I + \frac{\alpha_2}{\alpha_1} T(t_2 - t_1)], \text{ if } \alpha_1 \neq 0,$$

while (iv.1) and (iv.2) follow automatically from (iii).

Before proceeding further, let us notice that the assertions (ii.1), (iii.1)-(iii.2) and (iv.1)-(iv.2) still hold in the case of real spaces.

Proposition 9. *Suppose* $\Delta = \Delta(\alpha)$ *for some* $\alpha \in (0, \frac{\pi}{2})$, X *is a complex space and* $(T(t))_{t\in\Delta}$ *is a hypercyclic strongly continuous semigroup in* X. *Denote by* A_β *the infinitesimal generator of* $(T(te^{i\beta}))_{t\geq 0}$, $\beta \in (-\alpha, \alpha)$ *and suppose that* $A_\beta = e^{i\beta}A$, $\beta \in (-\alpha, \alpha)$. *Then* $\sigma_r(A) \cap (-\Delta(\frac{\pi}{2} - \alpha) \cup \Delta(\frac{\pi}{2} - \alpha)) = \emptyset$.

Proof. Suppose $\lambda \in \sigma_r(A) \cap (-\Delta(\frac{\pi}{2} - \alpha) \cup \Delta(\frac{\pi}{2} - \alpha))$. Since $A_\beta = e^{i\beta}A$, $\beta \in (-\alpha, \alpha)$, one obtains the existence of a functional $x^* \in X^* \setminus \{0\}$ such that $x^*(A_\beta x - \lambda e^{i\beta}x) = 0$ for all $x \in D(A)$ and $\beta \in (-\alpha, \alpha)$. Arguing as in the proof of Proposition 8, we have that $x^*(T(te^{i\beta})x - e^{\lambda e^{i\beta}t}x) = 0$, $t \geq 0$, $x \in X$, $\beta \in (-\alpha, \alpha)$. The strong continuity of $(T(t))_{t\in\Delta}$ implies:

$$x^*(T(z)x - e^{\lambda z}x) = 0, \ z \in \Delta, \ x \in X. \tag{21}$$

Suppose $x \in \mathrm{HC}(T)$. The surjectiveness of x^*, (21) and the proof of Proposition 8 give $\overline{\{e^{\lambda z}x^*(x) : z \in \Delta\}} = \mathbb{C}$. Exclusion of the trivial case $x^*(x) = 0$ yields $\overline{\{e^{\lambda z} : z \in \Delta\}} = \mathbb{C}$. This equality and the assumption $\lambda \in -\Delta(\frac{\pi}{2} - \alpha) \cup \Delta(\frac{\pi}{2} - \alpha)$ imply $\overline{\{e^z : z \in \mathbb{C}, \ \mathrm{Re}(z) \geq 0\}} = \mathbb{C}$ or $\overline{\{e^z : z \in \mathbb{C}, \ \mathrm{Re}(z) \leq 0\}} = \mathbb{C}$. This is a contradiction.

Notice that the equality $A_\beta = e^{i\beta}A$, $\beta \in (-\alpha, \alpha)$ appearing in the formulation of preceding proposition automatically holds if, in addition, $(T(t))_{t\in\Delta}$ is an analytic semigroup of angle α (cf. [64, Theorem P.3] and [65, Theorem 4.3]).

3 Weakly mixing semigroups

The most common tool for proving hypercyclicity of single operators presents the well-known Hypercyclicity Criterion which was discovered independently by C. Kitai [40, Theorem 1.4] and R. M. Gethner-J. H. Shapiro [33, Theorem 2.2]. It turned out that this criterion is equivalent to the corresponding ones given by J. Bès, A. Peris (cf. [10] and [57, Theorem 1.1]), L. Bernal-Gonzáles, K.-G. Grosse Erdmann [9], [37] and F. León-Saavedra [47]. We also refer the reader to [5], [12], [14], [18], [20] and [34]-[35]. Such criteria possess natural reformulations in the theory of operator semigroups (cf. [4], [8], [12], [14], [23], [29] and [39]). Motivated by K.-G. Grosse

Erdmann's collapse/blow-up definition of hypercyclicity for single operators and operator semigroups (cf. [8, Definition 2.1] and [9]), we introduce the Hypercyclicity Criterion for strongly continuous semigroups whose index set is, in general, an appropriate sector of the complex plane.

Definition 10. *Suppose $(T(t))_{t\in\Delta}$ is a strongly continuous semigroup in X. It is said that $(T(t))_{t\in\Delta}$ satisfies the Hypercyclicity Criterion iff there exist dense subsets $Y,\ Z$ of X and a sequence (t_n) in Δ such that:*

(○) $\lim\limits_{n\to\infty} T(t_n)y = 0,\ y \in Y$ *and*

(○○) *for every $z \in Z$ there exists a sequence (u_n) in X such that $\lim\limits_{n\to\infty} u_n = 0$ and that $\lim\limits_{n\to\infty} T(t_n)u_n = z$.*

The proof of following auxiliary lemma is classical.

Lemma 11. *Suppose that a strongly continuous semigroup $(T(t))_{t\in\Delta}$ satisfies the Hypercyclicity Criterion. Then:*

(i) $(T\underbrace{\oplus\cdots\oplus}_{k}T)(t))_{t\in\Delta}$ *satisfies the Hypercyclicity Criterion for all $k \in \mathbb{N}$.*

(ii) $(T\underbrace{\oplus\cdots\oplus}_{k}T)(t))_{t\in\Delta}$ *is topologically transitive for all $k \in \mathbb{N}$.*

The next theorem is a recollection of results obtained by J. A. Conejero, A. Peris [14], L. Bernal-Gonzáles, K.-G. Grosse Erdmann [9] and T. Kalmes [39]. Let us remind [14] that a *backwards orbit* of x under $(T(t))_{t\geq 0}$ is a family $\{x_t : t \geq 0\}$ of elements of X so that $x_0 = x$ and that $T(t)x_s = x_{s-t}$ for all $s \geq t \geq 0$.

Theorem 12. *([9], [14], [39]) Suppose $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in X. Then the following assertions are equivalent:*

(i) There exist dense subspaces $Y,\ Z$ of X, a strictly increasing sequence (t_n) in $(0,\infty)$ with $\lim\limits_{n\to\infty} t_n = \infty$ and a family $\{S(t) : Z \to X \mid t \geq 0\}$ of linear (not necessarily continuous) mappings satisfying:

(i.1) $\lim\limits_{n\to\infty} T(t_n)y = 0,\ y \in Y$ *and*

(i.2) $\lim\limits_{n\to\infty} S(t_n)z = 0,\ z \in Z$ *and* $T(t)S(t)z = z,\ t \geq 0,\ z \in Z$.

(ii) There exist dense subspaces $Y,\ Z$ of X, a strictly increasing sequence (t_n) in $(0,\infty)$ with $\lim\limits_{n\to\infty} t_n = \infty$ and a family $\{S(t) : Z \to X \mid t \geq 0\}$ of linear (not necessarily continuous) mappings satisfying:

(ii.1) $\lim\limits_{n\to\infty} T(t_n)y = 0,\ y \in Y$ *and*

(ii.2) every $z \in Z$ admits a backwards orbit $\{z_t : t \geq 0\}$ such that $\lim\limits_{n\to\infty} z_{t_n} = 0$.

(iii) $(T(t))_{t\geq 0}$ *satisfies the Hypercyclicity Criterion with a strictly increasing sequence* (t_n) *in* $(0,\infty)$ *which fulfills* $\lim_{n\to\infty} t_n = \infty$.

(iv) *For every pair of open non-empty subsets* U, V *of* X *and for every zero neighborhood* W *in* X *there exists* $t > 0$ *so that* $T(t)U \cap W \neq \emptyset$ *and* $T(t)W \cap V \neq \emptyset$.

(v) *For every pair of open non-empty subsets* U, V *of* X, *there exists* $t > 0$ *such that* $T(t)U \cap V \neq \emptyset$ *and* $T(t+1)U \cap V \neq \emptyset$.

(vi) *There exists* $\alpha > 0$ *such that for every pair of open non-empty subsets* U, V *of* X, *there exists* $t > 0$ *such that* $T(t)U \cap V \neq \emptyset$ *and* $T(t+\alpha)U \cap V \neq \emptyset$.

(vii) *If* $I \subset [0,\infty)$ *is syndetic, i.e., there exists* $K > 0$ *such that* $[t, t+K] \cap I \neq \emptyset$ *for all* $t \geq 0$, *then the family* $\{T \oplus T(t) : t \in I\}$ *is topologically transitive.*

(viii) *There exists* $K \in (0,\infty)$ *such that for every* $I \subset [0,\infty)$ *satisfying* $[t, t+K] \cap I \neq \emptyset$ *for all* $t \geq 0$, *the family* $\{T \oplus T(t) : t \in I\}$ *is topologically transitive.*

(ix) $(T(t_n))$ *is a hypercyclic sequence of operators for any strictly increasing sequence* (t_n) *in* $(0,\infty)$ *satisfying* $\lim_{n\to\infty} t_n = \infty$ *and* $\sup_{n\in\mathbb{N}}(t_{n+1} - t_n) < \infty$.

(x) *For every open, non-empty subsets* U, V_1, V_2 *of* X, *there exists* $t \in \Delta$ *such that* $T(t)U \cap V_1 \neq \emptyset$ *and* $T(t)U \cap V_2 \neq \emptyset$.

(xi) $(T(t))_{t\geq 0}$ *has a hereditarily hypercyclic subsequence* $(T(t_n))$.

(xii) $(T(t))_{t\geq 0}$ *is weakly mixing.*

(xiii) $(T(t))_{t\geq 0}$ *satisfies the Hypercyclicity Criterion.*

PROOF. By [14, Theorem 2.1], (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (xii), and by [8, Theorem 2.3-Theorem 2.5], (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (ix) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (xii). The equivalence (x) $\Leftrightarrow$ (xii) follows as in the proof of [39, Theorem 2.13]. Hence, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (ix) $\Leftrightarrow$ (x) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (xii). Arguing as in the proof of [39, Theorem 2.5], one obtains: (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) and (xii) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (vi). The proof of implication (vi) $\Rightarrow$ (xii) follows by making use of the argumentation given in the proofs of [35, Theorem 3.2] and [39, Theorem 2.5]; we will prove this implication for the sake of reader's convenience. It is evident that (vi) implies that $(T(t))_{t\geq 0}$ is topologically transitive. Hence, Theorem 1 shows that the set $\mathrm{HC}(T)$ is a dense G_δ-subset of X. Suppose now U_i, V_i, $i = 1, 2$ are open non-empty subsets of X and $v_1 \in \mathrm{HC}(T) \cap V_1$. So there exists $r_1 > 0$ such that $u_1 := T(r_1)v_1 \in U_1$. Since $\mathrm{R}(T(t_1))$ is dense in X (see Theorem 2(i), Proposition 8 and [17, Lemma 2.1]), there is an $\omega_2 \in E$ such that $u_2 := T(r_1)\omega_2 \in U_2$. Further on, let us assume $L(u_2, \delta) \subset U_2$, $L(v_2, \delta) \subset V_2$ and

$$T(r_1)(L(0,\delta') + L(0,\delta')) \subset L(0,\delta) \text{ for a suitable } \delta' \in (0,\delta). \tag{22}$$

An application of Proposition 8 gives $(T(\alpha) - I)v_1 \in \mathrm{HC}(T)$, and consequently, one can find positive real numbers p_1 and q_1 satisfying:

$$d(T(q_1)(T(\alpha) - I)v_1 - (\omega_2 - v_2), 0) < \delta' \text{ and} \tag{23}$$

$$d(T(p_1)v_1 - (v_2 - T(q_1)v_1), 0) < \delta'. \tag{24}$$

Set $y_2 := T(p_1)v_1 + T(q_1)v_1$ and $z_2 := T(p_1)u_1 + T(q_1 + \alpha)u_1$. Clearly, $z_2 = T(p_1 + r_1)v_1 + T(q_1 + \alpha + r_1)v_1$ and (24) implies that $y_2 \in V_2$. To prove that $z_2 \in U_2$, notice that

$$z_2 - u_2 = T(r_1)[(T(q_1)(T(\alpha) - I)v_1 - (\omega_2 - v_2)) + (T(p_1)v_1 - (v_2 - T(q_1)v_1))]$$

and that (22) implies

$$z_2 - u_2 \in T(r_1)(L(0, \delta') + L(0, \delta')) \subset L(0, \delta). \tag{25}$$

One can employ (vi) with $\tilde{U_k} = L(u_1, 2^{-k})$ and $\tilde{V_k} = L(v_1, 2^{-k})$ in order to obtain the existence of sequences (u_k) and $(\tilde{u_k})$ converging to u_1 and a sequence (t_k) in $(0, \infty)$ such that $(T(t_k)u_k)$ and $(T(t_k + \alpha)\tilde{u_k})$ converge to v_1. Since

$$\lim_{k\to\infty} T(t_k)(T(p_1)u_k + T(q_1 + \alpha)\tilde{u_k}) = T(p_1)v_1 + T(q_1)v_1 = y_2 \in V_2$$

and $\lim_{k\to\infty} T(p_1)u_k + T(q_1 + \alpha)\tilde{u_k} = T(p_1)u_1 + T(q_1 + \alpha)u_1 \in U_2$, one concludes that $T(t_k)U_i \cap V_i \neq \emptyset$, $i = 1, 2$ and that (xii) holds. The implication (iii) $\Rightarrow$ (xiii) is trivial and the implication (xiii) $\Rightarrow$ (xii) follows from an application of Lemma 11. This ends the proof of Theorem 12.

