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THE BROWDER AND WEYL SPECTRA OF AN OPERATOR AND ITS DIAGONAL

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Abstract

If $T \in B(\mathcal{X})$ is a Banach space operator and E is a closed T-invariant subspace of $\mathcal{X}$, then the restriction map $A = T|_E$ and the quotient map $B = T|_{\mathcal{X}/E}$ are well defined operators in $B(E)$ and $B(\mathcal{X}/E)$, respectively. It is proved that: (i) If $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, where σ_x is either the Weyl spectrum σ_w or the Weyl essential approximate point spectrum σ_{aw}, then $\sigma(T) = \sigma(A) \cup \sigma(B)$; (ii) if $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, and A^* has SVEP (the single–valued extension property), then $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B)$; (iii) if $\sigma(T) = \sigma(A) \cup \sigma(B)$, then a point λ is a pole (resp., finite rank pole) of the resolvent of T if and only if λ is a pole (resp., finite rank pole) of the resolvents of A and B. Letting σ_b and σ_{ab} denote, respectively, the Browder spectrum and the Browder essential approximate point spectrum, an operator $S \in B(\mathcal{X})$ satisfies Browder's theorem (resp., a-Browder's theorem) if $\sigma_w(S) = \sigma_b(S)$ (resp., $\sigma_{aw}(S) = \sigma_{ab}(S)$); S satisfies Weyl's theorem if $\sigma(S) \setminus \sigma_w(S) = \{\lambda \in \text{iso}\sigma(S) : 0 < \dim(S-\lambda)^{-1}(0) < \infty\}$. Recall that S is isoloid if $\lambda \in \text{iso}\sigma(S)$ implies $0 < \dim(S-\lambda)^{-1}(0)$. We prove that: (iv) if $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ (resp., $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$), then Browder's theorem (resp., a-Browder's theorem) transfers from A and B to T; (v) if $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, and A, B are isoloid, then Weyl's theorem transfers from A and B to T.

1 Introduction

Let $B(\mathcal{X})$ denote the algebra of operators (i.e., bounded linear transformations) on a Banach space $\mathcal{X}$ into itself. The problem of the relationship between the spectrum, and some of its more distinguished parts, of an (upper triangular) operator $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(\mathcal{X})$ and its diagonal (A, B) has been considered by a number of authors, amongst them [3, 4, 5, 8, 9, 11, 13, 14]. A related but more demanding

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problem which has been considered in the recent past is the following. Let $T \in B(\mathcal{X})$ and let E be a T-invariant closed subspace of $\mathcal{X}$. Then the restriction $A = T|_E$ and the quotient map $B = T|_{\mathcal{X}/E}$ are well defined operators in $B(E)$ and $B(\mathcal{X}/E)$, respectively. Following Barnes [3], let us call the pair (A, B) the diagonal of T. What is the relationship between the spectrum σ, the Fredholm spectrum σ_e, the Browder spectrum σ_b, the Weyl spectrum σ_w and the Weyl essential approximate point spectrum σ_{aw} of the operator T and its diagonal (A, B)? Evidently, if T is Fredholm, $T \in \Phi(\mathcal{X})$, then A is upper semi–Fredholm; identifying B^* with $T^*|_{E^\perp}$ it follows that B^* is upper semi–Fredholm, which implies that B is lower semi–Fredholm. It is not difficult to verify, [3, 6], that $\sigma_x(T) \cup \{\sigma_x(A) \cap \sigma_x(B)\} = \sigma_x(A) \cup \sigma_x(B)$ for $\sigma_x = \sigma$ or σ_e or σ_b. (Thus, if any two of T, A and B are invertible or Fredholm or Browder, then so is the third one.) The relationship between the Weyl spectrum, and the Weyl essential approximate point spectrum, of A, B and T is a bit more delicate. The equality $\sigma_x(T) \cup \{\sigma_x(A) \cap \sigma_x(B)\} = \sigma_x(A) \cup \sigma_x(B)$ fails for $\sigma_x = \sigma_w$ and $\sigma_x = \sigma_{aw}$; however, if the (Fredholm) indices satisfy the equality $\mathrm{ind}(T - \lambda) = \mathrm{ind}(A - \lambda) + \mathrm{ind}(B - \lambda)$, whenever the left hand side or the right hand side of the equality is finite, then $\sigma_x(T) \subseteq \sigma_x(A) \cup \sigma_x(B)$ for $\sigma_x = \sigma_w$ and σ_{aw}.

In this paper, we consider operators $T \in B(\mathcal{X})$ such that $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, where (A, B) is the diagonal of T, for $\sigma_x = \sigma_w$ or σ_{aw}, and prove that such operators satisfy $\sigma(T) = \sigma(A) \cup \sigma(B)$. In the case in which $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ and A^* has SVEP (*the single–valued extension property*) on the complement $\sigma_{aw}(T)^{\mathcal{C}}$ of $\sigma_{aw}(T)$ in the approximate point spectrum $\sigma_a(T)$ of T, it is seen that $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B)$. For an operator $S \in B(\mathcal{X})$, we say that S is polaroid (resp., a-polaroid) at a point λ in the complex plane $\mathbf{C}$ if (either λ is in the resolvent set $\rho(S) = \mathbf{C} \setminus \sigma(S)$ or) $\lambda \in \mathrm{iso}\sigma(S)$ is a pole of the resolvent of S (resp., $\lambda \in \mathrm{iso}\sigma_a(S)$, $(S - \lambda)\mathcal{X}$ is closed and the ascent $\mathrm{asc}(S - \lambda) < \infty$). Let $p(S)$, $p_0(S)$, $p^a(S)$, $p_0^a(S)$, $\pi_0(S)$ and $\pi_0^a(S)$ denote, respectively, the sets $p(S) = \{\lambda : S \text{ is polaroid at } \lambda\}$, $p_0(S) = \{\lambda \in p(S) : \dim(S - \lambda)^{-1}(0) < \infty\}$, $p^a(S) = \{\lambda : S \text{ is } a\text{-polaroid at } \lambda\}$, $p_0^a(S) = \{\lambda \in p^a(S) : \dim(S - \lambda)^{-1}(0) < \infty\}$, $\pi_0(S) = \{\lambda \in \mathrm{iso}\sigma(S) : 0 < \dim(S - \lambda)^{-1}(0) < \infty\}$ and $\pi_0^a(S) = \{\lambda \in \mathrm{iso}\sigma_a(S) : 0 < \dim(S - \lambda)^{-1}(0) < \infty\}$. If $\sigma(T) = \sigma(A) \cup \sigma(B)$, then $\lambda \in p(T) \Longleftrightarrow \lambda \in p(A) \bigcup p(B)$ $(= \{p(A) \cap \rho(B)\} \cup \{p(A) \cap p(B)\} \cup \{\rho(A) \cap p(B)\})$ and $\lambda \in p_0(T) \Longleftrightarrow \lambda \in p_0(A) \bigcup p_0(B)$; if $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, A and B are isoloid, $\sigma(A) \setminus \sigma_w(A) = \pi_0(A)$ and $\sigma(B) \setminus \sigma_w(B) = \pi_0(B)$, then $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$; if $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, $\sigma_a(A) \setminus \sigma_{aw}(A) = \pi_0^a(A)$, A and B are a-isoloid, $\sigma_a(B) \setminus \sigma_{aw}(B) = \pi_0^a(B)$ and A^* has SVEP on $\pi_0^a(T)$, then $\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_0^a(T)$.

