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BILIPSCHITZ MAPPINGS BETWEEN SECTORS IN PLANES AND QUASI-CONFORMALITY

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Abstract

We consider bilipschitz properties of conformal and quasiconformal mappings between sectors with respect to j-metric. Special attention is paid to the behaviour of the bilipschitz constant as the qc-constant K tends to 1.

1 Introduction

Quasiconformal mappings were introduced by H Grötzsch in 1928. Quasiconformal mappings in R^n are natural generalization of conformal functions of one complex variable. Their systematic study was begun by F. W. Gehring [1] and J. Väisälä [2] in 1961. Since then the theory has been actively studied [3, 4]. Quasiconformal mappings are characterized by the property that there exists a constant $C \geq 1$ such that the infinitesimally small spheres are mapped onto infinitesimally small ellipsoids with the ratio of the larger "semiaxis" to the smaller one bounded from above by C.
Quasiconformal mappings have a special subclass, so called bilipschitz maps.

Definition 1. A homeomoprhism $f : G \to fG$ satisfying

$$|x-y|/L \leq |f(x)-f(y)| \leq L|x-y|$$

for all $x,\ y \in G$ is called $L-$bilipschitz.

The distance ratio metric or j_G-metric in a proper subdomain G of the Euclidean space $R^n,\ n \geq 2$, is defined by

$$j_G(x,y) = \log\Big(1 + \frac{|x-y|}{min\{d(x),d(y)\}}\Big)$$

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where $d(x)$ is the Euclidean distance between x and ∂G. This metric was first introduced by F. W. Gehring and B. G. Osgood [5] and in the above form by M. Vuorinen [6].

The quasihyperbolic metric was introduced by F. W. Gehring and B. P. Palka [7]. For a domain $G \subsetneq R^n,\ n \geq 2$ we define the quasihyperbolic length of a rectifiable arc $\gamma \subset G$ by

$$l_k(\gamma) = \int_\gamma \frac{|dz|}{d(z, \partial G)},$$

where $d(z, \partial G)$ is the Euclidean distance between z and ∂G, and the quasihyperbolic metric by

$$k_G(x, y) = \inf_\gamma l_k(\gamma),$$

where the infimum is taken over all rectifiable curves in G joining x and y. By definition of j_G and k_G metrics it is easy to see that boundary ∂G defines the distances $k_G(x, y)$ and $j_G(x, y)$ for $x, y \in G$. F. W. Gehring and B. P. Palka showed [7] that

$$j_G(x, y) \leq k_G(x, y)$$

for all domains $G \subsetneq R^n$ and $x, y \in G$.

Definition 2. A domain $G \subsetneq R^n$ is said to be uniform, if there exists a number $A \geq 1$ such that

$$k_G(x, y) \leq A \cdot j_G(x, y)$$

for all $x, y \in G$.

Therefore on uniform domains we have the existence of a two-sided linear estimate of the quasihyperbolic metric in terms of the j_G-metric, so we can say that they are equivalent [8].

Note that the inverse of $K-$quasiconformal mapping is also $K-$quasiconformal mapping.

However, if $f : G \to G'$ is harmonic and $K-$quasiconformal mapping, it does not follow that $f^{-1} : G \to G$ is harmonic.

This fact explains why two-sided estimates are more difficult to prove for such mappings. We have the following theorem [9].

Theorem 1.1. *Suppose G and G' are proper domains in R^2. If $f : G \to G'$ is $K-$quasiconformal and harmonic, then it is bilipschitz with respect to quasihyperbolic metrics on G and G'.*

2 Mappings between plane sectors

Here the main result is obtained for the plane sectors:

$$S(a) = \{z : \ 0 < arg \ z < a\}$$

Since sector is uniform domain and j and k are equivalent on $S(a)$, the following theorem is a consequence of Theorem 1.1.