The situation is more complicated if $\Delta \neq [0, \infty)$ and $(T(t))_{t\in\Delta}$ is a strongly continuous semigroup in X. An insignificant modification of notion is being made to cover newly arisen situation: it is said that a subset I of Δ is *syndetic* if there exist a number $K > 0$ and a ray $R \subset \Delta$ starting at 0 so that:

$$\text{for every } t \in \Delta \text{ and } z \in R \text{ with } |z| \geq K \ : \ [t, t + z] \cap I \neq \emptyset. \tag{26}$$

Theorem 13. *Suppose $(T(t))_{t\in\Delta}$ is a strongly continuous semigroup in X. Consider the following assertions:*

(i) $(T(t))_{t\in\Delta}$ satisfies the Hypercyclicity Criterion.

(ii) For every pair of open non-empty sets U, $V \subset X$ and for every zero neighborhood W in X there exists $t \in \Delta\setminus\{0\}$ so that $T(t)U\cap W \neq \emptyset$ and $T(t)W\cap V \neq \emptyset$.

(iii) For every $s \in \Delta \setminus \{0\}$ and for every pair of open non-empty sets U, $V \subset X$, there exists $t \in \Delta \setminus \{0\}$ such that $T(t)U \cap V \neq \emptyset$ and $T(t + s)U \cap V \neq \emptyset$.

(iv) There exists $s \in \Delta \setminus \{0\}$ such that for every pair of open non-empty sets U, $V \subset X$, there exists $t \in \Delta \setminus \{0\}$ such that $T(t)U \cap V \neq \emptyset$ and that $T(t + s)U \cap V \neq \emptyset$.

(v) The family $\{T \oplus T(t) : t \in I\}$ is topologically transitive for every syndetic subset I of Δ.

(vi) There exist a number $K > 0$ and a ray $R \subset \Delta$ starting at 0 so that for every $I \subset \Delta$ satisfying (26), the family $\{T \oplus T(t) : t \in I\}$ is topologically transitive.

(vii) *For every open, non-empty sets* $U,\ V_1,\ V_2 \subset X$, *there exists* $t \in \Delta \setminus \{0\}$ *such that* $T(t)U \cap V_1 \neq \emptyset$ *and* $T(t)U \cap V_2 \neq \emptyset$.

(viii) $(T(t))_{t\in\Delta}$ *has a hereditarily hypercyclic subsequence* $(T(t_n))$.

(ix) $(T(t))_{t\in\Delta}$ *is weakly mixing.*

Then we have:

(a) *In the case* $\Delta = \mathbb{R}$, *(i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* $\Leftrightarrow$ *(iv)* $\Leftrightarrow$ *(v)* $\Leftrightarrow$ *(vi)* $\Leftrightarrow$ *(vii)* $\Leftrightarrow$ *(viii)* $\Leftrightarrow$ *(ix).*

(b) *Suppose* $\Delta = \mathbb{C}$. *Then the following holds: (i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(v)* $\Leftrightarrow$ *(vi)* $\Leftrightarrow$ *(vii)* $\Leftrightarrow$ *(viii)* $\Leftrightarrow$ *(ix) and (ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv). Suppose, in addition, that there exists* $\alpha \in \Delta \setminus \{0\}$ *such that* $R(T(\alpha) - I)$ *is dense in* X. *Then all assertions (i)-(ix) are mutually equivalent.*

(c) *Suppose* $\Delta = \Delta(\alpha)$, *for some* $\alpha \in (0, \frac{\pi}{2}]$. *Then (i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(v)* $\Leftrightarrow$ *(vi)* $\Leftrightarrow$ *(vii)* $\Leftrightarrow$ *(viii)* $\Leftrightarrow$ *(ix).*

(c1) *If, additionally,* $R(T(t))$ *is dense in* X *for every* $t \in \Delta$, *then (ii)* $\Rightarrow$ *(iii).*

(c2) *We have (ii)* $\Rightarrow$ *(iv).*

(c3) *(iv)* $\Rightarrow$ *(ix) under the additional assumption* $\overline{R(T(s) - I)} = X$.

(c4) *Suppose* $\overline{R(T(t))} = X,\ t \in \Delta$ *and* $\overline{R(T(s) - I)} = X$ *for some* $s \in \Delta$. *Then we have the equivalence of all assertions (i)-(ix).*

(d) *Suppose that* $(T(t))_{t\in\Delta}$ *is topologically transitive and that there exists a dense set of points* $x \in X$ *with bounded orbits* $\{T(t)x : t \in \Delta\}$. *Then (ii) holds so that* $(T(t))_{t\in\Delta}$ *is weakly mixing.*

(e) *Suppose* $\Delta \in \{[0,\infty),\ \mathbb{R}\}$ *and* $(T(t))_{t\in\Delta}$ *is chaotic. Then* $(T(t))_{t\in\Delta}$ *is weakly mixing.*

Proof. When $\Delta = \mathbb{R}$, one can employ the assertions (ii.1) and (iv.2) of Proposition 8, Lemma 11, the local equicontinuity of $(T(t))_{t\in\mathbb{R}}$ as well as the proofs of [39, Theorem 2.5] and Theorem 12 in order to prove that (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (ix). Using Lemma 11, we have that $(T\underbrace{\oplus \cdots \oplus}_{k} T)(t))_{t\in\mathbb{R}}$ is topologically transitive for all $k \in \mathbb{N}$ and the argumentation given in the proof of [39, Theorem 2.5] implies (i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv). The proof of (a) is completed if one shows (viii) $\Rightarrow$ (ix) $\Rightarrow$ (i) $\Rightarrow$ (viii). To prove (viii) $\Rightarrow$ (ix), we will slightly modify the proof of [10, Theorem 2.3]. Suppose that $U_i,\ V_i,\ i = 1, 2$ are open non-empty subsets of X and that $\overline{\{T(t_n)x : n \in \mathbb{N}\}} = X$; this implies that, for every $s \in \mathbb{R}$, $\overline{\{T(t_n)T(s)x : n \in \mathbb{N}\}} = X$. Hence, there exist $u_1 \in U_1$ and $n_1 \in \mathbb{N}$ such that $\overline{\{T(t_n)u_1 : n \in \mathbb{N}\}} = X$ and that $T(t_{n_1})U_1 \cap V_1 \neq \emptyset$. Using the fact that $(T(t_n))$ is hereditarily hypercyclic, one obtains inductively the existence of a strictly increasing sequence (n_k) in $\mathbb{N}$ satisfying $T(t_{n_k})U_1 \cap V_1 \neq \emptyset,\ k \in \mathbb{N}$. Now the hypercyclicity of $(T(t_{n_k}))$ gives the existence of a number $k_0 \in \mathbb{N}$ such that $T(t_{n_{k_0}})U_2 \cap V_2 \neq \emptyset$ and that $T(t_{n_{k_0}})U_1 \cap V_1 \neq \emptyset$. Therefore, $(T(t))_{t\in\mathbb{R}}$ is weakly mixing. The proof of (ix) $\Rightarrow$ (i) is essentially contained in that of [10, Lemma 2.5]. So, let $(x, y) \in \mathrm{HC}(T \oplus T)$ and $s \in \mathbb{R}$. Thereby, $x \in \mathrm{HC}(T)$ and $y \in \mathrm{HC}(T)$. Since $T(s)$ is bijective, $\mathrm{R}(T(s)) = X$ and this simply implies $(x, T(s)y) \in \mathrm{HC}(T \oplus T)$. As a consequence, we have that, for every open non-empty subset U of X, there exists $u \in U$ such that $(x, u) \in \mathrm{HC}(T \oplus T)$. Put now $Y = Z = \mathrm{Orb}(x, T)$ and $U_k = L(0, \frac{1}{k}),\ k \in \mathbb{N}$. An induction argument shows that there exist a sequence (u_k) in X and a sequence (t_k) in $\mathbb{R}$ so that:

$$u_k \in U_k,\ T(t_k)x \in U_k \text{ and } T(t_k)u_k \in x + U_k,\ k \in \mathbb{N}. \tag{27}$$

It is evident that (27) implies $\lim_{k\to\infty} T(t_k)x = 0$ and $\lim_{k\to\infty} T(t_k)y = 0,\ y \in Y$. If $z = T(t)x \in Z$ for some $t \in \Delta$, put $\tilde{u_k} = T(t)u_k,\ k \in \mathbb{N}$. Clearly, $\lim_{k\to\infty} \tilde{u_k} = 0$, $\lim_{k\to\infty} T(t_k)\tilde{u_k} = z$ and this yields (i). Suppose $(T(t))_{t\in\Delta}$ fulfills the Hypercyclicity Criterion with $Y,\ Z$ and (t_n). Then one can argue as in the proofs of Theorem 1 and Lemma 11 to conclude that $(T(t_n))$ is hereditarily hypercyclic, and consequently, (a) follows provided that $\Delta = \mathbb{R}$. Suppose $\Delta = \mathbb{C}$. Then the equivalence of (i), (vii), (viii) and (ix) can be proved as before. Let us prove that (ix) implies (ii). To do that, let us suppose that $U,\ V$ are open non-empty subsets of X, W is a zero neighborhood in X and $L(0, \varepsilon) \subset W$ for some $\varepsilon > 0$. Then the topological transitivity of $(T \oplus T(t))_{t\in\Delta}$ enables one to deduce the existence of a number $t \in \Delta$ satisfying $T(t)U \cap L(0, \varepsilon) \neq \emptyset$ and $T(t)L(0, \varepsilon) \cap V \neq \emptyset$. This clearly implies $T(t)U \cap W \neq \emptyset$, $T(t)W \cap V \neq \emptyset$ and (ii). The implication (ii) $\Rightarrow$ (i) can be proved by a simple modification of the proofs of [9, Theorems 3.3-3.4]. First of all, it is straightforward to see that (ii) implies the topological transitivity of $(T(t))_{t\in\Delta}$ (cf. also [34, Corollary 1.3]). Due to Theorem 1, $(T(t))_{t\in\Delta}$ has a dense G_δ-set of hypercyclic vectors. Let (U_k) be a base of open zero neighborhoods. Designate by P the set of all $x \in X$ such that there exist a sequence (u_n) in X and a sequence (t_n) in Δ satisfying $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} T(t_n)u_n = x$ and $\lim_{n\to\infty} T(t_n)x = 0$. Proceeding as in the proof of [9, Theorem 3.4], one obtains that

$$P = \bigcap_{k\in\mathbb{N}} \bigcup_{t\in\Delta} [T(t)^{-1}(U_k) \cap \{x \in X : T(t)U_k \cap (x + U_k) \neq \emptyset\}]$$

and that P is a dense G_δ-set of X. The intersection of two dense G_δ-sets in X is non-empty so that there exist $x \in \mathrm{HC}(T)$, a sequence (u_n) in X and a sequence (t_n) in Δ satisfying $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} T(t_n)u_n = x$ and $\lim_{n\to\infty} T(t_n)x = 0$. Put now $Y = Z = \mathrm{Orb}(x, T)$. Then one can check at once that the Hypercyclicity Criterion holds with $Y,\ Z$ and a sequence (t_n). This proves (i), and consequently, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix). In order to prove the implication (i) $\Rightarrow$ (v), let us suppose that $I \subset \Delta$ is syndetic and that $U_i,\ V_i,\ i = 1, 2$ are open non-empty subsets of X. The local equicontinuity of $(T(t))_{t\in R}$ implies the existence of a number $\delta > 0$ and open non-empty subsets $\tilde{U_i},\ i = 1, 2$ such that, for every $z \in R$ with $|z| \leq \delta$, $T(z)\tilde{U_i} \subset U_i,\ i = 1, 2$. Let $m \in \mathbb{N}$ and $z \in R$ satisfy $m\delta > K$ and $|z| = \delta$. Then