2 Results

We start by explaining the terminology already introduced, and by introducing further notation and terminology.

An operator $S \in B(\mathcal{X})$ is upper semi-Fredholm (resp., lower semi-Fredholm) at a complex number $\lambda \in \mathtt{C}$ if the range $(S - \lambda)\mathcal{X}$ is closed and $\alpha(S - \lambda) = \dim(S -$

$\lambda)^{-1}(0) < \infty$ (resp., $\beta(S-\lambda) = \dim(\mathcal{X}/(S-\lambda)\mathcal{X}) < \infty$). Let $\lambda \in \Phi_+(S)$ (resp., $\lambda \in \Phi_-(S)$) denote that S is upper semi-Fredholm (resp., lower semi-Fredholm) at λ. The operator S is Fredholm at λ, denoted $\lambda \in \Phi(S)$, if $\lambda \in \Phi_+(S) \cap \Phi_-(S)$. Let $\operatorname{ind}(S-\lambda) = \alpha(S-\lambda) - \beta(S-\lambda)$ denote the Fredholm index of $S-\lambda$. The ascent $\operatorname{asc}(S-\lambda)$ (resp., the descent $\operatorname{dsc}(S-\lambda)$) of $S-\lambda$ is the least non-negative integer n such that $(S-\lambda)^{-n}(0) = (S-\lambda)^{-(n+1)}(0)$ (resp., the least non-negative integer n such that $(S-\lambda)^n\mathcal{X} = (S-\lambda)^{n+1}\mathcal{X}$); if no such integer exists, then $\operatorname{asc}(S-\lambda)$ (resp., $\operatorname{dsc}(S-\lambda)$) is infinite. Let $\Phi_\pm^-(S) = \{\lambda : \lambda \in \Phi_\pm(S), \operatorname{ind}(S-\lambda) \leq 0\}$, $\Phi_\pm^+(S) = \{\lambda : \lambda \in \Phi_\pm(S), \operatorname{ind}(S-\lambda) \geq 0\}$ and $\Phi^0(S) = \{\lambda : \lambda \in \Phi(S), \operatorname{ind}(S-\lambda) = 0\}$. S is Browder (resp., Weyl) at λ if $\lambda \in \Phi(S)$ and $\operatorname{asc}(S-\lambda) = \operatorname{dsc}(S-\lambda) < \infty$ (resp., if $\lambda \in \Phi^0(S)$). Recall that a necessary and sufficient condition for $\lambda \in \mathbb{C}$ to belong to $p(S)$ is that $\operatorname{asc}(S-\lambda) = \operatorname{dsc}(S-\lambda) < \infty$; also, $\operatorname{asc}(S-\lambda) < \infty$ implies $\operatorname{ind}(S-\lambda) \leq 0$ and $\operatorname{dsc}(S-\lambda) < \infty$ implies $\operatorname{ind}(S-\lambda) \geq 0$. The Browder spectrum, the Weyl spectrum, the Browder essential approximate point spectrum $\sigma_{ab}(S)$ and the Weyl essential approximate point spectrum of S are, respectively, the sets $\sigma_b(S) = \{\lambda \in \mathbb{C} : S-\lambda \text{ is not Browder}\}$, $\sigma_w(S) = \{\lambda \in \mathbb{C} : S-\lambda \text{ is not Weyl}\}$, $\sigma_{ab}(S) = \{\lambda \in \sigma_a(S) : \lambda \notin \Phi_+(S) \text{ or } \operatorname{asc}(S-\lambda) = \infty\}$ and $\sigma_{aw}(S) = \{\lambda \in \mathbb{C} : \lambda \notin \Phi_+(S) \text{ or } \operatorname{ind}(S-\lambda) \not\leq 0\}$. An operator $S \in B(\mathcal{X})$ has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ which satisfies

$$(S-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator S has SVEP on its resolvent set $\rho(S) = \mathbb{C} \setminus \sigma(S)$; also S has SVEP at points $\lambda \in \operatorname{iso}\sigma(S)$. (Here $\operatorname{iso}\sigma(S)$ denotes the set of isolated points of $\sigma(S)$.) Let $\Xi(S)$ denote the set of $\lambda \in \mathbb{C}$ where S does not have SVEP: we say that S has SVEP if $\Xi(S) = \emptyset$. The *quasinilpotent part* $H_0(S-\lambda)$ and the *analytic core* $K(S-\lambda)$ of $(S-\lambda)$ are defined by

$$H_0(S-\lambda) = \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} ||(S-\lambda)^n x||^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(S-\lambda) = \{x \in \mathcal{X} : &\text{there exists a sequence } \{x_n\} \subset \mathcal{X} \quad \text{and} \quad \delta > 0 \\ &\text{for which } x = x_0, (S-\lambda)x_{n+1} = x_n \quad \text{and} \quad \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ...\}. \end{aligned}$$

We note that $H_0(S-\lambda)$ and $K(S-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(S-\lambda)$ such that $(S-\lambda)^{-p}(0) \subseteq H_0(S-\lambda)$ for all $p = 0, 1, 2, ...$ and $(S-\lambda)K(S-\lambda) = K(S-\lambda)$ [1]. Recall that if $\lambda \in \operatorname{iso}\sigma(S)$, then $\mathcal{X} = H_0(S-\lambda) \oplus K(S-\lambda)$ [1, Theorem 3.74].

Unless otherwise evident from the context, we assume in the following that $T \in B(\mathcal{X})$, E is a closed T-invariant subspace of $\mathcal{X}$, $A = T|_E$ and $B = T|_{\mathcal{X}/E}$. We write $\operatorname{iso}\sigma(A) \bigcup \operatorname{iso}\sigma(B)$ for $\{\operatorname{iso}\sigma(A) \cap \rho(B)\} \cup \{\operatorname{iso}\sigma(A) \cap \operatorname{iso}\sigma(B)\} \cup \{\rho(A) \cap \operatorname{iso}\sigma(B)\}$, where $\rho(\cdot) = \mathbb{C} \setminus \sigma(\cdot)$ is the resolvent set; the expressions $p_0(A) \bigcup p_0(B)$ and $\pi_0(A) \bigcup \pi_0(B)$ shall have a similar meaning. Henceforth, we shall write $A-\lambda$ for $A-\lambda I|_E$, $B-\lambda$ for $B-\lambda I|_{\mathcal{X}/E}$, $\sigma_w(\cdot)^{\mathcal{C}}$ for $\sigma(\cdot) \setminus \sigma_w(\cdot)$ and $\sigma_{aw}(\cdot)^{\mathcal{C}}$ for $\sigma_a(\cdot) \setminus \sigma_{aw}(\cdot)$.

