THE BROWDER AND WEYL SPECTRA
OF AN OPERATOR AND ITS DIAGONAL

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Abstract

If \( T \in B(\mathcal{X}) \) is a Banach space operator and \( E \) is a closed \( T \)-invariant subspace of \( \mathcal{X} \), then the restriction map \( A = T|_E \) and the quotient map \( B = T|_{\mathcal{X}/E} \) are well defined operators in \( B(E) \) and \( B(\mathcal{X}/E) \), respectively. It is proved that: (i) If \( \sigma_x(T) = \sigma_x(A) \cup \sigma_x(B) \), where \( \sigma_x \) is either the Weyl spectrum \( \sigma_w \) or the Weyl essential approximate point spectrum \( \sigma_{aw} \), then \( \sigma(T) = \sigma(A) \cup \sigma(B) \); (ii) if \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \), and \( A^* \) has SVEP (the single–valued extension property), then \( \sigma_x(T) = \sigma_x(A) \cup \sigma_x(B) \); (iii) if \( \sigma(T) = \sigma(A) \cup \sigma(B) \), then a point \( \lambda \) is a pole (resp., finite rank pole) of the resolvent of \( T \) if and only if \( \lambda \) is a pole (resp., finite rank pole) of the resolvents of \( A \) and \( B \). Letting \( \sigma_b \) and \( \sigma_{ab} \) denote, respectively, the Browder spectrum and the Browder essential approximate point spectrum, an operator \( S \in B(\mathcal{X}) \) satisfies Browder’s theorem (resp., \( a \)-Browder’s theorem) if \( \sigma_w(S) = \sigma_b(S) \) (resp., \( \sigma_{aw}(S) = \sigma_{ab}(S) \)); \( S \) satisfies Weyl’s theorem if \( \sigma(S) \setminus \sigma_w(S) = \{ \lambda \in \text{iso}\sigma(S) : 0 < \dim(S - \lambda)^{-1}(0) < \infty \} \). Recall that \( S \) is isoloid if \( \lambda \in \text{iso}\sigma(S) \) implies \( 0 < \dim(S - \lambda)^{-1}(0) \). We prove that: (iv) if \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) (resp., \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \)), then Browder’s theorem (resp., \( a \)-Browder’s theorem) transfers from \( A \) and \( B \) to \( T \); (v) if \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \), and \( A, B \) are isoloid, then Weyl’s theorem transfers from \( A \) and \( B \) to \( T \).

1 Introduction

Let \( B(\mathcal{X}) \) denote the algebra of operators (i.e., bounded linear transformations) on a Banach space \( \mathcal{X} \) into itself. The problem of the relationship between the spectrum, and some of its more distinguished parts, of an (upper triangular) operator \( T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(\mathcal{X}) \) and its diagonal \( (A, B) \) has been considered by a number of authors, amongst them [3, 4, 5, 8, 9, 11, 13, 14]. A related but more demanding
problem which has been considered in the recent past is the following. Let \( T \in B(\mathcal{X}) \) and let \( E \) be a \( T \)-invariant closed subspace of \( \mathcal{X} \). Then the restriction \( A = T|_E \) and the quotient map \( B = T|_E/\mathcal{X} \) are well defined operators in \( B(E) \) and \( B(\mathcal{X}/E) \), respectively. Following Barnes [3], let us call the pair \((A, B)\) the diagonal of \( T \). What is the relationship between the spectrum \( \sigma \), the Fredholm spectrum \( \sigma_s \), the Browder spectrum \( \sigma_b \), the Weyl spectrum \( \sigma_w \) and the Weyl essential approximate point spectrum \( \sigma_{aw} \) of the operator \( T \) and its diagonal \((A, B)\)?

Evidently, if \( T \) is Fredholm, \( T \in \Phi(\mathcal{X}) \), then \( A \) is upper semi-Fredholm; identifying \( B^* \) with \( T^*|_{\mathcal{X}/E} \), it follows that \( B^* \) is upper semi-Fredholm, which implies that \( B \) is lower semi-Fredholm. It is not difficult to verify, [3, 6], that \( \sigma_x(T) \cup \{ \sigma_x(A) \cap \sigma_x(B) \} = \sigma_x(A) \cup \sigma_x(B) \) for \( \sigma_x = \sigma = \sigma_b \). (Thus, if any two of \( T \), \( A \) and \( B \) are invertible or Fredholm or Browder, then so is the third one.) The relationship between the Weyl spectrum, and the Weyl essential approximate point spectrum, of \( A \) and \( B \) is a bit more delicate. The equality \( \sigma_x(T) \cup \{ \sigma_x(A) \cap \sigma_x(B) \} = \sigma_x(A) \cup \sigma_x(B) \) fails for \( \sigma_x = \sigma_w \) and \( \sigma_x = \sigma_{aw} \); however, if the (Fredholm) indices satisfy the equality \( \text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda) \), whenever the left hand side or the right hand side of the equality is finite, then \( \sigma_x(T) \subseteq \sigma_x(A) \cup \sigma_x(B) \) for \( \sigma_x = \sigma_w \) and \( \sigma_{aw} \).

In this paper, we consider operators \( T \in B(\mathcal{X}) \) such that \( \sigma_x(T) = \sigma_x(A) \cup \sigma_x(B) \), where \((A, B)\) is the diagonal of \( T \), for \( \sigma_x = \sigma_w \) or \( \sigma_{aw} \), and prove that such operators satisfy \( \sigma(T) = \sigma(A) \cup \sigma(B) \). In the case in which \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \) and \( A^* \) has SVEP \((\text{the single-valued extension property})\) on the complement \( \sigma_{aw}(T)^c \) of \( \sigma_{aw}(T) \) in the approximate point spectrum \( \sigma_{aw}(T) \), it is seen that \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \). For an operator \( S \in B(\mathcal{X}) \), we say that \( S \) is polaroid (resp., \( a \)-polaroid) at a point \( \lambda \) in the complex plane \( \mathbb{C} \) if either \( \lambda \) is in the resolvent set \( \rho(S) = \mathbb{C} \setminus \sigma(S) \) or \( \lambda \in \text{is} \sigma(S) \) is a pole of the resolvent of \( S \) (resp., \( \lambda \in \text{is} \sigma(S) \)), \( (S - \lambda) \mathcal{X} \) is closed and the ascent \( \text{asc}(S - \lambda) < \infty \). Let \( p(S), p_0(S), p^a(S), p_0^a(S), \pi_0(S) \) and \( \pi^a_0(S) \) denote, respectively, the sets \( p(S) = \{ \lambda : S \) is polaroid at \( \lambda \}, p_0(S) = \{ \lambda \in p(S) : \dim(S - \lambda)^{-1}(0) < \infty \}, p^a(S) = \{ \lambda : S \) is \( a \)-polaroid at \( \lambda \}, p_0^a(S) = \{ \lambda \in p^a(S) : \dim(S - \lambda)^{-1}(0) < \infty \}, \pi_0(S) = \{ \lambda \in \text{is} \sigma(S) : 0 < \dim(S - \lambda)^{-1}(0) < \infty \} \) and \( \pi^a_0(S) = \{ \lambda \in \text{is} \sigma(S) : 0 < \dim(S - \lambda)^{-1}(0) < \infty \} \). If \( \sigma(T) = \sigma(A) \cup \sigma(B) \), then \( \lambda \in p(T) \Leftrightarrow \lambda \in p(A) \cup p(B) = \{ p(A) \cap p(B) \cup (p(A) \cap p(B)) \} \) and \( \lambda \in p_0(T) \Leftrightarrow \lambda \in p_0(A) \cup p_0(B) ; \) if \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \), \( A \) and \( B \) are isoloid, \( \sigma(A) \setminus \sigma_w(A) = \pi_0(A) \) and \( \sigma(B) \setminus \sigma_w(B) = \pi_0(B) \), then \( \sigma(T) \setminus \sigma_{aw}(T) = \pi_0(T) ; \) if \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \), \( \sigma_a(A) \setminus \sigma_{aw}(A) = \pi^a_0(A) \), \( A \) and \( B \) are \( a \)-isoloid, \( \sigma(B) \setminus \sigma_{aw}(B) = \pi^a_0(B) \) and \( A^* \) has SVEP on \( \pi^a_0(T) \), then \( \sigma_a(T) \setminus \sigma_{aw}(T) = \pi^a_0(T) \).