Lemma 11 and the proof of [39, Theorem 2.5] give the existence of a number $t \in \Delta$ such that:

$$T(t+jz)\tilde{U_i} \cap V_i \neq \emptyset,\ i=1,2,\ 0 \leq j \leq m. \tag{28}$$

Since I is syndetic, one gets $[t, t+mz] \cap I \neq \emptyset$. Hence, there exist $j \in \{1, \cdot\cdot\cdot, m\}$ and $s \in \Delta$ such that $s \in [t+(j-1)z, t+jz] \cap I$. Taken together, this inclusion and (28) imply $t+jz-s \in R$, $|t+jz-s| \leq \delta$ and

$$\emptyset \neq T(t+jz)\tilde{U_i} \cap V_i = T(s)T(t+jz-s)\tilde{U_i} \cap V_i \subset T(s)U_i \cap V_i,\ i=1,2.$$

Since $s \in I$, the proof of (v) is completed. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow as in the proof of [39, Theorem 2.5], and moreover, the implications (v) $\Rightarrow$ (vi) $\Rightarrow$ (ix) are trivial. Therefore, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix). Notice also that the denseness of $\mathrm{R}(T(\alpha)-I)$ and the argumentation given in the proof of Theorem 12 imply the validity of (iv) $\Rightarrow$ (ix). This completes the proof of (b). Let us examine the case $\Delta = \Delta(\alpha)$, for some $\alpha \in (0, \frac{\pi}{2}]$. On the basis of proofs of (a) and (b), one gets (vii) $\Leftrightarrow$ (ix) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (ix), (i) $\Rightarrow$ (viii) and (i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (ix). Suppose that (viii) holds. An application of [9, Theorem 3.3] shows that $(T(t_n))$ satisfies the Hypercyclicity Criterion (cf. [9, Definition 1.1]) and one deduces the validity of (i). Hence, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix). In order to prove (c2), suppose first that $\alpha \in (0, \frac{\pi}{2})$. Then one can employ Theorem 2(i) to conclude that, for every $z \in (\partial\Delta) \setminus \{0\}$, $\overline{\mathrm{R}(T(z))} = E$ or $\overline{\mathrm{R}(T(\overline{z}))} = E$. In the case $\alpha = \frac{\pi}{2}$, notice only that for every $z \in (\partial\Delta)$, we have $\overline{\mathrm{R}(T(z))} = E$ and $\overline{\mathrm{R}(T(\overline{z}))} = E$. This follows from the fact that $\mathrm{Orb}(x, T) \subset \mathrm{R}(T(z)),\ x \in \mathrm{HC}(T)$. The argumentation given in the proof of [39, Theorem 2.5] can be applied again. In such a way, we obtain that (iv) holds with $s \in \{z, \overline{z}\}$. The remnant of proof of (c) follows by the use of arguments already given in the proofs of (a) and (b). To prove (d), notice that Theorem 1 implies that $(T(t))_{t\in\Delta}$ has a dense G_δ-set of hypercyclic vectors so that one can repeat literally the arguments given in the proof of [8, Theorem 2.4] to conclude that (ii) holds and (d) follows from a straightforward application of the assertions (a)-(c). The proof of (e) follows instantly from (d) and this completes the proof of theorem.

The assertion (e) of previous theorem is insignificantly improved in [27] where the authors showed that every chaotic semigroup $(T(t))_{t\geq 0}$ in a Banach space satisfies the Recurrent Hypercyclicity Criterion (cf. [8, Theorem 2.4] and [27, Corollary 6.2]). The next illustrative example shows that the Recurrent Hypercyclicity Criterion is strictly stronger than the Hypercyclicity Criterion.

Example 14. *Define two sequences $(s_n)_{n\in\mathbb{N}_0}$ and $(r_n)_{n\in\mathbb{N}}$ of positive real numbers by $s_0 := 0$, $s_1 := 1$, $r_1 := e$ and, for every $n \in \mathbb{N}$:*

$$(s_{n+1}, r_{n+1}) := \begin{cases} (k(1+s_n), er_n),\ n+1 = k^2 \text{ for some } k \in \mathbb{N}, \\ (1+s_n, e^{1+s_n}),\ \text{otherwise}. \end{cases} \tag{29}$$

It can be easily verified that $(s_n)_{n\in\mathbb{N}_0}$ and $(r_n)_{n\in\mathbb{N}}$ are strictly increasing sequences and that: $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} r_n = \infty$ and $\liminf_{n\to\infty} \frac{\ln(r_n)}{s_n} = 0$. Put now $\rho(0) := 1$ and

$\rho(s) := r_n e^{-s},\ s \in (s_{n-1}, s_n]$. Since $(r_n)_{n\in\mathbb{N}}$ is increasing and $r_1 > e$, one obtains that $\rho(s) \leq e^t \rho(t+s),\ t,\ s \geq 0$ and that ρ is an admissible weight function. Further on, one obtains inductively that:

$$r_n = \begin{cases} e^{s_n},\ n \neq k^2 \text{ for every } k \in \mathbb{N}, \\ e^{\frac{s_n}{k}},\ n = k^2 \text{ for some } k \in \mathbb{N}, \end{cases} \tag{30}$$

and that

$$s_{n^2-1} - s_{(n-1)^2} = 2n-2,\ n \in \mathbb{N}. \tag{31}$$

Clearly, $\rho(s) = r_n e^{-s} \geq r_n e^{-s_n},\ s \in (s_{n-1}, s_n],\ n \in \mathbb{N}$ and this inequality enables one to see that:

$$\rho(s) \geq 1, \quad \text{if } s \in (s_{n-1}, s_n] \text{ and } n \neq k^2 \text{ for every } k \in \mathbb{N}. \tag{32}$$

Suppose now $X = L^1_\rho([0,\infty), \mathbb{C})$ and notice that $\lim_{n\to\infty} \rho(s_{n^2}) = \lim_{n\to\infty} e^{s^2_n(\frac{1}{n}-1)} = 0$. Owing to Theorem 4.1 and [27, Preopostion 4.4], we have that the translation semigroup $(T(t))_{t\geq 0}$ (cf. Section 4 for the notion) fulfills the Hypercyclicity Criterion. Suppose $\varepsilon > 0$ and $(T(t))_{t\geq 0}$ fulfills the Recurrent Hypercyclicity Criterion. Due to [27, Theorem 4.6], one yields the existence of an increasing sequence (t_n) in $(0,\infty)$ and a number $L \in (0,\infty)$ satisfying:

$$\lim_{n\to\infty} t_n = \infty,\ t_{n+1} - t_n \leq L \text{ and } \rho(t_n) \leq \varepsilon,\ n \in \mathbb{N}. \tag{33}$$

Hence, (32) implies that there exist $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $t_n \in \bigcup_{i\geq k} (s_{i^2-1}, s_{i^2}]$. Furthermore, by (31), one has $\lim_{n\to\infty}(s_{n^2-1} - s_{(n-1)^2}) = \infty$ and the contradiction is obvious since (t_n) satisfies (33).

Let us notice that the equivalence of assertions (i), (v) and (ix) quoted in the formulation of Theorem 13 can be slightly strengthened by means of already given arguments:

Theorem 15. *Suppose $(T(t))_{t\in\Delta}$ is a strongly continuous semigroup in X. Then the following assertions are equivalent:*

(i) $(T(t))_{t\in\Delta}$ satisfies the Hypercyclicity Criterion.

(ii) The family $\{T \oplus T(t) : t \in I\}$ is topologically transitive for every subset I of Δ satisfying the next condition: There exist $K > 0$, $n \in \mathbb{N}$ and rays $R_i \subset \Delta,\ i = 1, \cdots, n$ starting at 0 so that for every $t \in \Delta \setminus \{0\}$ there exists $i \in \{1, \cdots, n\}$ such that for every $z \in R_i$ with $|z| \geq K$: $[t, t+z] \cap I \neq \emptyset$.

(iii) $(T(t))_{t\in\Delta}$ is weakly mixing.

Finally, we will prove the inheritance law for the Hypercyclicity Criterion.

Proposition 16. *Suppose $(T(t))_{t\in\Delta}$ is a strongly continuous semigroup in X and $t_0 \in \Delta \setminus \{0\}$. Then the following assertions are equivalent:*

(i) $(T(t))_{t\in\Delta}$ *is weakly mixing.*

(ii) *The family* $\{T \oplus T(z) : z \in \Delta,\ |z| = k|t_0| \text{ for some } k \in \mathbb{N}\}$ *is topologically transitive.*

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. Let us prove that (i) $\Rightarrow$ (ii). In the case $\Delta \in \{[0,\infty),\ \mathbb{R},\ \Delta(\alpha)\}$, $\alpha \in (0, \frac{\pi}{2})$, the proof follows from Theorem 13 and the fact that the set $I = \{z \in \Delta : |z| = k|t_0| \text{ for some } k \in \mathbb{N}\}$ is syndetic. We will show this only in the case $\Delta = \Delta(\alpha)$, where $\alpha \in (0, \frac{\pi}{2})$. Put $R = [0,\infty)$ and $K \geq \frac{2|t_0|}{\cos\alpha}$. Suppose, further, $t \in \Delta$, $|t| \in [k|t_0|, (k+1)|t_0|)$ for an appropriate $k \in \mathbb{N}_0$ as well as $z \in R$ and $|z| \geq K$. Clearly, $|\mathrm{Im}(t)| \leq \mathrm{Re}(t)\tan\alpha$ and $\mathrm{Re}(t) \geq k|t_0|\cos\alpha$. Hence, $|t+z|^2 \geq k^2|t_0|^2 + 2\mathrm{Re}(t)z + z^2 \geq k^2|t_0|^2 + 2k|t_0|z\cos\alpha \geq k^2|t_0|^2 + 2k|t_0|\frac{2|t_0|}{\cos\alpha}\cos\alpha \geq (k+1)^2|t_0|^2$. Accordingly, there exists $s \in [t, t+z]$ such that $|s| = (k+1)|t_0|$. This implies $[t, t+z] \cap I \neq \emptyset$. The rest of proof is a sophisticated application of Theorem 15 and we will sketch the proof only in the case $\Delta = \mathbb{C}$. Put $R_j := \{re^{i(2j-1)\frac{\pi}{4}} : r \geq 0\}$, $j = 1, 2, 3, 4$. Then the set $I = \{z \in \mathbb{C} : |z| = k|t_0| \text{ for some } k \in \mathbb{N}\}$ fulfills the condition quoted in the item (ii) of the previous theorem with $n = 4$, R_1, R_2, R_3, R_4 and $K > 4|t_0|$. Namely, if $\mathrm{Re}(z) \geq 0$, $\mathrm{Im}(z) \geq 0$ and $|z| \in [k|t_0|, (k+1)|t_0|)$ for an appropriate $k \in \mathbb{N}$, then we infer easily that for every $z \in R_1$ with $|z| \geq K$, the segment $[t, t+z]$ contains an element of I since $[t, t+z]$ intersects the circle $\{z \in \mathbb{C} : |z| = (k+1)|t_0|\}$. The other cases can be considered similarly.