It is well known that the equality $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, where $\sigma_x = \sigma$ or σ_b or σ_w or σ_{aw}, does not hold in general. If $\sigma_x = \sigma$ or σ_b, then $\sigma_x(T) \cup \{\sigma_x(A) \cap \sigma_x(B)\} = \sigma_x(A) \cup \sigma_x(B)$ [6]. This equality, however, fails if $\sigma_x = \sigma_w$ or σ_{aw}, as follows from the following examples. If we let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, ...) = (0, 0, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, ...),$$

$$B(x_1, x_2, x_3, ...) = (0, x_2, 0, x_4, ...)$$

and $T = A \oplus B$, then $\sigma_w(A) = \{0\}$, $\sigma_w(B) = \{0, 1\}$, $\sigma_w(T) = \{0\}$ and $\sigma_w(T) \cup \{\sigma_w(A) \cap \sigma_w(B)\} = \{0\} \subset \sigma_w(A) \cup \sigma_w(B)$. Again, if we let $A \in B(\ell^2)$ denote the forward unilateral shift, $B = A^*$ and define the unitary operator T by $T = \begin{pmatrix} A & 1 - AB \\ 0 & B \end{pmatrix}$, then $\sigma_{aw}(A)$ is the boundary $\partial\mathbf{D}$ of the closed unit disc $\mathbf{D}$, $\sigma_{aw}(B) = \mathbf{D}$, $\sigma_{aw}(T) = \partial\mathbf{D}$ and $\sigma_{aw}(T) \cup \{\sigma_{aw}(A) \cap \sigma_{aw}(B)\} = \partial\mathbf{D} \subset \sigma_{aw}(A) \cup \sigma_{aw}(B)$. If $\mathrm{ind}(T - \lambda) = \mathrm{ind}(A - \lambda) + \mathrm{ind}(B - \lambda)$ whenever either of the left hand side or the right hand side of the equality is defined (a hypothesis trivially satisfied by operators T with an upper triangular representation), then $\sigma_x(T) \subseteq \sigma_x(A) \cup \sigma_x(B)$ for $\sigma_x = \sigma_w$ or σ_{aw}. This follows from a straightforward argument when $\sigma_x = \sigma_w$ [6]; for the case in which $\sigma_x = \sigma_{aw}$ one argues as follows. Let $\lambda \notin \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Then $\lambda \in \Phi_+^-(A) \cap \Phi_+^-(B)$, $\alpha(T - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda) < \infty$ and $\lambda \in \Phi_+(T)$. We have two possibilities: either $\alpha(T - \lambda) < \beta(T - \lambda)$ or $\alpha(T - \lambda) \geq \beta(T - \lambda)$. If $\alpha(T - \lambda) < \beta(T - \lambda)$, then $\lambda \in \Phi_+^-(T) \iff \lambda \notin \sigma_{aw}(T)$. If, on the other hand, $\alpha(T - \lambda) \geq \beta(T - \lambda)$, then $\lambda \in \Phi(T)$. Since $\lambda \in \Phi(T)$ implies $\lambda \in \Phi_+(A) \cap \Phi_-(B)$, $\lambda \in \Phi(B)$, and hence also that $\lambda \in \Phi(A)$. But then $\mathrm{ind}(T - \lambda) = \mathrm{ind}(A - \lambda) + \mathrm{ind}(B - \lambda) \leq 0$; hence $\mathrm{ind}(T - \lambda) = 0$, which implies that $\lambda \notin \sigma_{aw}(T)$.

The equality $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, $\sigma_x = \sigma_w$ or σ_{aw}, fails to hold in general. However:

Lemma 2.1. *If $\mathrm{ind}(T - \lambda) = \mathrm{ind}(A - \lambda) + \mathrm{ind}(B - \lambda)$, then either of the hypotheses A and A^*, or A and B, or A^* and B^*, or B and B^* have SVEP on $\sigma_w(T)^{\mathcal{C}} = \sigma(T) \setminus \sigma_w(T)$ implies that $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$.*

Proof. We have to prove that $\sigma_w(T) \supset \sigma_w(A) \cup \sigma_w(B)$. If $\lambda \in \sigma_w(T)^{\mathcal{C}}$, then $\lambda \in \Phi_+(A) \cap \Phi_-(B)$ and $\mathrm{ind}(A - \lambda) + \mathrm{ind}(B - \lambda) = 0$. If either of the SVEP hypotheses holds, then $\mathrm{ind}(A - \lambda) = \mathrm{ind}(B - \lambda) = 0$ (see the argument of the proof of [10, Proposition 4.5]). This implies that $\lambda \notin \sigma_w(A) \cup \sigma_w(B)$. □

Again:

Lemma 2.2. *If $\mathrm{ind}(T - \lambda) = \mathrm{ind}(A - \lambda) + \mathrm{ind}(B - \lambda)$, A and A^* have SVEP on $\sigma_{aw}(T)^{\mathcal{C}} = \sigma_a(T) \setminus \sigma_{aw}(T)$, and $B - \lambda$ has closed range for all $\lambda \in \sigma_{aw}(T)^{\mathcal{C}}$, then $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$.*

Proof. We have to prove that $\sigma_{aw}(T) \supset \sigma_{aw}(A) \cup \sigma_{aw}(B)$. If $\lambda \in \sigma_{aw}(T)^{\mathcal{C}}$, then $\lambda \in \Phi_+(A)$ and $\mathrm{ind}(A-\lambda) + \mathrm{ind}(B-\lambda) \le 0$. Since A and A^* have SVEP at λ, $\mathrm{ind}(A-\lambda) = 0$, and so $\lambda \in \Phi^0(A)$. We (now borrow an argument from [4], proof of Proposition 8, (2) $\Longrightarrow$ (3), to) prove that $\alpha(B-\lambda) < \infty$; this, because $B-\lambda$ has closed range, would then imply that $\lambda \in \Phi_+^-(B)$ (and hence that $\sigma_{aw}(T) \supset \sigma_{aw}(A) \cup \sigma_{aw}(B)$). Start by observing that $\alpha(B-\lambda) = \dim(Y/E)$, where $Y = (T-\lambda)^{-1}[E] = \{x \in \mathcal{X} : (T-\lambda)x \in E\}$. Since $\beta(A-\lambda) < \infty$, there exists a finite dimensional subspace F of E such that $E = (T-\lambda)E \oplus F$. Take a $y \in Y$ (thus $(T-\lambda)y \in E$). Then there exist $e \in E$ and $f \in F$ such that $(T-\lambda)y = (T-\lambda)e + f$. But then $(T-\lambda)(y-e) = f$, i.e., $y \in (T-\lambda)^{-1}[F] + E$. Since F and $\alpha(T-\lambda)$ are finite dimensional, $(T-\lambda)^{-1}[F]$ is finite dimensional. Consequently, $Y \subseteq (T-\lambda)^{-1}[F] + E$, which implies that E has finite codimension in Y. $\square$

The interested reader is invited to consult [10] for (further) conditions implying the equality $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, $\sigma_x = \sigma_w$ or σ_{aw}, in the case in which the operator T has an upper triangular representation with diagonal (A, B).