## 2 Results

We start by explaining the terminology already introduced, and by introducing further notation and terminology.

An operator \( S \in B(\mathcal{X}) \) is upper semi-Fredholm (resp., lower semi-Fredholm) at a complex number \( \lambda \in \mathbb{C} \) if the range \((S - \lambda) \mathcal{X} \) is closed and \( \alpha(S - \lambda) = \dim(S - \lambda) \).
\( \lambda^{-1}(0) < \infty \) (resp., \( \beta(S - \lambda) = \dim(\mathcal{X}/(S - \lambda)\mathcal{X}) < \infty \)). Let \( \lambda \in \Phi_+ (S) \) (resp., \( \lambda \in \Phi_-(S) \)) denote that \( S \) is upper semi-Fredholm (resp., lower semi-Fredholm) at \( \lambda \). The operator \( S \) is Fredholm at \( \lambda \), denoted \( \lambda \in \Phi(S) \), if \( \lambda \in \Phi_+(S) \cap \Phi_-(S) \). Let \( \text{ind}(S - \lambda) = \alpha(S - \lambda) - \beta(S - \lambda) \) denote the Fredholm index of \( S - \lambda \). The ascent \( \text{asc}(S - \lambda) \) (resp., the descent \( \text{dsc}(S - \lambda) \)) of \( S - \lambda \) is the least non-negative integer \( n \) such that \( (S - \lambda)^{-n}(0) = (S - \lambda)^{-n+1}(0) \) (resp., the least non-negative integer \( n \) such that \( (S - \lambda)^n \mathcal{X} = (S - \lambda)^{n+1} \mathcal{X} \)); if no such integer exists, then \( \text{asc}(S - \lambda) \) (resp., \( \text{dsc}(S - \lambda) \)) is infinite. Let \( \Phi_\pm(S) = \{ \lambda : \lambda \in \Phi_+(S), \text{ind}(S - \lambda) \leq 0 \} \), \( \Phi_\pm(S) = \{ \lambda : \lambda \in \Phi_+(S), \text{ind}(S - \lambda) \geq 0 \} \) and \( \Phi^0(S) = \{ \lambda : \lambda \in \Phi(S), \text{ind}(S - \lambda) = 0 \} \). \( S \) is Browder (resp., Weyl) at \( \lambda \) if \( \lambda \in \Phi(S) \) and \( \text{asc}(S - \lambda) = \text{dsc}(S - \lambda) < \infty \) (resp., if \( \lambda \in \Phi^0(S) \)). Recall that a necessary and sufficient condition for \( \lambda \in \mathbb{C} \) to belong to \( p(S) \) is that \( \text{asc}(S - \lambda) = \text{dsc}(S - \lambda) < \infty \); also, \( \text{asc}(S - \lambda) < \infty \) implies \( \text{ind}(S - \lambda) \leq 0 \) and \( \text{dsc}(S - \lambda) < \infty \) implies \( \text{ind}(S - \lambda) \geq 0 \). The Browder spectrum, the Weyl spectrum, the Browder essential approximate point spectrum \( \sigma_{ab}(S) \) and the Weyl essential approximate point spectrum of \( S \) are, respectively, the sets \( \sigma_e(S) = \{ \lambda \in \mathbb{C} : S - \lambda \) is not Browder \}, \( \sigma_w(S) = \{ \lambda \in \mathbb{C} : S - \lambda \) is not Weyl \}, \( \sigma_{ab}(S) = \{ \lambda \in \sigma_e(S) : \lambda \notin \Phi_+(S) \) or \( \text{asc}(S - \lambda) = \infty \} \) and \( \sigma_{aw}(S) = \{ \lambda \in \mathbb{C} : \lambda \notin \Phi_+(S) \) or \( \text{ind}(S - \lambda) \leq 0 \} \). An operator \( S \in B(\mathcal{X}) \) has the single-valued extension property at \( \lambda_0 \in \mathbb{C} \), \( \text{SVEP} \) at \( \lambda_0 \), if for every open disc \( D_{\lambda_0} \) centered at \( \lambda_0 \) the only analytic function \( f : D_{\lambda_0} \to \mathcal{X} \) which satisfies

\[
(S - \lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in D_{\lambda_0}
\]

is the function \( f \equiv 0 \). Trivially, every operator \( S \) has \( \text{SVEP} \) on its resolvent set \( \rho(S) = \mathbb{C} \setminus \sigma(S) \); also \( S \) has \( \text{SVEP} \) at points \( \lambda \in \text{iso}(S) \). (Here \( \text{iso}(S) \) denotes the set of isolated points of \( \sigma(S) \).) Let \( \Xi(S) \) denote the set of \( \lambda \in \mathbb{C} \) where \( S \) does not have \( \text{SVEP} \); we say that \( S \) has \( \text{SVEP} \) if \( \Xi(S) = \emptyset \). The quasinilpotent part \( H_0(S - \lambda) \) and the analytic core \( K(S - \lambda) \) of \( (S - \lambda) \) are defined by

\[
H_0(S - \lambda) = \{ x \in \mathcal{X} : \lim_{n \to \infty} \|(S - \lambda)^n x\|^{\frac{1}{n}} = 0 \}
\]

and

\[
K(S - \lambda) = \{ x \in \mathcal{X} : \text{there exists a sequence } \{ x_n \} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which } x = x_0, (S - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \ldots \}.
\]

We note that \( H_0(S - \lambda) \) and \( K(S - \lambda) \) are (generally) non-closed hyperinvariant subspaces of \( (S - \lambda) \) such that \( (S - \lambda)^{-p}(0) \subseteq H_0(S - \lambda) \) for all \( p = 0, 1, 2, \ldots \) and \( (S - \lambda)K(S - \lambda) = K(S - \lambda) \) [1]. Recall that if \( \lambda \in \text{iso}(S) \), then \( \mathcal{X} = H_0(S - \lambda) \oplus K(S - \lambda) \) [1, Theorem 3.74].