4 S-hypercyclicity, chaoticity and topologically mixing properties of various kinds of strongly continuous semigroups

We start this section with the following recollection of the basic structural properties of function spaces used henceforth. A measurable function $\rho : \Delta \to (0,\infty)$ is said to be an *admissible weight function* if there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(t) \leq Me^{\omega|t'|}\rho(t+t')$ for all $t,\ t' \in \Delta$. For such a function ρ, we introduce the following Banach spaces:

$$L^p_\rho(\Delta, \mathbb{K}) := \{u : \Delta \to \mathbb{K} \mid u \text{ is measurable and } ||u||_p < \infty\},$$

where $p \in [1,\infty)$ and $||u||_p := (\int\limits_\Delta |u(t)\rho(t)|^p dt)^{1/p}$ as well as

$$C_{0,\rho}(\Delta, \mathbb{K}) := \{u : \Delta \to \mathbb{K} \mid u \text{ is continuous and } \lim_{t\to\infty} u(t)\rho(t) = 0\},$$

with $||u|| := \sup\limits_{t\in\Delta} |u(t)\rho(t)|$.

Suppose Ω is an open non-empty subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. A continuous mapping $\varphi : \Delta \times \Omega \to \Omega$ is called a *semiflow* ([38]-[39], [44]) if $\varphi(0,x) = x,\ x \in \Omega$,

$$\varphi(t+s, x) = \varphi(t, \varphi(s,x)),\ t,\ s \in \Delta,\ x \in \Omega \text{ and} \tag{34}$$

$$x \mapsto \varphi(t,x) \text{ is injective for all } t \in \Delta. \tag{35}$$

Denote by $\varphi(t,\cdot)^{-1}$ the inverse mapping of $\varphi(t,\cdot)$, i.e., $y = \varphi(t,x)^{-1}$ iff $x = \varphi(t,y)$, $t \in \Delta$. In what follows, we also deal with the Banach space $L^p_\rho(\Omega,\mathbb{K})$ where $\rho : \Omega \to (0,\infty)$ is a measurable function, $\rho^p : \Omega \to (0,\infty)$ is a locally integrable function and the norm of an element $f \in L^p_\rho(\Omega,\mathbb{K})$ is given by $||f||_p := (\int_\Omega |f(x)\rho(x)|^p dx)^{1/p}$. The Banach space $C_{0,\rho}(\Omega,\mathbb{K})$ consists of all continuous functions $f : \Omega \to \mathbb{K}$ satisfying that, for every $\varepsilon > 0$, $\{x \in \Omega : |f(x)|\rho(x) \geq \varepsilon\}$ is a compact subset of Ω; herein $\rho : \Omega \to (0,\infty)$ is an upper semicontinuous function and the norm of an element $f \in C_{0,\rho}(\Omega,\mathbb{K})$ is given by $||f|| := \sup_{x\in\Omega} |f(x)|\rho(x)$. Put, by common consent, $\sup_{x\in\emptyset} \rho(x) := 0$ and denote by $C(\Lambda,\mathbb{K})$ the $\mathbb{K}$-vector space consisting of all continuous functions from Λ into $\mathbb{K}$, where Λ is Δ or Ω. The Fréchet topology on $C(\Omega,\mathbb{K})$ is induced by the family of increasing seminorms: $||f||_n := \sup_{x\in K_n} |f(x)|$, $f \in C(\Omega,\mathbb{K})$, where (K_n) is a sequence of compact subsets of Ω satisfying $K_1 \subset K_2 \subset \cdot\cdot \subset K_n \subset \cdots$ and $\bigcup_{n\in\mathbb{N}} K_n = \Lambda$. Notice that $C_c(\Lambda,\mathbb{K})$ is dense in $L^p_\rho(\Lambda,\mathbb{K})$, and obviously, and $C_c(\Lambda,\mathbb{K})$ is dense in $C_{0,\rho}(\Lambda,\mathbb{K})$, too (cf. [54, Section 13]). The use of symbol ρ in the continuation of this section is clear from the context.

The following characterization of S-hypercyclic translation semigroups essentially follows from the argumentation given in the papers of W. Desch, W. Schappacher, G. F. Webb [28, Theorems 4.7-4.8] and J. A. Conejero, A. Peris [16, Theorems 5.11-5.12]. Notice only that the last equivalence in (ii) is a consequence of the proof of [44, Theorem 2.7].

Theorem 17. *Suppose $p \in [1,\infty)$, $\alpha \in (0,\frac{\pi}{2}]$, $\rho : \Delta \to (0,\infty)$ is an admissible weight function, $\tilde{X} \in \{L^p_\rho(\Delta,\mathbb{K}),\ C_{0,\rho}(\Delta,\mathbb{K})\}$ and the translation semigroup $(\tilde{T}(t))_{t\in\Delta}$ is given by*

$$(\tilde{T}(t)f)(x) := f(x+t),\ x,\ t \in \Delta,\ f \in \tilde{X}. \tag{36}$$

(i) Suppose $\Delta \in \{[0,\infty),\ \Delta(\alpha)\}$. The semigroup $(\tilde{T}(t))_{t\in\Delta}$ is S-topologically transitive if $\sup S = \infty$. In the case $\sup S < \infty$, $(\tilde{T}(t))_{t\in\Delta}$ is S-topologically transitive iff for every $\theta \in [0,\infty)$ there exist a sequence (t_j) in Δ satisfying $\lim_{j\to\infty} |t_j| = \infty$ and a sequence (a_j) in $S \setminus \{0\}$ such that:

$$\lim_{j\to\infty} \frac{1}{a_j}\rho(t_j+\theta) = 0 \tag{37}$$

iff $(\tilde{T}(t))_{t\in\Delta}$ is hypercyclic.

(ii) Suppose $\Delta \in \{\mathbb{R},\ \mathbb{C}\}$ and $S \subset [0,\infty)$. Then $(\tilde{T}(t))_{t\in\Delta}$ is S-topologically transitive iff for every $\theta \in (0,\infty)$ there exist a sequence (t_j) in Δ satisfying $\lim_{j\to\infty} |t_j| = \infty$ and a sequence (a_j) in $S \setminus \{0\}$ such that:

$$\lim_{j\to\infty} a_j\rho(-t_j+\theta) = \lim_{j\to\infty} \frac{1}{a_j}\rho(t_j+\theta) = 0. \tag{38}$$

If $S = [0, \infty)$, the above is also equivalent to the existence of a sequence (t_j) in Δ satisfying $\lim_{j\to\infty} |t_j| = \infty$ and

$$\lim_{j\to\infty} \rho(-t_j) \lim_{j\to\infty} \rho(t_j) = 0. \tag{39}$$

Fix a number $t \in \Delta$, a function $f : \Omega \to \mathbb{K}$, a semiflow $\varphi : \Delta \times \Omega \to \Omega$ and define after that a function $T_\varphi(t)f : \Omega \to \mathbb{K}$ by $(T_\varphi(t)f)(x) := f(\varphi(t, x)),\ x \in \Omega$. Then $T_\varphi(0)f = f$, $T_\varphi(t)T_\varphi(s)f = T_\varphi(s)T_\varphi(t)f = T_\varphi(t + s)f,\ t,\ s \in \Delta$ and Brouwer's theorem (cf. [25] and [38]) implies $C_c(\Omega) \subset T_\varphi(t)(C_c(\Omega))$. We refer the reader to [38, Theorem 2.1], resp. [38, Theorem 2.2], for the necessary and sufficient conditions stating when the composition operator $T_\varphi(t) : L^p_\rho(\Omega) \to L^p_\rho(\Omega)$, resp. $T_\varphi(t) : C_{0,\rho}(\Omega) \to C_{0,\rho}(\Omega)$, is well defined and continuous. The strong continuity of the semigroup $(T_\varphi(t))_{t\in\Delta}$ in $L^p_\rho(\Omega)$, resp. $C_{0,\rho}(\Omega)$, has been recently discussed in [38, Theorem 3.2, Theorem 3.4] and [44, Theorem 2.5, Theorem 2.6]. It is worthwhile to point out that such a property can be neglected from the formulation of next theorem whose proof follows from an essential application of [38, Theorem 4.3, Theorem 4.5] (cf. also [44, Theorem 2.7]).

Theorem 18. *Let $\varphi : \Delta \times \Omega \to \Omega$ be a semiflow and let $S \subset [0, \infty)$.*

(i) Suppose $(T_\varphi(t))_{t\in\Delta}$ is a strongly continuous semigroup in $L^p_\rho(\Omega)$. Then the following assertions are equivalent.

(i1) $(T_\varphi(t))_{t\in\Delta}$ is S-hypercyclic in $L^p_\rho(\Omega)$.

(i2) For every compact set $K \subset \Omega$ there exist a sequence (L_k) of measurable subsets of K, a sequence (t_k) in Δ and a sequence (c_k) in $S \setminus \{0\}$ such that:

$$\lim_{k\to\infty} \int_{K \setminus L_k} \rho^p(x)dx = 0 \ \text{and} \tag{40}$$

$$\lim_{k\to\infty} c_k^p \int_{\varphi(t_k,\cdot)^{-1}(L_k)} \rho^p(x)dx = \lim_{k\to\infty} \frac{1}{c_k^p} \int_{\varphi(t_k,L_k)} \rho^p(x)dx = 0. \tag{41}$$

In the case $S = [0, \infty)$, the above is also equivalent to the condition (i3), where:

(i3) For every compact set $K \subset \Omega$ there exist a sequence (L_k) of measurable subsets of K and a sequence (t_k) in Δ such that (40) holds and that

$$\lim_{k\to\infty} [\int_{\varphi(t_k,\cdot)^{-1}(L_k)} \rho^p(x)dx \ * \int_{\varphi(t_k,L_k)} \rho^p(x)dx] = 0. \tag{42}$$

(ii) Suppose $(T_\varphi(t))_{t\in\Delta}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega)$ and that for every compact set $K \subset \Omega$, we have $\inf_{x\in K} \rho(x) > 0$. Then the following assertions are equivalent.

(ii1) $(T_\varphi(t))_{t\in\Delta}$ *is S-hypercyclic in* $C_{0,\rho}(\Omega)$.

(ii2) *For every compact set* $K \subset \Omega$ *there exist a sequence* (t_k) *in* Δ *and a sequence* (c_k) *in* $S \setminus \{0\}$ *such that:*

$$\lim_{k\to\infty} c_k \sup_{x\in\varphi(t_k,\cdot)^{-1}(K)} \rho(x) = \lim_{k\to\infty} \frac{1}{c_k} \sup_{x\in\varphi(t_k,K)} \rho(x) = 0. \tag{43}$$

In the case $S = [0,\infty)$, *the above is also equivalent to the condition (ii3), where:*

(ii3) *For every compact set* $K \subset \Omega$ *there exists a sequence* (t_k) *in* Δ *such that:*

$$\lim_{k\to\infty} [\sup_{x\in\varphi(t_k,\cdot)^{-1}(K)} \rho(x) * \lim_{k\to\infty} \sup_{x\in\varphi(t_k,K)} \rho(x)] = 0. \tag{44}$$

Unfortunately, it is not clear whether Theorem 18 and the assertion (ii) of Theorem 17 remain true if S $\nsubseteq [0,\infty)$. Nevertheless, the argumentation given in the proofs of [48, Theorem 1, Corollary 4] implies the following important relationship between *positivity* and S-*hypercyclicity*:

Theorem 19. *Suppose* X *is a complex space,* $S \subset [0,\infty)$, $x \in X$, $(T(t))_{t\in\Delta}$ *is a locally equicontinuous semigroup in* X, $\{uv : u,\ v \in S\} \subset S$ *and there exists* $T \in L(X)$ *such that* $\overline{R(T-\lambda I)} = X,\ \lambda \in \mathbb{C}$ *and that* $TT(t) = T(t)T,\ t \in \Delta$. *Then* $\{cT(t)x : c \in S,\ t \in \Delta\}$ *is dense in* X *iff* $\{\lambda T(t)x : \lambda \in \mathbb{C},\ |\lambda| \in S,\ t \in \Delta\}$ *is dense in* X.