Below we consider operators T such that $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$ for $\sigma_x = \sigma_w$ or σ_{aw}. Such operators have some interesting properties, amongst them that $\sigma(T) = \sigma(A) \cup \sigma(B)$.

Theorem 2.3. *(i) If $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, where $\sigma_x = \sigma_w$ or σ_{aw}, then $\sigma(T) = \sigma(A) \cup \sigma(B)$.*

(ii) If $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ and A^* has SVEP on $\sigma_{aw}(T)^{\mathcal{C}}$, then $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B)$.

Proof. (i) We have to prove that $\sigma(A) \cup \sigma(B) \subseteq \sigma(T)$. Let $\lambda \notin \sigma(T)$. Then $\lambda \in \Phi^0(T)$, $\alpha(A-\lambda) \le \alpha(T-\lambda) = 0$, $\beta(B-\lambda) \le \beta(T-\lambda) = 0$ and $\lambda \in \Phi_+^-(A) \cap \Phi_-^+(B)$. Since $\lambda \notin \sigma_w(T)$, the hypothesis $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ implies that $\lambda \in \Phi^0(A) \cap \Phi^0(B)$. Hence $\alpha(A-\lambda) = \alpha(B-\lambda) = \beta(A-\lambda) = \beta(B-\lambda) = 0$, which implies that $\lambda \notin \sigma(A) \cup \sigma(B)$ ($\Longrightarrow \sigma(A) \cup \sigma(B) \subseteq \sigma(T)$). Now let $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Then $\lambda \notin \sigma_{aw}(T)$ implies that $\lambda \in \Phi_+^-(A) \cap \Phi_+^-(B)$. Already, $\lambda \in \Phi_-^+(B)$; hence $\lambda \in \Phi^0(B)$, which implies that $B-\lambda$ is invertible. This forces $A-\lambda$ to be invertible, leading us to the conclusion that $\lambda \notin \sigma(A) \cup \sigma(B)$. Once again, $\sigma(A) \cup \sigma(B) \subseteq \sigma(T)$.

(ii) If $\lambda \notin \sigma_a(A) \cup \sigma_a(B)$, then $\alpha(A-\lambda) = \alpha(B-\lambda) = 0$ and (since $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$) $\lambda \in \Phi_+^-(A) \cap \Phi_+^-(B)$. Since $\alpha(T-\lambda) \le \alpha(A-\lambda) + \alpha(B-\lambda)$ [4], we conclude that $\alpha(T-\lambda) = 0$. Recalling the isomorphisms $E^\perp \cong (\mathcal{X}/E)^*$ and $E^* \cong \mathcal{X}^*/E^\perp$, and identifying A^* with $T^*|_{\mathcal{X}^*/E^\perp}$ and B^* with $T^*|_{E^\perp}$, it follows that $\lambda \in \Phi_-^+(A^*) \cap \Phi_-^+(B^*)$. Hence $\beta(T^* - \lambda I^*) \le \beta(A^* - \lambda I^*|_{\mathcal{X}^*/E^\perp}) + \beta(B^* - \lambda I^*|_{E^\perp}) < \infty$ [4, Proposition 7]. This implies that $T^* - \lambda I^*$, and so also $T-\lambda$, has closed range. Already $\alpha(T-\lambda) = 0$; hence $\lambda \notin \sigma_a(T)$, which implies that $\sigma_a(T) \subseteq \sigma_a(A) \cup \sigma_a(B)$. For the reverse inclusion, let $\lambda \notin \sigma_a(T)$. Then $T-\lambda$ is left invertible and $\lambda \in \Phi_+^-(A) \cap \Phi_+^-(B)$, which implies that $A-\lambda$ is left invertible. Thus $A^* - \lambda(I|_E)^*$ is surjective. Since a surjective operator has SVEP at 0 if and only if it is injective

[1, Corollary 2.24], the hypothesis A^* has SVEP at λ implies that $A^* - \lambda(I|_E)^*$, and so also $A-\lambda$, is invertible. We prove next that $B-\lambda$ is left invertible. Let $(T-\lambda)^{-1}[E] = \{x \in \mathcal{X} : (T-\lambda)x \in E\}$. We prove that $(T-\lambda)^{-1}[E] = E$. Choose an $x \in \mathcal{X}$ such that $(T-\lambda)x \in E$. Then there exist $y, z \in E$ such that $(T-\lambda)x = y = (A-\lambda)z = ((T-\lambda)|_E)z = (T-\lambda)z$, i.e., $(T-\lambda)(x-z) = 0$. Since $T-\lambda$ is left invertible, $x = z$; consequently, $(T-\lambda)^{-1}[E] = E$. In view of this, we now have that $(B-\lambda)^{-1}(0) = \{x+E : (T-\lambda)x \in E\} = ((T-\lambda)^{-1}(0)+E)/E = (Y \oplus E)/E$, where Y is any subspace of $(T-\lambda)^{-1}(0)$ such that $(T-\lambda)^{-1}(0) = (A-\lambda)^{-1}(0) \oplus Y$. Since $(A-\lambda)^{-1}(0) = \{0\}$, $\alpha(B-\lambda) = \dim Y \leq \dim(T-\lambda)^{-1}(0) = 0$. Since $B-\lambda$ has closed range, we conclude that $B-\lambda$ is left invertible. Consequently, $\lambda \notin \sigma_a(A) \cup \sigma_a(B)$, which implies that $\sigma_a(A) \cup \sigma_a(B) \subseteq \sigma_a(T)$. $\square$

Remark 2.4. (i) The hypothesis $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ in Theorem 2.3(i) may be replaced by the (weaker) hypothesis that $\sigma_w(A)$ or $\sigma_w(B)$ (even, $\sigma_w(A) \cap \sigma_w(B)$) $\subseteq \sigma_w(T)$. Observe that if $\sigma_w(A) \subseteq \sigma_w(T)$, then $\lambda \notin \sigma_w(T) \Longrightarrow \lambda \notin \sigma_w(A)$. Thus, since $\lambda \notin \sigma(T)$ implies $\lambda \in \Phi_+(A) \cap \Phi_-(B)$ with $\alpha(A-\lambda) = \beta(B-\lambda) = 0$, it follows that $\alpha(A-\lambda) = \beta(A-\lambda) = 0$. Consequently, $T-\lambda$ and $A-\lambda$ are invertible; this forces $B-\lambda$ to be invertible.