Unless otherwise evident from the context, we assume in the following that \( T \in B(\mathcal{X}) \), \( E \) is a closed \( T \)-invariant subspace of \( \mathcal{X} \), \( A = T|_E \) and \( B = T|_{\mathcal{X}/E} \). We write \( \text{iso}(A) \cup \text{iso}(B) \) for \( \{ \text{iso}(A) \cap \rho(B) \} \cup \{ \text{iso}(A) \cap \sigma(B) \} \) and \( \{ \rho(A) \cap \text{iso}(B) \} \), where \( \rho(\cdot) = \mathbb{C} \setminus \sigma(\cdot) \) is the resolvent set; the expressions \( p_0(A) \cup p_0(B) \) and \( \tau_0(A) \cup \tau_0(B) \) shall have a similar meaning. Henceforth, we shall write \( A - \lambda I|_E \), \( B - \lambda I|_{\mathcal{X}/E} \), \( \sigma_w(\cdot)^C \) for \( \sigma(\cdot) \setminus \sigma_w(\cdot) \) and \( \sigma_{aw}(\cdot)^C \) for \( \sigma_a(\cdot) \setminus \sigma_{aw}(\cdot) \).
It is well known that the equality \( \sigma_x(T) = \sigma_x(A) \cup \sigma_x(B) \), where \( \sigma_x = \sigma \) or \( \sigma_b \) or \( \sigma_w \) or \( \sigma_{aw} \), does not hold in general. If \( \sigma_x = \sigma \) or \( \sigma_b \), then \( \sigma_x(T) \cup \{ \sigma_x(A) \cap \sigma_x(B) \} = \sigma_x(A) \cup \sigma_x(B) \) [6]. This equality, however, fails if \( \sigma_x = \sigma_w \) or \( \sigma_{aw} \), as follows from the following examples. If we let \( A, B \in B(\ell^2) \) be defined by

\[
A(x_1, x_2, x_3, \ldots) = (0, 0, 0, \tfrac{1}{2}x_2, 0, \tfrac{1}{3}x_3, \ldots),
\]

\[
B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, \ldots)
\]

and \( T = A \oplus B \), then \( \sigma_w(A) = \{ 0 \} \), \( \sigma_w(B) = \{ 0, 1 \} \), \( \sigma_w(T) = \{ 0 \} \) and \( \sigma_w(T) \cup \{ \sigma_w(A) \cap \sigma_w(B) \} = \{ 0 \} \subset \sigma_w(A) \cup \sigma_w(B) \). Again, if we let \( A \in B(\ell^2) \) denote the forward unilateral shift, \( B = A^* \) and define the unitary operator \( T \) by

\[
T = \begin{pmatrix} A & 1 - AB \\ 0 & B \end{pmatrix},
\]

then \( \sigma_{aw}(A) \) is the boundary \( \partial D \) of the closed unit disc \( D \), \( \sigma_{aw}(B) = D \), \( \sigma_{aw}(T) = \partial D \) and \( \sigma_{aw}(T) \cup \{ \sigma_{aw}(A) \cap \sigma_{aw}(B) \} = \partial D \subset \sigma_{aw}(A) \cup \sigma_{aw}(B) \). If \( \text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda) \) whenever either of the left hand side or the right hand side of the equality is defined (a hypothesis trivially satisfied by operators \( T \) with an upper triangular representation), then \( \sigma_x(T) \subseteq \sigma_x(A) \cup \sigma_x(B) \) for \( \sigma_x = \sigma_w \) or \( \sigma_{aw} \). This follows from a straightforward argument when \( \sigma_x = \sigma_w \) [6]; for the case in which \( \sigma_x = \sigma_{aw} \) one argues as follows. Let \( \lambda \notin \sigma_{aw}(A) \cup \sigma_{aw}(B) \). Then \( \lambda \in \Phi_+(A) \cap \Phi_+(B) \), \( \alpha(T - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda) < \infty \) and \( \lambda \in \Phi_+(T) \). We have two possibilities: either \( \alpha(T - \lambda) < \beta(T - \lambda) \) or \( \alpha(T - \lambda) \geq \beta(T - \lambda) \). If \( \alpha(T - \lambda) < \beta(T - \lambda) \), then \( \lambda \notin \Phi_+(T) \). If, on the other hand, \( \alpha(T - \lambda) \geq \beta(T - \lambda) \), then \( \lambda \in \Phi(T) \). Since \( \lambda \in \Phi(T) \) implies \( \lambda \in \Phi_+(A) \cap \Phi_-(B) \), \( \lambda \in \Phi(B) \), and hence also that \( \lambda \notin \Phi(A) \). But then \( \text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda) \leq 0 \); hence \( \text{ind}(T - \lambda) = 0 \), which implies that \( \lambda \notin \sigma_{aw}(T) \).

The equality \( \sigma_x(T) = \sigma_x(A) \cup \sigma_x(B) \), \( \sigma_x = \sigma_w \) or \( \sigma_{aw} \), fails to hold in general. However:

**Lemma 2.1.** If \( \text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda) \), then either of the hypotheses \( A \) and \( A^* \), or \( A \) and \( B \), or \( A^* \) and \( B^* \), or \( B \) and \( B^* \) have SVEP on \( \sigma_w(T)^c = \sigma(T) \setminus \sigma_w(T) \) implies that \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \).

**Proof.** We have to prove that \( \sigma_w(T) \supseteq \sigma_w(A) \cup \sigma_w(B) \). If \( \lambda \in \sigma_w(T)^c \), then \( \lambda \in \Phi_+(A) \cap \Phi_-(B) \) and \( \text{ind}(A - \lambda) + \text{ind}(B - \lambda) = 0 \). If either of the SVEP hypotheses holds, then \( \text{ind}(A - \lambda) = \text{ind}(B - \lambda) = 0 \) (see the argument of the proof of [10, Proposition 4.5]). This implies that \( \lambda \notin \sigma_w(A) \cup \sigma_w(B) \). \( \square \)

Again:

**Lemma 2.2.** If \( \text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda) \), \( A \) and \( A^* \) have SVEP on \( \sigma_{aw}(T)^c = \sigma_{aw}(T) \setminus \sigma_{aw}(T) \), and \( B - \lambda \) has closed range for all \( \lambda \in \sigma_{aw}(T)^c \), then \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \).
Proof. We have to prove that $\sigma_{aw}(T) \supset \sigma_{aw}(A) \cup \sigma_{aw}(B)$. If $\lambda \in \sigma_{aw}(T)^C$, then 
$\lambda \in \Phi_+(A)$ and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) \leq 0$. Since $A$ and $A^*$ have SVEP at $\lambda$, $\text{ind}(A - \lambda) = 0$, and so $\lambda \in \Phi^0(A)$. We (now borrow an argument from [4], proof of Proposition 8, (2) $\implies$ (3), to) prove that $\alpha(B - \lambda) < \infty$; this, because $B - \lambda$ has closed range, would then imply that $\lambda \in \Phi^+_w(B)$ (and hence that $\sigma_{aw}(T) \supset \sigma_{aw}(A) \cup \sigma_{aw}(B)$). Start by observing that $\alpha(B - \lambda) = \dim(Y/E)$, where 
$Y = (T - \lambda)^{-1}[E] = \{x \in X : (T - \lambda)x \in E\}$. Since $\alpha(B - \lambda) < \infty$, there exists a finite dimensional subspace $F$ of $E$ such that $E = (T - \lambda)E \oplus F$. Take a $y \in Y$ (thus 
$(T - \lambda)y = 0$). Then there exist $e \in E$ and $f \in F$ such that $(T - \lambda)y = (T - \lambda)e + f$. But then $(T - \lambda)(y - e) = f$, i.e., $y \in (T - \lambda)^{-1}[F] + E$. Since $F$ and $\alpha(T - \lambda)$ are finite dimensional, $(T - \lambda)^{-1}[F]$ is finite dimensional. Consequently, $Y \subseteq (T - \lambda)^{-1}[F] + E$, which implies that $E$ has finite codimension in $Y$. \qed

The interested reader is invited to consult [10] for (further) conditions implying the equality $\sigma_x(T) = \sigma_{aw}(A) \cup \sigma_x(B)$, $\sigma_x = \sigma_{aw}$ or $\sigma_{aw}$, in the case in which the operator $T$ has an upper triangular representation with diagonal $(A, B)$.

Below we consider operators $T$ such that $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$ for $\sigma_x = \sigma_{aw}$ or $\sigma_{aw}$. Such operators have some interesting properties, amongst them that $\sigma(T) = \sigma(A) \cup \sigma(B)$.

**Theorem 2.3.** (i) If $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, where $\sigma_x = \sigma_{aw}$ or $\sigma_{aw}$, then $\sigma(T) = \sigma(A) \cup \sigma(B)$.

(ii) If $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ and $A^*$ has SVEP on $\sigma_{aw}(T)^C$, then $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$.

**Proof.** (i) We have to prove that $\sigma(A) \cup \sigma(B) \subseteq \sigma(T)$. Let $\lambda \notin \sigma(T)$. Then 
$\lambda \in \Phi^0(T)$, so $\alpha(A - \lambda) \leq \alpha(T - \lambda) = 0$, $\beta(B - \lambda) \leq \beta(T - \lambda) = 0$ and $\lambda \in \Phi^-_w(A) \cap \Phi^+_w(B)$. Since $\lambda \notin \sigma_{aw}(T)$, the hypothesis $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies that $\lambda \in \Phi^0(A) \cap \Phi^0(B)$. Hence $\alpha(A - \lambda) = \alpha(B - \lambda) = \beta(A - \lambda) = \beta(B - \lambda) = 0$, which implies that $\lambda \notin \sigma(A) \cup \sigma(B)$ ($\implies \sigma(A) \cup \sigma(B) \subseteq \sigma(T)$). Now let $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Then $\lambda \notin \sigma_{aw}(T)$ implies that $\lambda \in \Phi^-_w(A) \cap \Phi^+_w(B)$. Already, $\lambda \in \Phi^+_w(B)$; hence $\lambda \in \Phi^0(B)$, which implies that $B - \lambda$ is invertible. This forces $A - \lambda$ to be invertible, leading us to the conclusion that $\lambda \notin \sigma(A) \cup \sigma(B)$. Once again, $\sigma(A) \cup \sigma(B) \subseteq \sigma(T)$.

(ii) If $\lambda \notin \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then $\alpha(A - \lambda) = \alpha(B - \lambda) = 0$ and (since $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$) $\lambda \in \Phi^-_w(A) \cap \Phi^-_w(B)$. Since $\alpha(T - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda)$ [4], we conclude that $\alpha(T - \lambda) = 0$. Recalling the isomorphisms $E^\perp \cong (X/E^*)^*$ and $E^* \cong X^*/E^\perp$, and identifying $A^*$ with $T^*|_{X^*/E^\perp}$ and $B^*$ with $T^*|_{E^\perp}$, it follows that 
$\lambda \in \Phi^+_w(A^*) \cap \Phi^+_w(B^*)$. Hence 
$\beta(T^* - \lambda I^*) \leq \beta(A^* - \lambda I^*|_{X^*/E^\perp} + B^* - \lambda I^*|_{E^\perp}) < \infty$ [4, Proposition 7]. This implies that $T^* - \lambda I^*$, and so also $T - \lambda$, has closed range. Already $\alpha(T - \lambda) = 0$; hence $\lambda \notin \sigma_{aw}(T)$, which implies that $\sigma_{aw}(T) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B)$. For the reverse inclusion, let $\lambda \notin \sigma_{aw}(T)$. Then $T - \lambda$ is left invertible and $\lambda \in \Phi^-_w(A) \cap \Phi^-_w(B)$, which implies that $A - \lambda$ is left invertible. Thus $A^* - \lambda I_{|E^\perp}$ is surjective. Since a surjective operator has SVEP at 0 if and only if it is injective...
The hypothesis $A^*$ has SVEP at $\lambda$ implies that $A^* - \lambda(I|_E)^*$, and so also $A - \lambda$, is invertible. We prove next that $B - \lambda$ is left invertible. Let $(T - \lambda)^{-1}[E] = \{x \in X : (T - \lambda)x \in E\}$. We prove that $(T - \lambda)^{-1}[E] = E$. Choose an $x \in X$ such that $(T - \lambda)x \in E$. Then there exist $y, z \in E$ such that $(T - \lambda)x = y = (A - \lambda)z = ((T - \lambda)|_E)z = (T - \lambda)z$, i.e., $(T - \lambda)(x - z) = 0$. Since $T - \lambda$ is left invertible, $x = z$; consequently, $(T - \lambda)^{-1}[E] = E$. In view of this, we now have that $(B - \lambda)^{-1}(0) = \{x + E : (T - \lambda)x \in E\} = ((T - \lambda)^{-1}(0) + E)/E = (Y \oplus E)/E$, where $Y$ is any subspace of $(T - \lambda)^{-1}(0)$ such that $(T - \lambda)^{-1}(0) = (A - \lambda)^{-1}(0) \oplus Y$. Since $(A - \lambda)^{-1}(0) = \{0\}$, $\alpha(B - \lambda) = \dim Y \leq \dim (T - \lambda)^{-1}(0) = 0$. Since $B - \lambda$ has closed range, we conclude that $B - \lambda$ is left invertible. Consequently, $\lambda \notin \sigma_u(A) \cup \sigma_u(B)$, which implies that $\sigma_u(A) \cup \sigma_u(B) \subseteq \sigma_u(T)$.