Motivated by investigation of T. Bermúdez, A. Bonilla, A. Peris [6], F. León-Saavedra, V. Müler [48] and F. León-Saavedra, A. Piqueras-Lerena [49], we raise the issue ($\mathbb{I}$ stands for the set of all irrational numbers):

Problem. Does there exist a supercyclic strongly continuous semigroup $(T(t))_{t\in\Delta}$ in X which is not positively supercyclic and satisfies

$$\emptyset \neq \sigma_p(T(t)^*) \subset (0,\infty)e^{2\pi i\mathbb{I}},\ t \in \Delta? \tag{45}$$

The analysis of introduced classes of strongly continuous semigroups in $L^p(\Delta,\mathbb{K})$ and $C_0(\Delta,\mathbb{K})$-type spaces requires some additional technical rearrangements. First of all, we state the following analogue of [63, Lemma 1].

Lemma 20. *Suppose* $p \in [1,\infty)$, $X \in \{L^p(\Delta,\mathbb{K}),\ C_0(\Delta,\mathbb{K})\}$, $g : \Delta\times\Delta \to \mathbb{K}$ *is continuous and* $(T(t)f)(x) := g(x,t)f(x+t),\ x,\ t \in \Delta,\ f \in X$. *If* $(T(t))_{t\in\Delta}$ *is a strongly continuous semigroup, then:*

(H1) $g(x,t+s) = g(x,t)g(x+t,s),\ x,\ t,\ s \in \Delta$,

(H2) $g(x,0) = 1,\ x \in \Delta$,

(H3) $g(x,t) \neq 0,\ x,\ t \in \Delta$ *and*

(H4) $g(t,s) = \frac{g(0,t+s)}{g(0,t)},\ t,\ s \in \Delta$.

Proof. Certainly, $(T(t+s)f)(x) = g(x,t+s)f(x+t+s)$ and $(T(t)T(s)f)(x) = g(x,t)(T(s)f)(x+t) = g(x,t)g(x+t,s)f(x+t+s),\ x,\ t,\ s \in \Delta,\ f \in X$. This simply implies (H1) while (H2) follows from $T(0) = I$. To prove (H3), suppose $g(x,t) = 0,\ x,\ t \in \Delta$. Since $g(x,0) = 1$ and g is continuous, we have the existence of a positive real number ε such that, for every $t' \in \Delta_\varepsilon$, $|g(x,t')| \geq \frac{1}{2}$. Therefore, $0 < \inf\{|t''| : t'' \in \Delta \text{ and } g(x,t'') = 0\} := r_0$. The continuity of g implies that there exists $t_0 \in \Delta$ such that $g(x,t_0) = 0$ and that $|t_0| = r_0$. Let $t_1 \in [0,t_0)$. Clearly, $t_0 - t_1 \in \Delta$ and, due to (H1), $g(x,t_0) = g(x,t_1)g(x+t_1,t_0-t_1)$. So $g(x+t_1,t_0-t_1) = 0$. Letting $t_1 \to t_0$, we obtain $g(x+t_0,0) = 0$ which contradicts (H2); (H4) is a simple consequence of (H1) and (H3).

Lemma 21. *Suppose $g : \Delta \times \Delta \to \mathbb{K}$ is continuous and satisfies (H1)-(H4). Put $\rho(t) := \frac{1}{|g(0,t)|}$, $t \in \Delta$. Then ρ is an admissible weight function iff there exist numbers $M \geq 1$ and $\omega \in \mathbb{R}$ so that $|g(t,t')| \leq Me^{\omega|t'|}$ for all $t,\ t' \in \Delta$.*

Proof. Suppose ρ is an admissible weight function. Then the existence of numbers $M \geq 1$ and $\omega \in \mathbb{R}$ satisfying $\frac{1}{|g(0,t)|} \leq Me^{\omega|t'|}\frac{1}{|g(0,t+t')|},\ t,\ t' \in \Delta$ is obvious. This implies $|\frac{g(0,t+t')}{g(0,t)}| \leq Me^{\omega|t'|},\ t,\ t' \in \Delta$, i.e., $|\frac{g(0,t)g(t,t')}{g(0,t)}| \leq Me^{\omega|t'|},\ t,\ t' \in \Delta$. Hence, $|g(t,t')| \leq Me^{\omega|t'|},\ t,\ t' \in \Delta$. The converse statement can be proved in a similar way.

Example 22. *(a) Suppose $\alpha \in (0, \frac{\pi}{2}]$, $\Delta \in \{\Delta(\alpha),\ \mathbb{C}\}$ as well as the continuous functions $g_1,\ g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{K}$ fulfill the conditions (H1)-(H2) for $x,\ t,\ s \in \mathbb{R}$. If $\Delta = \Delta(\alpha)$, then we assume that the function g_1 is defined and continuous on $[0,\infty) \times [0,\infty)$ and that satisfies (H1)-(H2) for $x,\ t,\ s \geq 0$. Put, for $x = x_1 + ix_2 \in \Delta$ and $t = t_1 + it_2 \in \Delta$, $g(x,t) := g_1(x_1,t_1)g_2(x_2,t_2)$. Then g satisfies (H1)-(H2) and the proof of Lemma 20 implies (H1)-(H4) for g. Suppose, further, that $h : \mathbb{R} \to \mathbb{K}$ is a bounded measurable function. Then the next functions (see also [63]) satisfy (H1)-(H4):*

(1) $g_i(x_i,t_i) = e^{\int_{x_i}^{x_i+t_i} h(s)ds}$, $i = 1,2$,

(2) $g_1(x_1,t_1) = (1 + \frac{t_1}{x_1+a})^b,\ a > 0,\ b \in \mathbb{C}$ ($1^b = 1$) and $g_2(x_2,t_2) = (1 + \frac{t_2}{x_2+a})^n,\ a \in \mathbb{C} \setminus \mathbb{R},\ n \in \mathbb{N}_0$, if $\mathbb{K} = \mathbb{C}$ and $\Delta = \Delta(\alpha)$.

However, for any function $g(x,t) = g_1(x_1,t_1)g_2(x_2,t_2)$, where g_1, resp., g_2, is of the form (1)-(2), there exist appropriate constants $M \geq 1$ and $\omega \in \mathbb{R}$ so that $|g(t,t')| \leq Me^{\omega|t'|}$ for all $t,\ t' \in \Delta$. Owing to Lemma 21, the mapping $\rho(t) = \frac{1}{|g(0,t)|},\ t \in \Delta$ is an admissible weight function.

(b) Suppose $g : \Delta \times \Delta \to \mathbb{K}$ is continuous and satisfies (H1)-(H4). Set, for every $p \in [1,\infty)$, $g_p(x,t) := g^p(x,t)$ and $\tilde{g_p}(x,t) := |g^p(x,t)|,\ x,\ t \in \Delta$, where $1^p = 1$. Then $\tilde{g_p}$ is continuous and satisfies (H1)-(H4); the same conclusion holds for g_p provided that $\mathbb{K} = \mathbb{R}$ since, in this case, we have $g(x,t) > 0,\ x,\ t \in \Delta$.

In general, g_p does not satisfy (H1) if $\mathbb{K} = \mathbb{C}$ and $p \notin \mathbb{N}$. A counterexample can be simply constructed; just put $g(x,t) := e^{it},\ x,\ t \in \mathbb{C}$ and notice that $g_p(x,-\pi) \neq g_p(x,-\frac{\pi}{2})g_p(x-\frac{\pi}{2},-\frac{\pi}{2})$ if $x \in \mathbb{C}$ and $p \notin \mathbb{N}$. Nevertheless, Lemma 21 enables one to see that the admissability of $t \to \frac{1}{|g(0,t)|},\ t \in \Delta$ implies the admissability of $t \to \frac{1}{|g_p(0,t)|},\ t \in \Delta$.

(c) *Suppose $f \in C([0,\infty) : \mathbb{K})$ and $g : \Delta \times \Delta \to \mathbb{K}$ is defined by $g(x,t) := e^{f(|x|)-f(|x+t|)},\ x,\ t \in \Delta$. Then g is continuous and satisfies (H1)-(H4). Notice also that the Lipschitz continuity of f implies that $t \mapsto \frac{1}{|g(0,t)|},\ t \in \Delta$ is an admissible weight function.*

From now on, we assume that the continuous function $g : \Delta\times\Delta \to \mathbb{K}$ satisfies (H1)-(H4) and that the mapping $\rho(t) = \frac{1}{|g(0,t)|},\ t \in \Delta$ is an admissible weight function. Set $g_n(x,t) := g^n(x,t),\ x,\ t \in \Delta,\ n \in \mathbb{N}_0$ and $\rho_n(t) := \rho^n(t),\ t \in \Delta,\ n \in \mathbb{N}_0$. Due to previous example, one gets that, for every $n \in \mathbb{N}_0$, the continuous function g_n satisfies (H1)-(H4) and that $t \mapsto \rho_n(t),\ t \in \Delta$ is an admissible weight function.

Theorem 23. *Suppose $i,\ j \in \mathbb{N}_0,\ p \in [1,\infty)$ and X_i is either $L^p_{\rho_i}(\Delta,\mathbb{K})$ or $C_{0,\rho_i}(\Delta,\mathbb{K})$. Define $(T_j(t))_{t\in\Delta} \subset L(X_i)$ and $(\widetilde{T_{i+j}}(t))_{t\in\Delta} \subset L(X_{i+j})$ by:*

$$(T_j(t)f)(x) := g_j(x,t)f(x+t),\ x,\ t \in \Delta,\ f \in X_i \text{ and:} \tag{46}$$

$$(\widetilde{T_{i+j}}(t)\tilde{f})(x) := \tilde{f}(x+t),\ x,\ t \in \Delta,\ \tilde{f} \in X_{i+j}. \tag{47}$$

Then the operator family $(T_j(t))_{t\in\Delta}$ is strongly continuous semigroup in X_i and the following holds:

(i) *$(T_j(t))_{t\in\Delta}$ is S-hypercyclic in X_i iff $(\widetilde{T_{i+j}}(t))_{t\in\Delta}$ is S-hypercyclic in X_{i+j}.*

(ii) *$(T_j(t))_{t\in\Delta}$ is chaotic in X_i iff $(\widetilde{T_{i+j}}(t))_{t\in\Delta}$ is chaotic in X_{i+j}.*

(iii) *$(T_j(t))_{t\in\Delta}$ is topologically mixing in X_i iff $(\widetilde{T_{i+j}}(t))_{t\in\Delta}$ is topologically mixing in X_{i+j}.*

Proof. Define the mapping $\varphi_{i,j} : X_{i+j} \to X_i$ by:

$$\varphi_{i,j}(\tilde{f})(\tau) := \frac{1}{g_j(0,\tau)}\tilde{f}(\tau),\ \tilde{f} \in X_{i+j},\ \tau \in \Delta. \tag{48}$$

It can be easily seen that $\varphi_{i,j}$ is an isometric isomorphism. Furthermore:

$$\varphi_{i,j} \circ \widetilde{T_{i+j}}(t) = T_j(t) \circ \varphi_{i,j} \text{ and } \varphi^{-1}_{i,j} \circ T_j(t) = \widetilde{T_{i+j}}(t) \circ \varphi^{-1}_{i,j},\ t \in \Delta, \tag{49}$$

the operator family $(\widetilde{T_{i+j}}(t))_{t\in\Delta}$ is a strongly continuous semigroup in X_{i+j} (cf. [16], [28]) and the existence of numbers $M \geq 1$ and $\omega \in \mathbb{R}$ satisfying $||T_j(t)|| \leq M^{i+j}e^{\omega(i+j)|t|},\ t \in \Delta$ follows from Lemma 21 and the admissability of ρ_i. Since g_j satisfies (H1)-(H4), one yields that $(T_j(t))_{t\in\Delta}$ is a semigroup in X_i; the strong continuity of $(T_j(t))_{t\in\Delta}$ can be simply proved as follows. Suppose $f \in X_i$. Then $\lim_{t\to 0,\ t\in\Delta} T_j(t)f = f$ is equivalent to

$$\lim_{t\to 0,\ t\in\Delta} \varphi_{i,j}^{-1} T_j(t)f = \varphi_{i,j}^{-1} f,$$

i.e., to $\lim_{t\to 0,\ t\in\Delta} \widetilde{T_{i+j}}(t)\varphi_{i,j}^{-1} f = \varphi_{i,j}^{-1} f$. The last statement holds because $(\widetilde{T_{i+j}}(t))_{t\in\Delta}$ is a strongly continuous semigroup. Hence, $(T_j(t))_{t\in\Delta}$ is a strongly continuous semigroup in X_i. The statements (i) and (ii) follow from the following observations: $f \in \mathrm{HC}_{\mathrm{S}}(T_j)$ iff $\varphi_{i,j}^{-1} f \in \mathrm{HC}_{\mathrm{S}}(\widetilde{T_{i+j}})$ and $f \in X_j$ is a periodic point for $(T_j(t))_{t\in\Delta}$ iff $\varphi_{i,j}^{-1} f \in X_{i+j}$ is a periodic point for $(\widetilde{T_{i+j}}(t))_{t\in\Delta}$. The assertion (iii) can be justified analogously.