(ii) The hypothesis $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ in Theorem 2.3 may be replaced by the hypothesis that $\sigma_{aw}(B) \subseteq \sigma_{aw}(T)$. Thus, if $\lambda \notin \sigma(T)$, then $\sigma_{aw}(B) \subseteq \sigma_{aw}(T)$ implies that $\lambda \in \Phi_-(B)$, $\beta(B-\lambda) = 0$ and $\lambda \in \Phi_+^-(B)$. But then $\alpha(B-\lambda) = \beta(B-\lambda) = 0$ and $B-\lambda$ is invertible; since $T-\lambda$ is invertible, it follows that $A-\lambda$ is invertible. Again, let $\lambda \notin \sigma_a(T)$ and $\sigma_{aw}(B) \subseteq \sigma_{aw}(T)$. Then the SVEP hypothesis on A^* implies that $A-\lambda$ is invertible, and this in turn implies that $\alpha(B-\lambda) = 0$. Since $\lambda \notin \sigma_{aw}(B)$, $B-\lambda$ has closed range; hence $B-\lambda$ is left invertible.

For an operator $S \in B(\mathcal{X})$, let $\mathrm{acc}\sigma(S)$ denote the points of accumulation of $\sigma(S)$. S satisfies Browder's theorem (or, condition), Bt for short, if $\mathrm{acc}\sigma(S) \subseteq \sigma_w(S)$; S satisfies a-Browder's theorem (or, condition), $a-Bt$ for short, if $\mathrm{acc}\sigma_a(S) \subseteq \sigma_{aw}(S)$. The following implications are well known [1, 7, 10, 14]:

S satisfies $Bt \Longleftrightarrow S^*$ satisfies $Bt \Longleftrightarrow \sigma_b(S) = \sigma_w(S) \Longleftrightarrow \sigma(S) \setminus \sigma_w(S) = p_0(S) \Longleftrightarrow S$ has SVEP on $\sigma_w(S)^{\mathcal{C}}$;

S satisfies $a-Bt \Longleftrightarrow \sigma_{ab}(S) = \sigma_{aw}(S) \Longleftrightarrow \sigma_a(S) \setminus \sigma_{aw}(S) = p_0^a(S) \Longleftrightarrow S$ has SVEP on $\sigma_{aw}(S)^{\mathcal{C}}$;

$a-Bt \Longrightarrow Bt$, but the converse is generally false.

Bt, much less $a-Bt$, does not transfer from A and B to T: consider the operator $T = A \oplus B$, where $A \in B(\ell^2)$ is the forward unilateral shift and $B = A^*$ (when it is seen that A and B satisfy Bt but T does not). The following theorem gives a sufficient condition for the transfer of Bt (resp., $a-Bt$) from A and B to T.

Theorem 2.5. *(i) If $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, then A and B satisfy Bt implies T satisfies Bt.*

(ii) If $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then A and B satisfy $a-Bt$ implies T satisfies $a-Bt$.

Proof. (i) We prove that $\sigma_w(T)=\sigma_b(T)$: since $\sigma_w(T)\subseteq\sigma_b(T)$ for every operator T, it would suffice to prove the reverse inclusion. Let $\lambda\notin\sigma_w(T)$. Then the hypothesis $\sigma_w(T)=\sigma_w(A)\cup\sigma_w(B)$ implies that $\lambda\in\Phi^0(A)\cap\Phi^0(B)$. Since A and B satisfy Bt, it follows that $\lambda\in p_0(A)\bigcup p_0(B)$. Consequently, $\operatorname{asc}(T-\lambda)\leq\operatorname{asc}(A-\lambda)+\operatorname{asc}(B-\lambda)<\infty$ and $\operatorname{dsc}(T-\lambda)\leq\operatorname{dsc}(A-\lambda)+\operatorname{dsc}(B-\lambda)<\infty$ [15, Exercise 7, Page 293]. Evidently, $\alpha(T-\lambda)\leq\alpha(A-\lambda)+\alpha(B-\lambda)<\infty$. Hence $\lambda\notin\sigma_b(T)$.

(ii) We prove that $\sigma_{aw}(T)=\sigma_{ab}(T)$: since $\sigma_{aw}(T)\subseteq\sigma_{ab}(T)$ for every operator T, it would suffice to prove the reverse inclusion. Let $\lambda\notin\sigma_{aw}(T)$. Then (the hypothesis $\sigma_{aw}(T)=\sigma_{aw}(A)\cup\sigma_{aw}(B)$ implies that) $\lambda\in\Phi_+^-(A)\cap\Phi_+^-(B)$. The hypothesis A and B satisfy $a-Bt$ implies that A has SVEP on $\sigma_{aw}(A)^{\mathcal{C}}$ and B has SVEP on $\sigma_{aw}(B)^{\mathcal{C}}$. Recall, [1, Theorem 3.16], that if an operator S has SVEP at a point $\mu\in\Phi_+(S)$, then $\operatorname{asc}(S-\mu)<\infty$. Thus $\operatorname{asc}(T-\lambda)\leq\operatorname{asc}(A-\lambda)+\operatorname{asc}(B-\lambda)<\infty$. Evidently, $\lambda\in\Phi_+(T)$; hence $\lambda\notin\sigma_{ab}(T)$. □

Remark 2.6. The hypothesis $\sigma_w(T)=\sigma_w(A)\cup\sigma_w(B)$ is not sufficient for T satisfies Bt to imply A and B satisfy Bt. To see this, let $T=\begin{pmatrix}A & C\\ 0 & B\end{pmatrix}$, where $A=U\in B(\ell^2)$ is the forward unilateral shift, $C=(1-UU^*\ \ 0)$ and $B=U^*\oplus U$. Then $\sigma(T)=\sigma_w(T)=\sigma_w(A)\cup\sigma_w(B)$ is the closed unit disc $\mathbf{D}$, $p_0(T)=\emptyset$, $\sigma(B)=\mathbf{D}$, $\sigma_w(B)=\partial\mathbf{D}$, both T and A satisfy Bt but B does not satisfy Bt.