**Remark 2.4.** (i) The hypothesis $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ in Theorem 2.3(i) may be replaced by the (weaker) hypothesis that $\sigma_w(A)$ or $\sigma_w(B)$ (even, $\sigma_w(A) \cap \sigma_w(B)$) $\subseteq \sigma_w(T)$. Observe that if $\sigma_w(A) \subseteq \sigma_w(T)$, then $\lambda \notin \sigma_w(T) \Rightarrow \lambda \notin \sigma_w(A)$. Thus, since $\lambda \notin \sigma(T)$ implies $\lambda \in \Phi_+(A) \cap \Phi_-(B)$ with $\alpha(A - \lambda) = \beta(B - \lambda) = 0$, it follows that $\alpha(A - \lambda) = \beta(B - \lambda) = 0$. Consequently, $T - \lambda$ and $A - \lambda$ are invertible; this forces $B - \lambda$ to be invertible.

(ii) The hypothesis $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ in Theorem 2.3 may be replaced by the hypothesis that $\sigma_{aw}(B) \subseteq \sigma_{aw}(T)$. Thus, if $\lambda \notin \sigma(T)$, then $\sigma_{aw}(B) \subseteq \sigma_{aw}(T)$ implies that $\lambda \in \Phi_-(B)$, $\beta(B - \lambda) = 0$ and $\lambda \in \Phi_+(B)$. But then $\alpha(B - \lambda) = \beta(B - \lambda) = 0$ and $B - \lambda$ is invertible; since $T - \lambda$ is invertible, it follows that $A - \lambda$ is invertible. Again, let $\lambda \notin \sigma_u(T)$ and $\sigma_{aw}(B) \subseteq \sigma_{aw}(T)$. Then the SVEP hypothesis on $A^*$ implies that $A - \lambda$ is invertible, and this in turn implies that $\alpha(B - \lambda) = 0$. Since $\lambda \notin \sigma_{aw}(B)$, $B - \lambda$ has closed range; hence $B - \lambda$ is left invertible.

For an operator $S \in B(X)$, let $\text{acc}\sigma(S)$ denote the points of accumulation of $\sigma(S)$. $S$ satisfies Browder’s theorem (or, condition), $Bt$ for short, if $\text{acc}\sigma(S) \subseteq \sigma_u(S)$; $S$ satisfies a-Browder’s theorem (or, condition), $a-Bt$ for short, if $\text{acc}\sigma_a(S) \subseteq \sigma_{aw}(S)$. The following implications are well known [1, 7, 10, 14]:

1. $S$ satisfies $Bt \iff S^*$ satisfies $Bt \iff \sigma_u(S) = \sigma_w(S) \iff \sigma(S) \setminus \sigma_u(S) = p_0(S) \iff S$ has SVEP on $\sigma_u(S)^c$;

2. $S$ satisfies $a-Bt \iff \sigma_{aw}(S) = \sigma_{aw}(S) \iff \sigma_a(S) \setminus \sigma_{aw}(S) = p_0^a(S) \iff S$ has SVEP on $\sigma_{aw}(S)^c$;

3. $a-Bt \implies Bt$, but the converse is generally false.

$Bt$, much less $a-Bt$, does not transfer from $A$ and $B$ to $T$: consider the operator $T = A \oplus B$, where $A \in B(\ell^2)$ is the forward unilateral shift and $B = A^*$ (when it is seen that $A$ and $B$ satisfy $Bt$ but $T$ does not). The following theorem gives a sufficient condition for the transfer of $Bt$ (resp., $a-Bt$) from $A$ and $B$ to $T$.

**Theorem 2.5.** (i) If $\sigma_w(T) = \sigma_u(A) \cup \sigma_u(B)$, then $A$ and $B$ satisfy $Bt$ implies $T$ satisfies $Bt$.

(ii) If $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then $A$ and $B$ satisfy $a-Bt$ implies $T$ satisfies $a-Bt$. 
Proof. (i) We prove that \( \sigma_w(T) = \sigma_b(T) \): since \( \sigma_w(T) \subseteq \sigma_b(T) \) for every operator \( T \), it would suffice to prove the reverse inclusion. Let \( \lambda \notin \sigma_w(T) \). Then the hypothesis \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) implies that \( \lambda \in \Phi^0(A) \cap \Phi^0(B) \). Since \( A \) and \( B \) satisfy \( Bt \), it follows that \( \lambda \in p_0(A) \cup p_0(B) \). Consequently, \( \text{asc}(T - \lambda) \leq \text{asc}(A - \lambda) + \text{asc}(B - \lambda) < \infty \) and \( \text{dsc}(T - \lambda) \leq \text{dsc}(A - \lambda) + \text{dsc}(B - \lambda) < \infty \) [15, Exercise 7, Page 293]. Evidently, \( \alpha(T - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda) < \infty \). Hence \( \lambda \notin \sigma_b(T) \).

(ii) We prove that \( \sigma_{aw}(T) = \sigma_{ab}(T) \): since \( \sigma_{aw}(T) \subseteq \sigma_{ab}(T) \) for every operator \( T \), it would suffice to prove the reverse inclusion. Let \( \lambda \notin \sigma_{aw}(T) \). Then (the hypothesis \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \) implies that) \( \lambda \in \Phi^0_+(A) \cap \Phi^0_+(B) \). The hypothesis \( A \) and \( B \) satisfy \( a - Bt \) implies that \( A \) has SVEP on \( \sigma_{aw}(A)^c \) and \( B \) has SVEP on \( \sigma_{aw}(B)^c \). Recall, [1, Theorem 3.16], that if an operator \( S \) has SVEP at a point \( \mu \in \Phi_+(S) \), then \( \text{asc}(S - \mu) < \infty \). Thus \( \text{asc}(T - \lambda) \leq \text{asc}(A - \lambda) + \text{asc}(B - \lambda) < \infty \). Evidently, \( \lambda \in \Phi_+(T) \); hence \( \lambda \notin \sigma_{ab}(T) \).

Remark 2.6. The hypothesis \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) is not sufficient for \( T \) satisfies \( Bt \) to imply \( A \) and \( B \) satisfy \( Bt \). To see this, let \( T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \), where \( A = U \in B(\ell^2) \) is the forward unilateral shift, \( C = (1 - UU^*) \) and \( B = U^* \oplus U \). Then \( \sigma(T) = \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) is the closed unit disc \( D, \ p_0(T) = \emptyset, \ \sigma(B) = D, \ \sigma_w(B) = \partial D, \) both \( T \) and \( A \) satisfy \( Bt \) but \( B \) does not satisfy \( Bt \).