Concerning chaoticity of strongly continuous translation semigroups, we would like to point out that all structural results proved by J. A. Conejero and A. Peris in [16] still hold in the case $\alpha = \frac{\pi}{2}$ which is also allowed in our research. The only vital change compared with the case $\alpha \in (0, \frac{\pi}{2})$ is the construction of periodic points given in the proof of implication (3) $\Rightarrow$ (1) of [16, Corollary 1]. This construction must be technically rearranged by the use of appropriate rectangles and enables one to avoid the overlapping of corresponding sectors $kt + \Delta_{|t|},\ k \in \mathbb{N}$ appearing in the proof of cited result. Taking into consideration this observation as well as Lemmas 20-21, Theorem 17 and Theorem 23, we immediately have the following.

Theorem 24. *Suppose $p \in [1,\infty)$, $\alpha \in (0, \frac{\pi}{2}]$, $i,\ j \in \mathbb{N}_0$, $i + j > 0$, $\Delta \in \{\Delta(\alpha),\ \mathbb{C}\}$ and consider the strongly continuous semigroups $(T_j(t))_{t\in\Delta}$ in $X_i \in \{L^p_{\rho_i}(\Delta,\mathbb{K}),\ C_{0,\rho_i}(\Delta,\mathbb{K})\}$ (see Theorem 23). Then:*

(i) The semigroup $(T_j(t))_{t\in\Delta}$, given by (46), is chaotic in $L^p_{\rho_i}(\Delta,\mathbb{K})$ iff for every $\theta \in [0,\infty)$, there exists $t \in \Delta \setminus \{0\}$ such that:

$$\sum_{k=0}^{\infty} \frac{1}{|g(0,\theta + kt)|^{p(i+j)}} < \infty \text{ if } \Delta \neq \mathbb{C},\ \textit{resp.,} \tag{50}$$

there exists $t \in \Delta \setminus \{0\}$ such that:

$$\sum_{k=-\infty}^{\infty} \frac{1}{|g(0,kt)|^{p(i+j)}} < \infty \text{ if } \Delta = \mathbb{C}. \tag{51}$$

(ii) The semigroup $(T_j(t))_{t\in\Delta}$ is chaotic in $L^p_{\rho_i}(\Delta,\mathbb{K})$ iff there exists a ray $R \subset \Delta$ starting at 0 such that, for every $m \in \mathbb{N}$:

$$\int_{F_{R,m}} \frac{dt}{|g(0,t)|^{p(i+j)}} < \infty \text{ if } \Delta \neq \mathbb{C}, \quad \textit{resp.,} \tag{52}$$

$$\int_{F_{\pm R,m}} \frac{dt}{|g(0,t)|^{p(i+j)}} < \infty \text{ if } \Delta = \mathbb{C}. \tag{53}$$

(iii) *Suppose that a ray $R \subset \Delta$ is not contained in the boundary of Δ and that $0 \in R$. The following conditions are equivalent and any of them implies that the semigroup $(T_j(t))_{t\in\Delta}$ is chaotic in $L^p_{\rho_i}(\Delta, \mathbb{K})$:*

(iii.1) *There exists $t \in R \setminus \{0\}$ such that*

$$\sum_{k=0}^{\infty} \frac{1}{|g(0,kt)|^{p(i+j)}} < \infty \text{ if } \Delta \neq \mathbb{C}, \text{ resp.,} \tag{54}$$

$$\sum_{k=-\infty}^{\infty} \frac{1}{|g(0,kt)|^{p(i+j)}} < \infty \text{ if } \Delta = \mathbb{C}. \tag{55}$$

(iii.2) $\int_{F_{R,1}} \frac{dt}{|g(0,t)|^{p(i+j)}} < \infty$ *if* $\Delta \neq \mathbb{C}$, *resp.,* $\int_{F_{\pm R,1}} \frac{dt}{|g(0,t)|^{p(i+j)}} < \infty$ *if* $\Delta = \mathbb{C}$.

(iii.3) *The restriction $(T_j(t))_{t\in R}$ of the semigroup $(T_j(t))_{t\in\Delta}$ to the ray R admits a non-trivial periodic point.*

(iv) *Suppose $i_1,\ i_2,\ j_1,\ j_2 \in \mathbb{N}_0$, $i_1 + j_1 > 0$ and $i_2 + j_2 > 0$. Then the semigroup $(T_{j_1}(t))_{t\in\Delta}$ is chaotic in $C_{0,\rho_{i_1}}(\Delta, \mathbb{K})$ iff the semigroup $(T_{j_2}(t))_{t\in\Delta}$ is chaotic in $C_{0,\rho_{i_2}}(\Delta, \mathbb{K})$. If $\Delta \neq \mathbb{C}$, then the semigroup $(T_{j_1}(t))_{t\in\Delta}$ is chaotic in $C_{0,\rho_{i_1}}(\Delta, \mathbb{K})$ iff for every $\theta \in [0, \infty)$, there exists $t \in \Delta \setminus \{0\}$ such that:*

$$\lim_{k\to\infty} |g(0, \theta + kt)| = \infty. \tag{56}$$

The semigroup $(T_{j_1}(t))_{t\in\mathbb{C}}$ is chaotic in $C_{0,\rho_{i_1}}(\mathbb{C}, \mathbb{K})$ iff there exists $t \in \Delta\setminus\{0\}$ such that:

$$\lim_{k\to\infty} |g(0, kt)| = \lim_{k\to\infty} |g(0, -kt)| = \infty. \tag{57}$$

(v) *Suppose that a ray $R \subset \Delta$ is not contained in the boundary of Δ and that $0 \in R$. The following conditions are equivalent and any of them implies that the semigroup $(T_j(t))_{t\in\Delta}$ is chaotic in $C_{0,\rho_i}(\Delta, \mathbb{K})$:*

(v.1) $\lim_{z\to\infty,\ z\in R} |g(0, z)| = \infty$, *if* $\Delta \neq \mathbb{C}$, *resp.,* $\lim_{z\to\infty,\ z\in\pm R} |g(0, z)| = \infty$, *if* $\Delta = \mathbb{C}$.

(v.2) *The restriction $(T_j(t))_{t\in R}$ of the semigroup $(T_j(t))_{t\in\Delta}$ to the ray R admits a non-trivial periodic point.*

(vi) *Suppose $i_1,\ i_2,\ j_1,\ j_2 \in \mathbb{N}_0$, $i_1 + j_1 > 0$ and $i_2 + j_2 > 0$. Then the semigroup $(T_{j_1}(t))_{t\in\Delta(\alpha)}$, $\alpha \in (0, \frac{\pi}{2}]$ is always positively supercyclic in X_{i_1}. Moreover, the semigroup $(T_{j_1}(t))_{t\in\mathbb{C}}$ is positively supercyclic in X_{i_1} iff the semigroup $(T_{j_2}(t))_{t\in\mathbb{C}}$ is positively supercyclic in X_{i_2} iff there exists a sequence (t_n) in $\mathbb{C}$ such that $\lim_{n\to\infty} t_n = \infty$ and that:*

$$\lim_{n\to\infty} |g(0, t_n) g(0, -t_n)| = \infty. \tag{58}$$

(vii) Suppose $i_1 \in \mathbb{N}_0$ and $j_1 \in \mathbb{N}_0$. If $\Delta = [0, \infty)$ or $\Delta = \Delta(\alpha)$ for an appropriate $\alpha \in (0, \frac{\pi}{2}]$, then the semigroup $(T_{j_1}(t))_{t \in \Delta}$ is S-hypercyclic in X_{i_1} iff for every $\theta \in [0, \infty)$, there exist a sequence (t_n) in $\Delta \setminus \{0\}$ and a sequence (a_n) in $S \setminus \{0\}$ such that $\lim_{n \to \infty} |t_n| = \infty$ and that:

$$\lim_{n \to \infty} a_n^{\frac{1}{i_1 + j_1}} |g(0, \theta + t_n)| = \infty. \tag{59}$$

Suppose $\Delta \in \{\mathbb{R}, \mathbb{C}\}$ and $S \subset [0, \infty)$. Then the semigroup $(T_{j_1}(t))_{t \in \mathbb{C}}$ is S-hypercyclic in X_{i_1} iff for every $\theta \in (0, \infty)$, there exist a sequence (t_n) in $\Delta \setminus \{0\}$ and a sequence (a_n) in $S \setminus \{0\}$ such that $\lim_{n \to \infty} |t_n| = \infty$ and that:

$$\lim_{n \to \infty} a_n^{\frac{(-1)}{i_1 + j_1}} |g(0, \theta - t_n)| = \lim_{n \to \infty} a_n^{\frac{1}{i_1 + j_1}} |g(0, \theta + t_n)| = \infty. \tag{60}$$

In particular, the hypotheses $i_1 + j_1 > 0$ and $i_2 + j_2 > 0$ imply that $(T_{j_1}(t))_{t \in \Delta}$ is hypercyclic in X_{i_1} iff $(T_{j_2}(t))_{t \in \Delta}$ is hypercyclic in X_{i_2}.

The standard proof of following theorem is omitted (cf. also Theorem 23 and [38, Section 4]).

Theorem 25. *Suppose $a : \Omega \to \mathbb{K} \setminus \{0\}$ is continuous, $g : \Omega \times \Delta \to \mathbb{K} \setminus \{0\}$ is given by $g(x, t) := \frac{a(x)}{a(\varphi(t,x))}$, $x \in \Omega$, $t \in \Delta$, $\varphi : \Omega \times \Delta \to \Omega$ is a semiflow and $(T_\varphi(t))_{t \in \Delta}$ is a strongly continuous semigroup in $C_{0,\rho}(\Omega, \mathbb{K})$, resp. $L^p_\rho(\Omega, \mathbb{K})$. Set, for every $x \in \Omega$, $t \in \Delta$ and $f \in C_{0, \frac{\rho}{|a|}}(\Omega, \mathbb{K})$, resp., $f \in L^p_{\frac{\rho}{|a|}}(\Omega, \mathbb{K})$:*

$$(T_{g,\varphi}(t)f)(x) := g(x, t) f(\varphi(t, x)). \tag{61}$$

Then $(T_{g,\varphi}(t))_{t \in \Delta}$ is a strongly continuous semigroup in $C_{0, \frac{\rho}{|a|}}(\Omega, \mathbb{K})$, resp. $L^p_{\frac{\rho}{|a|}}(\Omega, \mathbb{K})$, and $(T_\varphi(t))_{t \in \Delta}$ is S-hypercyclic, resp. chaotic, topologically mixing, in $C_{0,\rho}(\Omega, \mathbb{K})$, resp. $L^p_\rho(\Omega, \mathbb{K})$ iff $(T_{g,\varphi}(t))_{t \in \Delta}$ is S-hypercyclic, resp. chaotic, topologically mixing, in $C_{0, \frac{\rho}{|a|}}(\Omega, \mathbb{K})$, resp. $L^p_{\frac{\rho}{|a|}}(\Omega, \mathbb{K})$.