Recall that an operator $S\in B(\mathcal{X})$ is polaroid at a point λ, $\lambda\in p(S)$ (resp., a-polaroid at λ, $\lambda\in p^a(S)$) if (either $\lambda\in\rho(S)$ or) $\lambda\in\operatorname{iso}\sigma(S)$ is a pole of the resolvent of S (resp., $\lambda\in\operatorname{iso}\sigma_a(S)$, $(S-\lambda)\mathcal{X}$ is closed and $\operatorname{asc}(S-\lambda)<\infty$) [11, 12]; we say that S is polaroid (resp., a-polaroid) if $\{\lambda:\lambda\in\operatorname{iso}\sigma(S)\}=p(S)$ (resp., $\{\lambda:\lambda\in\operatorname{iso}\sigma_a(S)\}=p^a(S)$). The following theorem relates the polaroid points of T, A and B satisfying $\sigma(T)=\sigma(A)\cup\sigma(B)$.

Theorem 2.7. *If $\sigma(T)=\sigma(A)\cup\sigma(B)$, then T is polaroid at a point λ if and only if A and B are polaroid at λ.*

Proof. Let $\lambda\in\operatorname{iso}\sigma(T)$. Then $\lambda\in\operatorname{iso}\sigma(A)\bigcup\operatorname{iso}\sigma(B)$. If A and B are polaroid at λ, then the inequalities $\operatorname{asc}(T-\lambda)\leq\operatorname{asc}(A-\lambda)+\operatorname{asc}(B-\lambda)$ and $\operatorname{dsc}(T-\lambda)\leq\operatorname{dsc}(A-\lambda)+\operatorname{dsc}(B-\lambda)$ imply that T is polaroid at λ. Conversely, assume that T is polaroid at λ. Then $\operatorname{dsc}(B-\lambda)\leq\operatorname{dsc}(T-\lambda)<\infty$ and $\operatorname{asc}(A-\lambda)\leq\operatorname{asc}(T-\lambda)<\infty$. Since B has SVEP at λ, $\operatorname{asc}(B-\lambda)<\infty$ [1, Theorem 3.81]. This implies that $\lambda\in p(B)$. The hypothesis that λ is a pole of the resolvent of T implies that $H_0(T-\lambda)=(T-\lambda)^{-p}(0)$ for some integer $p\geq 1$. Since

$$\begin{aligned}H_0(A-\lambda) &= H_0((T-\lambda)|_E)\subseteq(T-\lambda)^{-p}(0)\cap E=((T-\lambda)^{-p}(0)|_E)\\ &= (A-\lambda)^{-p}(0)\subseteq H_0(A-\lambda),\end{aligned}$$

it follows that $H_0(A-\lambda)=(A-\lambda)^{-p}(0)$. Since $\lambda\in\operatorname{iso}\sigma(A)$,

$$E=H_0(A-\lambda)\oplus K(A-\lambda)=(A-\lambda)^{-p}(0)\oplus K(A-\lambda),$$

from which it follows that

$$E=(A-\lambda)^{-p}(0)\oplus(A-\lambda)^p(E),$$

i.e., $\lambda \in p(A)$. $\square$

Remark 2.8. Apparently, if $\sigma(T) = \sigma(A) \cup \sigma(B)$, then A and B polaroid implies T polaroid. The implication T is polaroid implies A and B are polaroid is however false (even if one assumes that $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$). Let $T = A \oplus B \in B(\ell^2 \oplus \ell^2)$, where A is the forward unilateral shift and B is a quasinilpotent. Then $\sigma(T) = \sigma_w(T) = \sigma(A) = \sigma_w(A)$ is the closed unit disc, and T and A are (vacuously) polaroid. However, $\sigma(B) = \sigma_w(B) = \{0\}$ and B is not polaroid at 0. In the presence of $\sigma(T) = \sigma(A) \cup \sigma(B)$, a sufficient condition for T polaroid to imply A and B polaroid is that the sets $\text{acc}\sigma(T) \cap \text{iso}\sigma(A)$ and $\text{acc}\sigma(T) \cap \text{iso}\sigma(B)$ are empty: this condition is however not necessary, as follows from a consideration of the operator $T = A \oplus B \in B(\ell^2 \oplus \ell^2)$, where A is the forward unilateral shift and B is a nilpotent.

The operator S is said to be finitely polaroid at a point λ if $\lambda \in p_0(S)$. The following corollary generalizes [2, Theorems 1 and 2].

Corollary 2.9. *If $\sigma(T) = \sigma(A) \cup \sigma(B)$, then T is finitely polaroid at (a point) λ if and only if A and B are finitely polaroid at λ.*

Proof. Since $\alpha(T - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda)$ whenever $\alpha(A - \lambda)$ and $\alpha(B - \lambda)$ are finite [4, Proposition 7], Theorem 2.7 implies that T is finitely polaroid at λ whenever A and B are finitely polaroid at λ. Conversely, if T is finitely polaroid at λ, then $\lambda \in \text{iso}\sigma(T)$ implies $\lambda \in \Phi(T)$. Hence, if $\lambda \in p_0(T)$, then $\lambda \in \Phi_+(A) \cap \Phi_-(B)$ and λ is a pole of the resolvents of A and B (or, λ is in the resolvent set of A and/or B). Thus $\lambda \in \Phi^0(A) \cap \Phi^0(B)$, which implies that λ is a finite rank pole of the resolvents of A and B. $\square$

The sufficiency part of Corollary 2.9 extends to finitely a-polaroid operators.

Proposition 2.10. *If $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B)$ and A, B are finitely a-polaroid at a point λ, then T is finitely a-polaroid at λ.*

Proof. The hypothesis $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B)$ implies that if $\lambda \in \text{iso}\sigma_a(T)$, then $\lambda \in \text{iso}\sigma_a(A) \bigcup \text{iso}\sigma_a(B)$. Thus, if A, B are finitely polaroid at λ and $\lambda \in \text{iso}\sigma_a(T)$, then $\lambda \in \Phi_+(A) \cap \Phi_+(B)$, $\text{asc}(A - \lambda) < \infty$ and $\text{asc}(B - \lambda) < \infty$. But then $\lambda \in \Phi_+(T)$ and $\text{asc}(T - \lambda) < \infty$, i.e., $\lambda \in p_0^a(T)$. $\square$

$S \in B(\mathcal{X})$ satisfies Weyl's theorem (or, condition), Wt for short, if $\sigma(S) \setminus \sigma_w(S) = \pi_0(S)$; S satisfies a-Weyl's theorem (or, condition), $a - Wt$ for short, if $\sigma_a(S) \setminus \sigma_{aw}(S) = \pi_0^a(S)$. A necessary and sufficient condition for S to satisfy Wt (resp., $a - Wt$) is that S satisfies Bt (resp., $a - Bt$) and S is polaroid on $\pi_0(S)$ (resp., S is a-polaroid on $\pi_0^a(S)$) [10, Theorem 4.3]. It is well known that $a - Wt \Longrightarrow Wt$; the reverse implication is generally false.