Theorem 2.1. *Any harmonic $K-$quasiconformal mapping $\varphi : \ S(a) \to S(b)$ is bilipschitz also with respect to j metric. (Conformal mapping is a special case).*

Note that the problem of characterizing bilipschitz mappings for several classes of mappings and domains was suggested in [10, pp.322-323] for many different metrics including the distance ratio metric. Hence it is of interest to study the sharpness of the above result. Here we are interested in the following question:

Conjecture 1. For $a, \ b \in (0, \pi)$ and $K \geq 1$ there exists a constant C such that $C \to 1$ when $a \to b$ and $K \to 1$ and for every $K-$quasiconformal mapping f : $S(a) \to S(b)$ we have

$$j_{S(b)}(f(a), \ f(b)) \leq C \cdot j_{S(a)}(a, b).$$

We will show below that this plausible conjecture is in fact false.

In some special cases one can get an explicit constant C.

We note here that map $\omega : \ S(\alpha) \to S(\beta)$ given by $\omega(z) = z^k$, $k = \beta/\alpha$, satisfies

$$\frac{j_{S(\beta)}(\omega(z_1), \omega(z_2))}{j_{S(\alpha)}(z_1, z_2)} \in \left[\frac{1}{C}, C\right], \ C = C(\alpha, \beta)$$

if $|z_1| = |z_2|$.

We choose two points in $S(\alpha)$ which are at the same distance from zero (on the same arc): $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$, we can suppose that $0 < \theta_1 < \theta_2 < \alpha$. Then we have $\omega(z_1) = r^k e^{ik\theta_1}$ and $\omega(z_2) = r^k e^{ik\theta_2}$. Let $\delta_1 = \theta_1$ and $\delta_2 = \alpha - \theta_2$, then we have:

$$j_{S(\alpha)}(z_1, z_2) = \log(1 + a), \ a = \frac{r|e^{i\theta_1} - e^{i\theta_2}|}{min\{r \sin \delta_1, r \sin \delta_2\}}$$

$$j_{S(\beta)}(\omega(z_1), \omega(z_2)) = \log(1 + b), \ b = \frac{r^k|e^{ik\theta_1} - e^{ik\theta_2}|}{min\{r^k \sin k\delta_1, r^k \sin k\delta_2\}}$$

Without loss of generality suppose $\delta_1 \leq \delta_2$. Then $\delta_1 \leq \dfrac{\alpha}{2}$ and also $k\delta_1 \leq \dfrac{\beta}{2}$. Then

$$a = \frac{|e^{i\theta_1} - e^{i\theta_2}|}{\sin \delta_1} \quad , \quad b = \frac{|e^{ik\theta_1} - e^{ik\theta_2}|}{\sin k\delta_1}$$

First notice that

$|e^{ik\theta_1} - e^{ik\theta_2}| \leq k|e^{i\theta_1} - e^{i\theta_2}|$ (Lagrange's theorem). (1)
$((z^k)' = kz^{k-1},\ |(z^k)'| \leq k$ in unit disc).

Further, function $\Phi(x) = \dfrac{\sin kx}{\sin x}$ is strictly positive for $0 < x \leq \dfrac{\pi}{2k}$ (for $k > 1$ function Φ is decreasing) and by prolongation by continuity $\Phi(0) = k$, so on that interval attains strictly positive minimum $C = C(k)$. In the case $k > 1$

$$\Phi_{min}(x) = \Phi\Big(\frac{\pi}{2k}\Big) = \frac{1}{\sin \frac{\pi}{2k}}.$$

So, $\Phi(x) \geq C$, i.e.

$\sin k\delta_1 \geq C{\cdot}\sin \delta_1$. (2)

Since $k\delta_1 \leq \dfrac{\beta}{2} \leq \dfrac{\pi}{2}$ $(\beta \leq \pi)$ we can suppose that $0 < \delta_1 \leq \dfrac{\pi}{2k}$.