Recall that an operator \( S \in B(X) \) is polaroid at a point \( \lambda, \lambda \in \rho(S) \) (resp., \( a \)-polaroid at \( \lambda, \lambda \in \rho^a(S) \)) if (either \( \lambda \in \rho(S) \) or) \( \lambda \in \sigma(S) \) is a pole of the resolvent of \( S \) (resp., \( \lambda \in \sigma(S) \)), \( (S - \lambda)X \) is closed and \( \text{asc}(S - \lambda) < \infty \) [11, 12]; we say that \( S \) is polaroid (resp., \( a \)-polaroid) if \( \{ \lambda : \lambda \in \sigma(S) \} = \rho(S) \) (resp., \( \{ \lambda : \lambda \in \sigma(S) \} = \rho^a(S) \)). The following theorem relates the polaroid points of \( T, A \) and \( B \) satisfying \( \sigma(T) = \sigma(A) \cup \sigma(B) \).

Theorem 2.7. If \( \sigma(T) = \sigma(A) \cup \sigma(B) \), then \( T \) is polaroid at a point \( \lambda \) if and only if \( A \) and \( B \) are polaroid at \( \lambda \).

Proof. Let \( \lambda \in \sigma(S) \). Then \( \lambda \in \sigma(A) \cup \sigma(B) \). If \( A \) and \( B \) are polaroid at \( \lambda \), then the inequalities \( \text{asc}(T - \lambda) \leq \text{asc}(A - \lambda) + \text{asc}(B - \lambda) \) and \( \text{dsc}(T - \lambda) \leq \text{dsc}(A - \lambda) + \text{dsc}(B - \lambda) \) imply that \( T \) is polaroid at \( \lambda \). Conversely, assume that \( T \) is polaroid at \( \lambda \). Then \( \text{asc}(B - \lambda) \leq \text{asc}(T - \lambda) < \infty \) and \( \text{dsc}(A - \lambda) \leq \text{dsc}(T - \lambda) < \infty \). Since \( B \) has SVEP at \( \lambda, \ \text{asc}(B - \lambda) < \infty \) [1, Theorem 3.81]. This implies that \( \lambda \in \rho(B) \). The hypothesis that \( \lambda \) is a pole of the resolvent of \( T \) implies that \( H_0(T - \lambda) = (T - \lambda)^{-p}(0) \) for some integer \( p \geq 1 \). Since

\[
H_0(A - \lambda) = H_0((T - \lambda)|_E) \subseteq (T - \lambda)^{-p}(0) \cap E = (\lambda^p(0)) \subseteq H_0(A - \lambda),
\]

it follows that \( H_0(A - \lambda) = (A - \lambda)^{-p}(0) \). Since \( \lambda \in \sigma(A) \),

\[
E = H_0(A - \lambda) \oplus K(A - \lambda) = (A - \lambda)^{-p}(0) \oplus K(A - \lambda),
\]

from which it follows that

\[
E = (A - \lambda)^{-p}(0) \oplus (A - \lambda)^p(E),
\]
i.e., $\lambda \in p(A)$. □

**Remark 2.8.** Apparently, if $\sigma(T) = \sigma(A) \cup \sigma(B)$, then $A$ and $B$ polaroid implies $T$ polaroid. The implication $T$ is polaroid implies $A$ and $B$ are polaroid is however false (even if one assumes that $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$). Let $T = A \oplus B \in B(\ell^2 \oplus \ell^2)$, where $A$ is the forward unilateral shift and $B$ is a quasinilpotent. Then $\sigma(T) = \sigma_w(T) = \sigma(A) = \sigma_w(A)$ is the closed unit disc, and $T$ and $A$ are (vacuously) polaroid. However, $\sigma(B) = \sigma_w(B) = \{0\}$ and $B$ is not polaroid at 0. In the presence of $\sigma(T) = \sigma(A) \cup \sigma(B)$, a sufficient condition for $T$ polaroid to imply $A$ and $B$ polaroid is that the sets $\text{acc}(T) \cap \text{iso}(A)$ and $\text{acc}(T) \cap \text{iso}(B)$ are empty: this condition is however not necessary, as follows from a consideration of the operator $T = A \oplus B \in B(\ell^2 \oplus \ell^2)$, where $A$ is the forward unilateral shift and $B$ is a nilpotent.

The operator $S$ is said to be finitely polaroid at a point $\lambda$ if $\lambda \in p_0(S)$. The following corollary generalizes [2, Theorems 1 and 2].

**Corollary 2.9.** If $\sigma(T) = \sigma(A) \cup \sigma(B)$, then $T$ is finitely polaroid at (a point) $\lambda$ if and only if $A$ and $B$ are finitely polaroid at $\lambda$.

**Proof.** Since $\alpha(T - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda)$ whenever $\alpha(A - \lambda)$ and $\alpha(B - \lambda)$ are finite [4, Proposition 7], Theorem 2.7 implies that $T$ is finitely polaroid at $\lambda$ whenever $A$ and $B$ are finitely polaroid at $\lambda$. Conversely, if $T$ is finitely polaroid at $\lambda$, then $\lambda \in \text{iso}(T)$ implies $\lambda \in \Phi(T)$. Hence, if $\lambda \in p_0(T)$, then $\lambda \in \Phi_+(A) \cap \Phi_-(B)$ and $\lambda$ is a pole of the resolvents of $A$ and $B$ (or, $\lambda$ is in the resolvent set of $A$ and/or $B$). Thus $\lambda \in \Phi^0(A) \cap \Phi^0(B)$, which implies that $\lambda$ is a finite rank pole of the resolvents of $A$ and $B$. □

The sufficiency part of Corollary 2.9 extends to finitely $a$-polaroid operators.

**Proposition 2.10.** If $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B)$ and $A$, $B$ are finitely $a$-polaroid at a point $\lambda$, then $T$ is finitely $a$-polaroid at $\lambda$.

**Proof.** The hypothesis $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B)$ implies that if $\lambda \in \text{iso}_a(T)$ then $\lambda \in \text{iso}_a(A) \cup \text{iso}_a(B)$. Thus, if $A$, $B$ are finitely polaroid at $\lambda$ and $\lambda \in \text{iso}_a(T)$, then $\lambda \in \Phi_+(A) \cap \Phi_-(B)$, $\text{asc}(A - \lambda) < \infty$ and $\text{asc}(B - \lambda) < \infty$. But then $\lambda \in \Phi_+(T)$ and $\text{asc}(T - \lambda) < \infty$, i.e., $\lambda \in p^a_0(T)$. □

$S \in B(\mathcal{X})$ satisfies Weyl’s theorem (or, condition), $Wt$ for short, if $\sigma(S) \setminus \sigma_w(S) = \pi_0(S)$; $S$ satisfies $a$-Weyl’s theorem (or, condition), $aWt$ for short, if $\sigma_a(S) \setminus \sigma_{aw}(S) = \pi^a_0(S)$. A necessary and sufficient condition for $S$ to satisfy $Wt$ (resp., $aWt$) is that $S$ satisfies $Bt$ (resp., $aBt$) and $S$ is polaroid on $\pi_0(S)$ (resp., $S$ is $a$-polaroid on $\pi^a_0(S)$) [10, Theorem 4.3]. It is well known that $aWt \iff Wt$; the reverse implication is generally false.