The hypothesis A and B satisfy Wt (or, $a - Wt$) is neither necessary nor sufficient for T to satisfy Wt (resp., $a - Wt$). Thus, if A and $B \in B(\ell^2)$ are the operators $A(x_1, x_2, x_3, ...) = (0, 0, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, ...)$ and $B(x_1, x_2, x_3, ...) = (0, x_2, 0, x_4, ...)$, then $\sigma(A) = \sigma_w(A) = \pi_0(A) = \{0\}$, $\sigma(B) = \sigma_a(B) = \sigma_w(B) = \sigma_{aw}(B) = \{0, 1\}$, $\pi_0^a(B) = p_0^a(B) = \emptyset$, A does not satisfy Wt but both B and $T = A \oplus B$ satisfy

$a-Wt$. Again, if B is the operator above, and A and $C \in B(\ell^2)$ are the operators $A(x_1, x_2, x_3, ...) = (0, x_1, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, ...)$ and $C(x_1, x_2, x_3, ...) = (x_1, 0, x_2, 0, x_3, ...)$, then A and B satisfy $a-Wt$, but $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ does not satisfy Wt (since $\sigma(T) = \sigma_w(T) = \{0, 1\}$ and $\pi_0(T) = \{0\}$). Observe that neither of the equalities $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ and $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ holds for the operators of the examples above. The following theorem proves that the hypothesis A and B satisfy Wt is sufficient for T to satisfy Wt if $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ and A, B are isoloid. Recall that the operator S is isoloid (resp., a-isoloid) if the isolated points of $\sigma(S)$ (resp., $\sigma_a(S)$) are eigenvalues of S.

Theorem 2.11. *Suppose that $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$. If A, B are isoloid and satisfy Wt, then T satisfies Wt.*

Proof. Evidently, A and B satisfy Bt. Hence, see Theorem 2.5(i), T satisfies Bt, i.e., $\sigma(T) \setminus \sigma_w(T) = p_0(T)$. Since $p_0(T) \subseteq \pi_0(T)$, to complete the proof it would suffice to prove the reverse inclusion. Let $\lambda \in \pi_0(T)$. Recall from Theorem 2.3(i) that the hypothesis $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ implies $\sigma(T) = \sigma(A) \cup \sigma(B)$. Hence $\lambda \in \mathrm{iso}\sigma(A) \bigcup \mathrm{iso}\sigma(B)$. Clearly, $\alpha(A-\lambda) = \dim\{(T-\lambda)^{-1}(0) \cap E\} < \infty$. Since A is isoloid, we may assume that $\lambda \in \pi_0(A)$; hence, since A satisfies Wt, $\lambda \in p_0(A)$. Evidently, $\beta(A-\lambda) < \infty$. Arguing as in the proof of Lemma 2.2 it is seen that $\alpha(B-\lambda) < \infty$. Since B is isoloid and satisfies Bt, $\lambda \in p_0(B)$ (or, $\lambda \in \rho(B)$). Applying Theorem 2.7 it follows that $\lambda \in p_0(T)$. Hence $\pi_0(T) \subseteq p_0(T)$. $\square$

The operator T of Remark 2.6 satisfies $\sigma(T) = \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B) = \mathbf{D}$, $\sigma(A) = \sigma_w(A) = \mathbf{D}$ and $\pi_0(T) = \pi_0(A) = \emptyset$. Hence both T and A satisfy Wt. However, since B does not satisfy Bt, it does not satisfy Wt: the condition $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ is not sufficient for T satisfies Wt to imply A and B satisfy Wt.

Remark 2.12. The hypothesis that A and B satisfy Wt in Theorem 2.11 may be replaced by the hypotheses that A and B satisfy Bt, A is polaroid on $\pi_0(A)$ and B is polaroid on $\pi_0(B)$. A tightening of the hypotheses of Theorem 2.11 is possible in the case in which either $\mathcal{X} = \mathcal{H}$ is a Hilbert space or the subspace E is complemented in $\mathcal{X}$. In such a case, T has an upper triangular representation $T = \begin{pmatrix} A & C \\ 0 & B_1 \end{pmatrix}$, where B_1 is similar to B. Let $T_0 = A \oplus B_1$. Then $\sigma(T) \subseteq \sigma(T_0) = \sigma(A) \cup \sigma(B_1)$ and $\sigma_w(T) \subseteq \sigma_w(T_0) \subseteq \sigma_w(A) \cup \sigma_w(B_1)$. This, since $\sigma_x(B_1) = \sigma_x(B)$ for $\sigma_x = \sigma$ or σ_w, implies that if $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma_x(T_0) = \sigma_x(T)$ for $\sigma_x = \sigma$ or σ_w.

Proposition 2.13. (cf. *[9, Theorem 3.7]) Let T_0 and T be defined as above. If $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, T_0 satisfies Wt and A is polaroid on $\pi_0(T)$, then T satisfies Wt.*

Proof. Apparently, $\lambda \in p_0(T_0)$ if and only if $\lambda \in p_0(A) \bigcup p_0(B)$ ($\Longleftrightarrow \lambda \in p_0(A) \bigcup p_0(B_1)$) if and only if $\lambda \in p_0(T)$; hence $p_0(T_0) = p_0(T)$. Since T_0 satisfies Wt,

$$\sigma(T) \setminus \sigma_w(T) = \sigma(T_0) \setminus \sigma_w(T_0) = p_0(T_0) = \pi_0(T_0) = p_0(T) \subseteq \pi_0(T).$$

Let $\lambda \in \pi_0(T)$. Then $\lambda \in \mathrm{iso}\sigma(A)\bigcup \mathrm{iso}\sigma(B_1)$. Since A is polaroid on $\pi_0(T)$, $\lambda \in p_0(A) = \pi_0(A)$. Arguing as in the proof of Theorem 2.11, it is seen that $0 \leq \alpha(B_1 - \lambda) < \infty$, and hence that $\lambda \in \pi_0(A) \cap \pi_0(B_1)$. Since T_0 satisfies Wt, $\lambda \in \pi_0(T_0)$. Thus $\pi_0(T) = \pi_0(T_0)$, and T satisfies Wt. $\square$

An easy argument shows that if any two of T_0, A and B satisfy Wt, then so does the third one. Again, if A is isoloid and satisfies Wt, then $\lambda \in \pi_0(T)$ implies $\lambda \in p_0(A) = \pi_0(A)$ (implies A is polaroid on $\pi_0(T)$.) Hence Proposition 2.13 implies Theorem 2.4 of [13].