Example 26. *(i) Suppose $j \in \mathbb{N}$, $\Delta = \Delta(\alpha)$ for some $\alpha \in (0, \frac{\pi}{2}]$, $g(x, t) = (1 + \frac{t_1}{x_1 + a})^b e^{\int_{x_2}^{x_2 + t_2} h(s)ds}$, where $a > 0$, $b \in \mathbb{C}$ ($1^b = 1$) and $h : \mathbb{R} \to \mathbb{C}$ is a bounded measurable function. Due to Theorem 24, the semigroup $(T_j(t))_{t \in \Delta}$, given by (46), is chaotic in $X = C_0(\Delta, \mathbb{C})$ iff for every $\theta \in [0, \infty)$, there exists $t = t_1 + it_2 \in \Delta \setminus \{0\}$ so that:*

$$\lim_{k \to \infty} (1 + \frac{\theta + kt_1}{a})^{Re(b)} e^{\int_0^{kt_2} Re(h(s))ds} = +\infty. \tag{62}$$

If $Re(b) > 0$, then one can choose $t = 1$ in order to see that the semigroup $(T_j(t))_{t \in \Delta}$ is chaotic. Suppose now $Re(b) = 0$; then it is straightforward to *verify that the semigroup* $(T_j(t))_{t\in\Delta}$ *is chaotic iff* $\int\limits_0^{+\infty} Re(h(s))ds = +\infty$ *or* $\int\limits_{-\infty}^{0} Re(h(s))ds = -\infty$. *The case* $Re(b) < 0$ *is non-trivial. For example, if* $h(s) = \frac{d}{ds}\ln(s^{2n}+1),\ s \in \mathbb{R}$, *then* $(T_j(t))_{t\in\Delta}$ *is chaotic iff* $Re(b) > -2n$. *Let us suppose now* $p \in [1,\infty)$ *and* $X = L^p(\Delta,\mathbb{K})$. *Then the semigroup* $(T_j(t))_{t\in\Delta}$ *is chaotic in* $L^p(\Delta,\mathbb{K})$ *iff for every* $\theta \in [0,\infty)$, *there exists* $t = t_1+it_2 \in \Delta\setminus\{0\}$ *so that:*

$$\sum_{k=1}^{\infty} \frac{1}{(1+\frac{\theta+kt_1}{a})^{jpRe(b)} e^{jp\int\limits_0^{kt_2} Re(h(s))ds}} < \infty. \tag{63}$$

Hence, the chaoticity of $(T_j(t))_{t\in\Delta}$ *depends on* j. *For the sake of brevity, we consider only the case* $j = 1$. *If* $Re(b) > \frac{1}{p}$, *then one can choose* $t = 1$ *in order to conclude that* $(T_1(t))_{t\in\Delta}$ *is chaotic. The situation is more complicated when* $Re(b) \leq \frac{1}{p}$. *To illustrate this, suppose* $h(s) = \frac{d}{ds}\ln(\ln(s^2+2)),\ s \in \mathbb{R}$. *In the case* $Re(b) = \frac{1}{p}$ *and* $p > 1$, *the semigroup* $(T_1(t))_{t\in\Delta}$ *is chaotic since* $\int\limits_2^{\infty} \frac{d\xi}{\xi \ln^p \xi} < \infty$; *analogously, the semigroup* $(T_1(t))_{t\in\Delta}$ *is not chaotic in the case* $p = 1$ *and* $Re(b) = 1$. *Finally, suppose* $Re(b) < \frac{1}{p}$ *and* $h(s) = \frac{d}{ds}\ln(s^{2n}+1),\ s \in \mathbb{R}$. *Then* $(T_1(t))_{t\in\Delta}$ *is chaotic iff* $p(Re(b)+2n) > 1$.

(ii) Suppose $S \subset [0,\infty)$, $\Delta = \mathbb{K} = \mathbb{C}$, $p \in [1,\infty)$ *and* $\rho_i : \mathbb{R} \to (0,\infty)$ *is an admissible weight function,* $i = 1, 2$. *Define* $\rho(t_1+it_2) := \rho_1(t_1)\rho_2(t_2),\ t_1,\ t_2 \in \mathbb{R}$. *Then it is straightforward to verify that* $\rho : \Delta \to (0,\infty)$ *is an admissible weight function. Suppose* $\tilde{X} \in \{L^p_\rho(\Delta,\mathbb{K}),\ C_{0,\rho}(\Delta,\mathbb{K})\}$; *then Theorem 17 implies that the semigroup* $(\tilde{T}(t))_{t\in\mathbb{C}}$, *given by (36), is positively supercyclic in* $\tilde{X}$ *iff there exist two real sequences* (a_n) *and* (b_n) *such that* $\lim\limits_{n\to\infty}(a_n^2+b_n^2) = \infty$ *and that:*

$$\lim_{n\to\infty} \rho_1(-a_n)\rho_1(a_n)\rho_2(-b_n)\rho_2(b_n) = 0. \tag{64}$$

The S-hypercyclicity of $(\tilde{T}(t))_{t\in\mathbb{C}}$ *can be easily described by means of Theorem 17. For example, put* $\rho_2(t) := 1$ *and* $\rho_1(t) := e^{-\int\limits_0^t h(s)ds},\ t \in \mathbb{R}$, *where* $h : \mathbb{R} \to \mathbb{R}$ *is a bounded measurable function satisfying* $\int\limits_0^{+\infty} h(s)ds = +\infty$ *and* $\int\limits_{-\infty}^{0} |h(s)|ds < \infty$. *By Theorem 17, we have that* $(\tilde{T}(t))_{t\in\mathbb{C}}$ *is not hypercyclic and that* $(\tilde{T}(t))_{t\in\mathbb{C}}$ *is S-hypercyclic iff* $infS = 0$ *or* $supS = \infty$. *We end (ii) with the following adaptation of [62, Example 1]. Put* $\Delta = \Delta(\alpha)$ *for some* $\alpha \in (0, \frac{\pi}{2}]$, $\tilde{X} = C_{0,\rho}(\Delta,\mathbb{K})$ *and* $\rho(t_1+it_2) := e^{-(t_1+1)\cos(\ln(t_1+1))+1},\ t_1+it_2 \in \Delta$. *Notice that* ρ *is an admissible weight function and that the translation semigroup* $(\tilde{T}(t))_{t\in\Delta}$ *is hypercyclic but not chaotic.*

(iii) *Suppose* $S \subset [0,\infty)$, $\Delta = [0,\infty)$ *and* $\Omega = (0,\infty)^n$, $n \in \mathbb{N}$. *Define a semiflow* $\varphi : \Delta \times \Omega \to \Omega$ *by* $\varphi(t,x_1,\cdots,x_n) := (e^t x_1,\cdots,e^t x_n)$ *and a continuous function* $\rho : \Omega \to (0,\infty)$ *by* $\rho(x_1,\cdots,x_n) := \frac{1}{1+x_1^2+\cdots+x_n^2}$, $t \in \Delta$, $(x_1,\cdots,x_n) \in \Omega$. *Owing to [38, Theorem 3.7] and Theorem 18, one gets that* $(T_\varphi(t))_{t\geq 0}$ *is a non-hypercyclic strongly continuous semigroup in* $C_{0,\rho}(\Omega,\mathbb{K})$ *and that* $(T_\varphi(t))_{t\geq 0}$ *is S-hypercyclic in* $C_{0,\rho}(\Omega,\mathbb{K})$ *iff* $\inf S = 0$, *and this in particular shows that the concepts of hypercyclicity, resp. positive supercyclicity, and S-hypercyclicity do not coincide if S is bounded, resp. unbounded. Suppose, further,* $n = 1$ *and* $a : \Omega \to \mathbb{R} \setminus \{0\}$ *is continuously differentiable. The semigroup solution of the following partial differential equation in* $C_{0,\frac{\rho}{|a|}}(\Omega,\mathbb{K})$:

$$u_t = xu_x - x\frac{a'(x)}{a(x)}u,\ t \geq 0,\ u(0,x) = f(x),\ x \in \Omega,$$

is given by

$$(T_{g,\varphi}(t)f)(x) := \frac{a(x)}{a(e^t x)}f(e^t x),\ t \geq 0,\ x \in \Omega.$$

By Theorem 25, we have that $(T_{g,\varphi}(t))_{t\geq 0}$ *is S-hypercyclic in* $C_{0,\frac{\rho}{|a|}}(\Omega,\mathbb{K})$ *iff* $(T_\varphi(t))_{t\geq 0}$ *is S-hypercyclic in* $C_{0,\rho}(\Omega,\mathbb{K})$ *iff* $\inf S = 0$.

(iv) *Suppose* $p \in [1,\infty)$, $C \geq 0$, $\omega \in \mathbb{R}$, $\Delta = \Delta(\alpha)$ *for some* $\alpha \in (0,\frac{\pi}{2})$ *and* $f_n : [n,n+1] \to \mathbb{R}$ *is a function of bounded variation for all* $n \in \mathbb{N}_0$. *Suppose, in addition, that for every* $m,\ n \in \mathbb{N}_0$ *with* $m > n$ *and for every* $t \in [n,n+1]$:

$$f_n(t) \leq C + \omega(m-n) + f_m(t+m-n) \text{ and} \tag{65}$$

$$V := \sup_{n\in\mathbb{N}_0} V_n^{n+1}(f_n) < \infty. \tag{66}$$

Define $\rho : \Delta \to (0,\infty)$ *by* $\rho(t) := e^{f_{\lfloor |t| \rfloor}(|t|)}$, $t \in \Delta$. *Let* $t,\ x \in \Delta$, $|t| \in [n,n+1)$ *and let* $|t+x| \in [m,m+1)$ *for some* $m,\ n \in \mathbb{N}_0$ *with* $m \geq n$. *The assumption* $m = n$ *immediately implies that* $\rho(t) \leq e^V \rho(t+x)$. *Suppose now that* $m > n$; *then one gets* $|x| \geq |t+x| - |t| \geq m-n-1$ *and:*

$$\rho(t) = e^{f_n(|t|)} \leq e^{C+\omega(m-n)+f_m(|t|+(m-n))} \leq e^{(C+V)+\omega(m-n)+f_m(|t+x|)}$$

$$\leq e^{(C+V+|\omega|)+|\omega||x|+f_m(|t+x|)} = e^{(C+V+|\omega|)}e^{|\omega||x|}\rho(t+x).$$

Hence, $\rho(t) \leq e^{(C+V+|\omega|)}e^{|\omega||x|}\rho(t+x)$, $t,\ x \in \Delta$ *and* ρ *is an admissible weight function. Let us consider the next special case:*

$$f_n(t) := f_0(t-n) - a_n\omega,\ t \in [n,n+1],\ n \in \mathbb{N}, \tag{67}$$

where $\omega > 0$ *and* (a_n) *is a sequence of real numbers satisfying:*

$$1 + a_n \geq a_{n+1},\ n \in \mathbb{N}_0. \tag{68}$$

Notice that (68) forces $f_n(t) \leq \omega + f_{n+1}(t+1),\ n \in \mathbb{N}_0,\ t \in [n, n+1]$ *and that an induction argument shows the validity of (65) with* $C = 0$. *Furthermore,* $V_n^{n+1}(f_n) = V_0^1(f_0),\ n \in \mathbb{N}$, *(66) holds and it is straightforward to see that the translation semigroup* $(\tilde{T}(t))_{t\in\Delta}$ *is hypercyclic in* $\tilde{X} \in \{L^p_\rho(\Delta, \mathbb{K}),\ C_{0,\rho}(\Delta, \mathbb{K})\}$ *iff* $\limsup_{n\to\infty} a_n = +\infty$. *Let us prove that* $(\tilde{T}(t))_{t\in\Delta}$ *is chaotic in* $C_{0,\rho}(\Delta, \mathbb{K})$ *iff there exists* $t \in (0,\infty)$ *such that* $\lim_{n\to\infty} a_{\lfloor nt \rfloor} = +\infty$ *iff* $\lim_{n\to\infty} a_n = +\infty$. *Indeed, suppose that* $(\tilde{T}(t))_{t\in\Delta}$ *is chaotic. According to [16, Theorem 5], we have the existence of a complex number* $t_0 \in \Delta \setminus \{0\}$ *satisfying* $\lim_{n\to\infty} f_{\lfloor n|t_0| \rfloor}(n|t_0|) = -\infty$. *Put* $t = |t_0|$ *and observe that the boundedness of* f_0 *implies* $\lim_{n\to\infty} a_{\lfloor nt \rfloor} = +\infty$. *Let us suppose now* $\theta \geq 0,\ t > 0,\ \lim_{n\to\infty} a_{\lfloor nt \rfloor} = +\infty$ *and* $\theta \in [kt, (k+1)t)$ *for some* $k \in \mathbb{N}_0$. *Owing to (68),*

$$(kt - \theta + 2) + a_{\lfloor nt+\theta \rfloor} \geq a_{\lfloor (n+k)t \rfloor}. \tag{69}$$