The hypothesis $A$ and $B$ satisfy $Wt$ (or, $aWt$) is neither necessary nor sufficient for $T$ to satisfy $Wt$ (resp., $aWt$). Thus, if $A$ and $B \in B(\ell^2)$ are the operators $A(x_1, x_2, x_3, \ldots) = (0, 0, 0, \frac{1}{3}x_2, 0, \frac{1}{5}x_3, \ldots)$ and $B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, \ldots)$, then $\sigma(A) = \sigma_w(A) = \pi_0(A) = \{0\}$, $\sigma(B) = \sigma_w(B) = \sigma_{aw}(B) = \{0, 1\}$, $\pi^a_0(B) = p^a_0(B) = \emptyset$, $A$ does not satisfy $Wt$ but both $B$ and $T = A \oplus B$ satisfy
Suppose that \( a - Wt \). Again, if \( B \) is the operator above, and \( A \) and \( C \in B(\ell^2) \) are the operators \( A(x_1, x_2, x_3, \ldots) = (0, x_1, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, \ldots) \) and \( C(x_1, x_2, x_3, \ldots) = (x_1, 0, x_2, 0, x_3, \ldots) \), then \( A \) and \( B \) satisfy \( a - Wt \), but \( T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) does not satisfy \( Wt \) (since \( \sigma(T) = \sigma_w(T) = \{0, 1\} \) and \( \pi_0(T) = \{0\} \)). Observe that neither of the equalities \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) and \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \) holds for the operators of the examples above. The following theorem proves that the hypothesis \( A \) and \( B \) satisfy \( Wt \) is sufficient for \( T \) to satisfy \( Wt \) if \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) and \( A, B \) are isoloid. Recall that the operator \( S \) is isoloid (resp., \( a \)-isoloid) if the isolated points of \( \sigma(S) \) (resp., \( \sigma_a(S) \)) are eigenvalues of \( S \).

**Theorem 2.11.** Suppose that \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \). If \( A, B \) are isoloid and satisfy \( Wt \), then \( T \) satisfies \( Wt \).

**Proof.** Evidently, \( A \) and \( B \) satisfy \( Bt \). Hence, see Theorem 2.5(i), \( T \) satisfies \( Bt \), i.e., \( \sigma(T) \setminus \sigma_w(T) = p_0(T) \). Since \( p_0(T) \subseteq \pi_0(T) \), to complete the proof it would suffice to prove the reverse inclusion. Let \( \lambda \in \pi_0(T) \). Recall from Theorem 2.3(i) that the hypothesis \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) implies \( \sigma(T) = \sigma(A) \cup \sigma(B) \). Hence \( \lambda \in \sigma(A) \cup \sigma(B) \). Clearly, \( \alpha(A - \lambda) = \dim \{ (T - \lambda)^{-1}(0) \cap E \} < \infty \). Since \( A \) is isoloid, we may assume that \( \lambda \in \pi_0(A) \); hence, since \( A \) satisfies \( Wt \), \( \lambda \in p_0(A) \). Evidently, \( \beta(A - \lambda) < \infty \). Arguing as in the proof of Lemma 2.2 it is seen that \( \alpha(B - \lambda) < \infty \). Since \( B \) is isoloid and satisfies \( Bt \), \( \lambda \in p_0(B) \) (or, \( \lambda \in \rho(B) \)). Applying Theorem 2.7 it follows that \( \lambda \in p_0(T) \). Hence \( \pi_0(T) \subseteq p_0(T) \). \( \square \)

The operator \( T \) of Remark 2.6 satisfies \( \sigma(T) = \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) = D \), \( \sigma(A) = \sigma_w(A) = D \) and \( \pi_0(T) = \pi_0(A) = \emptyset \). Hence both \( T \) and \( A \) satisfy \( Wt \). However, since \( B \) does not satisfy \( Bt \), it does not satisfy \( Wt \): the condition \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \) is not sufficient for \( T \) satisfies \( Wt \) to imply \( A \) and \( B \) satisfy \( Wt \).

**Remark 2.12.** The hypothesis that \( A \) and \( B \) satisfy \( Wt \) in Theorem 2.11 may be replaced by the hypotheses that \( A \) and \( B \) satisfy \( Bt \), \( A \) is polaroid on \( \pi_0(A) \) and \( B \) is polaroid on \( \pi_0(B) \). A tightening of the hypotheses of Theorem 2.11 is possible in the case in which either \( X = H \) is a Hilbert space or the subspace \( E \) is complemented in \( X \). In such a case, \( T \) has an upper triangular representation \( T = \begin{pmatrix} A & C \\ 0 & B_1 \end{pmatrix} \), where \( B_1 \) is similar to \( B \). Let \( T_0 = A \oplus B_1 \). Then \( \sigma(T) \subseteq \sigma(T_0) = \sigma(A) \cup \sigma(B_1) \) and \( \sigma_w(T) \subseteq \sigma_w(T_0) \subseteq \sigma_w(A) \cup \sigma_w(B_1) \). This, since \( \sigma_x(B_1) = \sigma_x(B) \) for \( \sigma_x = \sigma \) or \( \sigma_w \), implies that if \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \), then \( \sigma_x(T_0) = \sigma_x(T) \) for all \( \sigma \).

**Proposition 2.13.** (cf. [9, Theorem 3.7]) Let \( T_0 \) and \( T \) be defined as above. If \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) \), \( T_0 \) satisfies \( Wt \) and \( A \) is polaroid on \( \pi_0(T) \), then \( T \) satisfies \( Wt \).

**Proof.** Apparently, \( \lambda \in p_0(T_0) \) if and only if \( \lambda \in p_0(A) \cup p_0(B) \) \( \iff \lambda \in p_0(A) \cup p_0(B_1) \) if and only if \( \lambda \in p_0(T) \); hence \( p_0(T_0) = p_0(T) \). Since \( T_0 \) satisfies \( Wt \),
\[
\sigma(T) \setminus \sigma_w(T) = \sigma(T_0) \setminus \sigma_w(T_0) = p_0(T_0) = \pi_0(T_0) = p_0(T) \subseteq \pi_0(T).
\]
Let $\lambda \in \pi_0(T)$. Then $\lambda \in \text{iso}(A) \cup \text{iso}(B_1)$. Since $A$ is polaroid on $\pi_0(T)$, $\lambda \in p_0(A) = \pi_0(A)$. Arguing as in the proof of Theorem 2.11, it is seen that $0 \leq \alpha(B_1 - \lambda) < \infty$, and hence that $\lambda \in \pi_0(A) \cap \pi_0(B_1)$. Since $T_0$ satisfies $Wt$, $\lambda \in \pi_0(T_0)$. Thus $\pi_0(T) = \pi_0(T_0)$, and $T$ satisfies $Wt$. \qed

An easy argument shows that if any two of $T_0$, $A$ and $B$ satisfy $Wt$, then so does the third one. Again, if $A$ is isoloid and satisfies $Wt$, then $\lambda \in \pi_0(T)$ implies $\lambda \in p_0(A) = \pi_0(A)$ (implies $A$ is polaroid on $\pi_0(T)$.) Hence Proposition 2.13 implies Theorem 2.4 of [13].