Combining (1) and (2) we have

$$b \leq \frac{k}{C} a \qquad (b \leq \frac{k}{\sin \frac{\pi}{2k}} a,\ k > 1).$$

Now we apply similar argument to a function $\Psi(x) = \dfrac{\log(1 + tx)}{\log(1 + x)},\ x > 0.$

We see that $\Psi(x)$ is strictly positive on $(0, +\infty)$ and has finite and strictly positive limits at points 0 and $+\infty$:
$\overline{\Psi}(0) = t$ and $\Psi(+\infty) = 1$, so it attains its infimum which is strictly positive, denote it by $m = m(t)$. So, we have

$$\log(1 + tx) \leq m \log(1 + x).$$

For $k > 1,\ t = \dfrac{k}{\sin \frac{\pi}{2k}} > 1$, so we can apply Bernoulli's inequality $\log\Big(1 + \dfrac{k}{\sin \frac{\pi}{2k}} x\Big) \leq \dfrac{k}{\sin \frac{\pi}{2k}} \log(1 + x)$.

Finally,

$$j_{S(\beta)}(\omega(z_1), \omega(z_2)) = \log(1 + b) \leq \log(1 + \frac{k}{C} a) \leq m\Big(\frac{k}{C}\Big) j_{s(\alpha)}(z_1, z_2)$$

where for $k > 1$ $m\left(\frac{k}{C}\right) = \frac{k}{\sin\frac{\pi}{2k}}$.

If we apply this proof for function $\omega^{-1}(g) = g^{1/k}$ we will get

$$\frac{j_{S(\beta)}(\omega(z_1), \omega(z_2))}{j_{S(\alpha)}(z_1, z_2)} \in \left[\frac{1}{m}, m\right].$$

For $k > 1$ we have

$$\frac{j_{S(\beta)}(\omega(z_1), \omega(z_2))}{j_{S(\alpha)}(z_1, z_2)} \in \left[\frac{\sin\frac{\pi}{2k}}{k}, \frac{k}{\sin\frac{\pi}{2k}}\right].$$

So, in this special case we get $C(k) = \left[\frac{\sin\left(\frac{\pi}{2k}\right)}{k}\right]^{-1}$, but only under additional assumption $|z_1| = |z_2|$, with $k = \beta/\alpha$.

However, the conjecture is not true in general, due to the following counterexample:

Example 2.1. *Let $S = S(\pi/2)$ and let φ be the inversion of s with respect to unit circle $C = \{z|\ |z| = 1\}$. Let $z_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $z_2 = (\sqrt{3}, 1), \omega_1 = \varphi(z_1) = z_1$ and $\omega_2 = \varphi(z_2) = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right)$.*

Then a simple calculation shows that

$$j(z_1, z_2) \neq j(\omega_1, \omega_2).$$

Note that φ is harmonic and anticonformal, so $R \circ \varphi : S\left(\frac{\pi}{2}\right) \to S\left(\frac{\pi}{2}\right)$ is a conformal map, where R is reflection with respect to the line $x = y$.

Of course $j(z_1, z_2) \neq j(R \circ \varphi(z_1), R \circ \varphi(z_2))$ which shows that our conjecture is false.

References

[1] F. W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. 101 (1961), 499-519.

[2] J. Väisälä, *On quasiconformal mappings in space*, Am. Acad. Sci. Fenn. Ser AI 298 (1961), 1-36.

[3] J. Väisälä, *Lectures on $n-$Dimensional Quasiconformal Mappings*, Lecture Notes in Math., Vol. 229, Springer-Verlag Berlin, 1971.

[4] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, Berlin, 1988.

[5] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric* - J. Analyse Math. 36 (1979), 50-74.

[6] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Analyse Math. 45 (1985), 69-115.

[7] F. W. Gehring and B. P. Palka, *Quasiconformally homogenous domains* - J. Analyse Math. 30 (1976), 172-199.

[8] V. Manojlović, *On conformally invariant external problems*, Applicable Analysis and Discrete mathematics 3 (1) (2009).

[9] V. Manojlović, *Bi-Lipschicity of quasiconformal harmonic mappings in the plane*, Filomat 23:1 (2009), 85-89.

[10] M. Vuorinen, *Metrics and quasiregular mappings*, in Quasiconformal mappings and their applications, Proceedings of the International Workshop (IWQCMA05) held in Chennai, December 27, 2005–January 1, 2006. Edited by S. Ponnusamy, T. Sugawa and M. Vuorinen, 291–325, Narosa, New Delhi, 2007, ISBN: 978-81-7319-807-6; 81-7319-807-1

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