Therefore, $\lim_{n\to\infty} a_{\lfloor nt+\theta \rfloor} = +\infty,\ \lim_{n\to\infty} \rho(\theta + nt) = 0$ *and this yields that the condition 1. given in the formulation of [16, Theorem 5] holds with* $R = [0,\infty)$. *In conclusion, one gets that* $(\tilde{T}(t))_{t\in\Delta}$ *is chaotic. Keeping in mind (68), we have that the existence of a number* $t \in (0,\infty)$ *satisfying* $\lim_{n\to\infty} a_{\lfloor nt \rfloor} = +\infty$ *is obviously equivalent with* $\lim_{n\to\infty} a_n = +\infty$. *Analogously,* $(\tilde{T}(t))_{t\in\Delta}$ *is chaotic in* $L^p_\rho(\Delta, \mathbb{K})$ *iff there exists* $t \in (0,\infty)$ *such that* $\sum_{n=1}^{\infty} e^{-p\omega a_{\lfloor nt \rfloor}} < \infty$. *Suppose, for the time being,* $p\omega \leq 1$ *and* $a_n = \ln(n+1),\ n \in \mathbb{N}_0$. *Then* $(\tilde{T}(t))_{t\in\Delta}$ *is chaotic in* $C_{0,\rho}(\Delta, \mathbb{K})$ *but* $(\tilde{T}(t))_{t\in\Delta}$ *is not chaotic in* $L^p_\rho(\Delta, \mathbb{K})$. *Finally, suppose that* (a_n) *satisfies (68),* $\limsup_{n\to\infty} a_n = +\infty$ *and* $\lim_{n\to\infty} a_n \neq +\infty$. *Then* $(\tilde{T}(t))_{t\in\Delta}$ *is hypercyclic in* $\tilde{X}$ *but* $(\tilde{T}(t))_{t\in\Delta}$ *is not chaotic in* $\tilde{X}$.

(v) Suppose $\Delta = [0,\infty),\ \Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\},\ p > 0,\ q \in \mathbb{R}$ *and*

$$\varphi(t,x,y) := e^{pt}(x\cos qt - y\sin qt, x\sin qt + y\cos qt),\ t \geq 0,\ (x,y) \in \Omega.$$

One can simply verify that $\varphi : \Delta \times \Omega \to \Omega$ *is a semiflow and that, by putting* $z = x + iy$, *we have* $\varphi(t,z) = e^{t(p+iq)}z,\ z \in \Omega,\ k \in \mathbb{N}$. *Proceeding as in [44, Example 4], one obtains that* $(T_\varphi(t))_{t\geq 0}$ *is topologically mixing in* $C(\Omega, \mathbb{K})$. *We will prove that* $(T_\varphi(t))_{t\geq 0}$ *is chaotic in* $C(\Omega, \mathbb{K})$. *Indeed, suppose* $f \in C_c(\Omega, \mathbb{K})$, $a > a_0 > 1$, *supp*$f \subset \{z \in \Omega : a_0 \leq |z| \leq a\}$, (a_n) *is a strictly increasing sequence in* (a,∞) *satisfying* $\lim_{n\to\infty} a_n = \infty$ *and* $t_n := \frac{\ln a_n}{p},\ n \in \mathbb{N}$. *Define, for every* $n \in \mathbb{N}$, *a function* $f_n : \Omega \to \mathbb{K}$ *as follows. Fix a number* $z \in \Omega$ *and suppose that* $|z| \in [e^{kt_np}, e^{(k+1)t_np}) = [e^{ka_n}, e^{(k+1)a_n})$ *for some* $k \in \mathbb{N}_0$. *Put now* $f_n(z) := f(e^{-kt_np}ze^{-ikqt_n})$. *By construction,* $f_n \in C(\Omega, \mathbb{K})$, $f_n(z) = f(z),\ z \in \Omega,\ |z| \leq a_n$ *and* $f_n(\varphi(t_n,z)) = f_n(z),\ z \in \Omega,\ n \in \mathbb{N}$. *Thereby,* f_n *is a* t_n*-periodic point of* $(T_\varphi(t))_{t\geq 0}$ *and* $\lim_{n\to\infty} f_n = f$ *in* $C(\Omega, \mathbb{K})$.

In the next example, we identify $\mathbb{C}$ and $\Delta(\alpha),\ \alpha \in (0, \frac{\pi}{2}]$ with corresponding subsets of $\mathbb{R}^2$.

Example 27. *Let $m \in \mathbb{N}_0$ and let $C^m(\Delta, \mathbb{K})$ denote the vector space of all functions $\varphi : \Delta \to \mathbb{K}$ which are m times continuously differentiable in Δ° and whose partial derivatives $D^\alpha \varphi$ can be extended continuously throughout Δ; if $|\alpha| \leq m$ and $\varphi \in C^m(\Delta, \mathbb{K})$, then we also denote by $D^\alpha \varphi$ the extended partial derivative on Δ. Set $C^\infty(\Delta, \mathbb{K}) := \bigcap_{m \in \mathbb{N}} C^m(\Delta, \mathbb{K})$. The Fréchet topology on $C^m(\Delta, \mathbb{K})$, resp. $C^\infty(\Delta, \mathbb{K})$, defines the system of increasing seminorms:*

$$p_n(f) := \sup_{\tau \in \Delta_n} \sup_{|\alpha| \leq m} |D^\alpha f(\tau)|, \ f \in C^m(\Delta, \mathbb{K}), \ \textit{resp.}$$

$$p_n(f) := \sup_{\tau \in \Delta_n} \sup_{|\alpha| \leq n} |D^\alpha f(\tau)|, \ f \in C^\infty(\Delta, \mathbb{K}), \ n \in \mathbb{N}.$$

Suppose, further, $X = C^m(\Delta, \mathbb{K})$ for some $m \in \mathbb{N}_0$ or $X = C^\infty(\Delta, \mathbb{K})$. It can be easily verified that the translation semigroup $(T(t))_{t \in \Delta}$ is a locally equicontinuous semigroup in X. We will prove that $(T(t))_{t \in \Delta}$ is chaotic by means of concrete construction of periodic points. Suppose that $\Delta = \Delta(\alpha)$ for some $\alpha \in (0, \frac{\pi}{2}]$ and that a C^∞ function $\varphi_\theta : \mathbb{R}^2 \to \mathbb{R}$ satisfies, for every $\theta \in (0, \infty)$:

$$\varphi_\theta(\tau) = \begin{cases} 1, & |\tau| \leq \theta, \\ 0, & |\tau| \geq \theta + 1. \end{cases}$$

Define, for every $f \in X$ and $n \in \mathbb{N}$, the function $f_n : \Delta \to \mathbb{K}$ by $f_n(\tau) := f(\tau)\varphi_n(\tau), \ \tau \in \Delta$. Clearly, $f_n \in X$ and $\lim_{n \to \infty} f_n = f$ in X. Hence, the set $\{f \in X : supp f \text{ is a compact subset of } \Delta\}$ is dense in X which implies that X_0 is dense in X. Let us prove that X_∞ is also dense in X. Suppose $g \in X$ and $supp g \subset \Delta_\theta$ for some $\theta > 0$. The well known extension type theorems for continuously differentiable functions and the Whitney extension theorem (see [54, p. 350], [58, p. 305-306] and [61]) imply that there exists a C^m (C^∞) function $\tilde{g} : \mathbb{R}^2 \to \mathbb{K}$ such that $\tilde{g}(\tau) = g(\tau), \ \tau \in \Delta_{\theta+1}$. Define now, for all $t \in \Delta$ with $|t| > 2\theta + 2$,

$$g_t(\tau) := \begin{cases} \tilde{g}(\tau - t)\varphi_{\theta+1}(\tau - t), & \tau \in B(t, \theta + 1) \cap \Delta, \\ 0, & \textit{otherwise.} \end{cases} \tag{70}$$

It is evident that $g_t \in X$ for all $t \in \Delta$ with $|t| > 2\theta + 2$ and that there exists $n_0 \in \mathbb{N}$ such that $g_n \in X$, $T(n)g_n = g, \ n \geq n_0$ and that $\lim_{n \to \infty} g_n = 0$ in X. This implies that X_∞ is dense in X. Using similar argumentation and [42, Theorem 2.6, p. 47], one obtains that the set of all polynomials with rational coefficients is sequentially dense in X; especially, X is separable and Theorem 4 yields that $(T(t))_{t \in \Delta}$ is topologically transitive. To prove that the set of all periodic points of $(T(t))_{t \in \Delta}$ is dense in X, let us define, for all sufficiently large numbers $n \in \mathbb{N}$, the function $v_n : \Delta \to \mathbb{K}$ by setting:

$$v_n(\tau) := \begin{cases} g(\tau), & \tau \in \Delta_\theta, \\ \tilde{g}(\tau - nk)\varphi_{\theta+1}(\tau - nk), & \tau \in \bigcup_{k \in \mathbb{N}} (nk + B(0, \theta + 1)), \\ 0, & \textit{otherwise.} \end{cases} \tag{71}$$

We infer that $T(n)v_n = v_n,\ n \geq n_0$ *and that* $\lim_{n\to\infty} v_n = g$ *in* X. *This ends the proof in the case* $\Delta = \Delta(\alpha)$ *while the proof in the case* $\Delta = [0,\infty)$ *follows by making use of E. Borel's theorem (cf. [54, p. 324]) and mollification. The proof in the case* $\Delta \in \{\mathbb{R},\ \mathbb{C}\}$ *is much easier.*
It is also worth noting that $(T(t))_{t\in\Delta}$ *is topologically mixing. We will show this only in the case* $\Delta = \Delta(\alpha)$. *So, fix an* $\varepsilon > 0$ *and a function* $g \in X$ *with suppg* $\subset \Delta_\theta$ *for some* $\theta > 0$. *Suppose now that* $r \in (2\theta + 2, \infty)$ *and that* $2^{\theta+1-r} < \frac{\varepsilon}{4}$. *Then* $T(t)g_t = g$ *and* $d(0, g_t) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(g_t)}{1+p_n(g_t)} = \sum_{n=\lceil r-\theta-1\rceil}^{\infty} \frac{1}{2^n} \frac{p_n(g_t)}{1+p_n(g_t)} \leq \frac{1}{2^{\lceil r-\theta-1\rceil -1}} < \varepsilon$ *for all* $t \in \Delta \setminus \Delta_r$. *Since* $\lim_{t\to\infty,\ t\in\Delta} T(t)f = 0$ *for all* $f \in X$ *with compact support, one can proceed as in the proofs of [28, Theorems 2.2-2.3] in order to deduce that, for every* $f,\ g \in X$ *and* $\varepsilon > 0$, *there exists* $r > 0$ *such that for every* $t \in \Delta \setminus \Delta_r$ *there exists* $v \in X$ *so that* $d(f, v) < \varepsilon$ *and* $d(g, T(t)v) < \varepsilon$. *The last statement simply implies that* $(T(t))_{t\in\Delta}$ *is topologically mixing. Finally, let us notice that the infinitesimal generator* A *of* $(T(t))_{t\in\Delta}$, *where* $\Delta \in \{[0,\infty),\ \mathbb{R}\}$, *satisfies* $\sigma_p(A) = \mathbb{K}$.

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