If $A^*$ and $B^*$ have SVEP, then $A$ and $B$ satisfy $a - Bt$ [10, Corollary 3.5], $\sigma_a(A) = \sigma(A)$ and $\sigma_a(B) = \sigma(B)$ [1, Corollary 2.45], and (this follows from a straightforward argument) $\sigma_{aw}(A) = \sigma_w(A)$ and $\sigma_{aw}(B) = \sigma_w(B)$. If, furthermore, $\text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda)$, then it is seen that $\sigma_w(T) = (\sigma_{aw}(T)) = \sigma_w(A) \cup \sigma_w(B) = (\sigma_{aw}(A) \cup \sigma_{aw}(B))$. The following theorem generalizes [9, Theorem 3.11].

**Theorem 2.14.** If $\text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B - \lambda)$, $A^*$ and $B^*$ have SVEP, $A$ is polaroid on $\pi_0(T)$ and $B$ is polaroid on $\pi_0(B)$, then $T$ satisfies $a - Wt$.

**Proof.** Theorem 2.3 implies that $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B) = \sigma(A) \cup \sigma(B) = \sigma(T)$. Evidently, $T$ satisfies $a - Bt$; indeed

$$\sigma_a(T) \setminus \sigma_{aw}(T) = \sigma(T) \setminus \sigma_w(T) = p_0(T) = p_0^B(T) \cup \pi_0(T) = \pi_0(T).$$

Observe that if $\lambda \in \pi_0(T)$, then $\lambda \in p_0(A) = p_0^B(A)$ (or, $\lambda \in \rho(A)$) and $\lambda \in p_0(B) = p_0^B(B)$ (or, $\lambda \in \rho(B)$). Hence $\lambda \in \pi_0(T) \implies \lambda \in \pi_0(T)$, which completes the proof. \qed

**Remark 2.15.** The example of the operator $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(\ell^2 \oplus \ell^2)$, where $A(x_1, x_2, x_3, \ldots) = (0, x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, \ldots), B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, 0, \ldots)$ and $C(x_1, x_2, x_3, \ldots) = (0, 0, x_2, 0, x_3, \ldots)$, shows that the hypothesis $A$ is polaroid on $\pi_0(T)$ in Theorem 2.14 can not be replaced by the hypothesis that $A$ is polaroid on $\pi_0(A)$. Observe that all the hypotheses of the theorem are satisfied, except for the fact that $\pi_0(T) = \{0\}$ and $0 \notin \rho(A)$; $T$ does not satisfy $a - Wt$, even $Wt$.

**Remark 2.16.** Let $E$ be complemented in $X$, so that $T_0$ and $T$ have the representations of Remark 2.12. Since $B$ has SVEP at a point if and only if $B_1$ has SVEP at the point, and since $\text{ind}(T - \lambda) = \text{ind}(A - \lambda) + \text{ind}(B_1 - \lambda)$), either of the conditions $A$ and $A^*$, or $A$ and $B$, or $A^*$ and $B^*$, or $B$ and $B^*$ have SVEP on $\sigma_w(T)^C$ implies that $\sigma_w(T_0) = \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B_1)$ (see Lemma 2.1). Again, either of the conditions $A$ and $A^*$, or $A$ and $B$, have SVEP on $\sigma_{aw}(T)^C$ implies that $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B_1)$ [9, Theorem 4.12(ii)]. (Observe that if $A^*$ and $B^*$ have SVEP on $\sigma_{aw}(T)^C$, then $\sigma_{aw}(T) = \sigma_{aw}(A) = \sigma_{aw}(A) \cup \sigma_{aw}(B_1) = \sigma_{aw}(A) \cup \sigma_{aw}(B_1)$.)

Evidently, $\sigma_{aw}(T) \subseteq \sigma_{aw}(T_0)$; if $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B_1)$, then the following implications show that $\sigma_{aw}(T_0) \subseteq \sigma_{aw}(T)$ (so that $\sigma_{aw}(T) = \sigma_{aw}(T_0)$):

$$\lambda \notin \sigma_{aw}(T) \implies \lambda \in \Phi_+(A) \cap \Phi_+(B_1) \implies \lambda \notin \Phi_+(T_0) \implies \lambda \notin \sigma_{aw}(T_0).$$
It is known, [9, Theorems 5.1 and 5.7], that: (i) If \( \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B_1) \), then the equivalence \( T \) satisfies \( W_t \Leftrightarrow T_0 \) satisfies \( W_t \) holds if and only if \( \pi_0(T) = \pi_0(T_0) \); (ii) If \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B_1) \) and \( A^* \) has SVEP on \( \sigma_{aw}(T)^c \), then the equivalence \( T \) satisfies \( a-Wt \Leftrightarrow T_0 \) satisfies \( a-Wt \) holds if and only if \( \pi_0^a(T) = \pi_0^a(T_0) \).

We prove next an analogue of Theorem 2.11 for operators \( T \) satisfying \( a-Wt \).

**Theorem 2.17.** If \( \sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \), \( A^* \) has SVEP on \( \pi_0^a(T) \), \( A \) and \( B \) are \( a \)-isoloid, and both \( A \) and \( B \) satisfy \( a-Wt \), then \( T \) satisfies \( a-Wt \).

**Proof.** The hypotheses imply that \( \sigma_a(A) \setminus \sigma_{aw}(A) = p_0^a(A) = \pi_0^a(A) \), \( \sigma_a(B) \setminus \sigma_{aw}(B) = p_0^a(B) = \pi_0^a(B) \), and \( \sigma_a(T) \setminus \sigma_{aw}(T) = p_0^a(T) \subseteq \pi_0^a(T) \) (see Theorem 2.5). Since \( \sigma_{au}(T)^c \subseteq \pi_0^a(T) \), \( A^* \) has SVEP on \( \sigma_{au}(T)^c \); hence \( \sigma_a(T) = \sigma_a(A) \cup \sigma_a(B) \) (by Theorem 2.3). Thus to complete the proof, we have to prove that \( \pi_0^a(T) \subseteq p_0^a(T) \). Let \( \lambda \in \pi_0^a(T) \). Then \( \lambda \in \text{iso}_a(T) = \text{iso}_a(A) \cup \text{iso}_{aw}(B) \) and \( \alpha(T-\lambda) < \infty \). Evidently, \( \alpha(A-\lambda) < \infty \); hence (since \( A \) is \( a \)-isoloid) \( \lambda \in \pi_0^a(A) \) (or, \( \lambda \in \rho(A) \)). Since \( A^* \) has SVEP at \( \lambda \), \( \lambda \in p_0^a(A) \), so that \( \beta(A-\lambda) < \infty \). Arguing as before, it is seen that \( \lambda \in p_0^a(B) \) (or, \( \lambda \in \rho(B) \)). Applying Proposition 2.10 we conclude that \( \lambda \in p_0^a(T) \). \( \square \)

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**References**


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