If A^* and B^* have SVEP, then A and B satisfy $a - Bt$ [10, Corollary 3.5], $\sigma_a(A) = \sigma(A)$ and $\sigma_a(B) = \sigma(B)$ [1, Corollary 2.45], and (this follows from a straightforward argument) $\sigma_{aw}(A) = \sigma_w(A)$ and $\sigma_{aw}(B) = \sigma_w(B)$. If, furthermore, $\mathrm{ind}(T - \lambda) = \mathrm{ind}(A - \lambda) + \mathrm{ind}(B - \lambda)$, then it is seen that $\sigma_w(T) =(\sigma_{aw}(T))= \sigma_w(A)\cup\sigma_w(B)$ $(= \sigma_{aw}(A)\cup\sigma_{aw}(B))$. The following theorem generalizes [9, Theorem 3.11].

Theorem 2.14. *If $ind(T - \lambda) = ind(A - \lambda) + ind(B - \lambda)$, A^* and B^* have SVEP, A is polaroid on $\pi_0(T)$ and B is polaroid on $\pi_0(B)$, then T satisfies $a - Wt$.*

Proof. Theorem 2.3 implies that $\sigma_a(T) = \sigma_a(A) \cup \sigma_a(B) = \sigma(A) \cup \sigma(B) = \sigma(T)$. Evidently, T satisfies $a - Bt$; indeed

$$\sigma_a(T) \setminus \sigma_{aw}(T) = \sigma(T) \setminus \sigma_w(T) = p_0(T) = p_0^a(T) \subseteq \pi_0^a(T) = \pi_0(T).$$

Observe that if $\lambda \in \pi_0(T)$, then $\lambda \in p_0(A) = p_0^a(A)$ (or, $\lambda \in \rho(A)$) and $\lambda \in p_0(B) = p_0^a(B)$ (or, $\lambda \in \rho(B)$). Hence $\lambda \in \pi_0(T) \Longrightarrow \lambda \in p_0^a(T)$, which completes the proof. $\square$

Remark 2.15. The example of the operator $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(\ell^2 \oplus \ell^2)$, where $A(x_1, x_2, x_3, ...) = (0, x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, ...)$, $B(x_1, x_2, x_3, ...) = (0, x_2, 0, x_4, 0, ...)$ and $C(x_1, x_2, x_3, ...) = (0, 0, x_2, 0, x_3, ...)$, shows that the hypothesis A is polaroid on $\pi_0(T)$ in Theorem 2.14 can not be replaced by the hypothesis that A is polaroid on $\pi_0(A)$. Observe that all the hypotheses of the theorem are satisfied, except for the fact that $\pi_0(T) = \{0\}$ and $0 \notin p(A)$: T does not satisfy $a - Wt$, even Wt.

Remark 2.16. Let E be complemented in $\mathcal{X}$, so that T_0 and T have the representations of Remark 2.12. Since B has SVEP at a point if and only if B_1 has SVEP at the point, and since $\mathrm{ind}(T - \lambda) = \mathrm{ind}(A - \lambda) + \mathrm{ind}(B_1 - \lambda)$, either of the conditions A and A^*, or A and B, or A^* and B^*, or B and B^* have SVEP on $\sigma_w(T)^{\mathcal{C}}$ implies that $\sigma_w(T_0) = \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B_1)$ (see Lemma 2.1). Again, either of the conditions A and A^*, or A and B, have SVEP on $\sigma_{aw}(T)^{\mathcal{C}}$ implies that $\sigma_{aw}(T) = \sigma_{aw}(A)\cup\sigma_{aw}(B_1)$ [9, Theorem 4.12(ii)]. (Observe that if A^* and B^* have SVEP on $\sigma_{aw}(T)^{\mathcal{C}}$, then $\sigma_{aw}(T) = \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B_1) = \sigma_{aw}(A) \cup \sigma_{aw}(B_1)$.) Evidently, $\sigma_{aw}(T) \subseteq \sigma_{aw}(T_0)$; if $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B_1)$, then the following implications show that $\sigma_{aw}(T_0) \subseteq \sigma_{aw}(T)$ (so that $\sigma_{aw}(T) = \sigma_{aw}(T_0)$):

$$\lambda \notin \sigma_{aw}(T) \Longrightarrow \lambda \in \Phi_+^-(A) \cap \Phi_+^-(B_1) \Longrightarrow \lambda \in \Phi_+^-(T_0) \Longrightarrow \lambda \notin \sigma_{aw}(T_0).$$

It is known, [9, Theorems 5.1 and 5.7], that: (i) If $\sigma_w(T)=\sigma_w(A)\cup\sigma_w(B_1)$, then the equivalence T satisfies $Wt \Longleftrightarrow T_0$ satisfies Wt holds if and only if $\pi_0(T)=\pi_0(T_0)$; (ii) If $\sigma_{aw}(T)=\sigma_{aw}(A)\cup\sigma_{aw}(B_1)$ and A^* has SVEP on $\sigma_{aw}(T)^{\mathcal{C}}$, then the equivalence T satisfies $a-Wt \Longleftrightarrow T_0$ satisfies $a-Wt$ holds if and only if $\pi_0^a(T)=\pi_0^a(T_0)$.

We prove next an analogue of Theorem 2.11 for operators T satisfying $a-Wt$.

Theorem 2.17. *If $\sigma_{aw}(T)=\sigma_{aw}(A)\cup\sigma_{aw}(B)$, A^* has SVEP on $\pi_0^a(T)$, A and B are a-isoloid, and both A and B satisfy $a-Wt$, then T satisfies $a-Wt$.*

Proof. The hypotheses imply that $\sigma_a(A)\setminus\sigma_{aw}(A)=p_0^a(A)=\pi_0^a(A)$, $\sigma_a(B)\setminus\sigma_{aw}(B)=p_0^a(B)=\pi_0^a(B)$, and $\sigma_a(T)\setminus\sigma_{aw}(T)=p_0^a(T)\subseteq\pi_0^a(T)$ (see Theorem 2.5). Since $\sigma_{aw}(T)^{\mathcal{C}}\subseteq\pi_0^a(T)$, A^* has SVEP on $\sigma_{aw}(T)^{\mathcal{C}}$; hence $\sigma_a(T)=\sigma_a(A)\cup\sigma_a(B)$ (by Theorem 2.3). Thus to complete the proof, we have to prove that $\pi_0^a(T)\subseteq p_0^a(T)$. Let $\lambda\in\pi_0^a(T)$. Then $\lambda\in \mathrm{iso}\sigma_a(T)=\mathrm{iso}\sigma_a(A)\bigcup\mathrm{iso}\sigma_a(B)$ and $\alpha(T-\lambda)<\infty$. Evidently, $\alpha(A-\lambda)<\infty$; hence (since A is a-isoloid) $\lambda\in\pi_0^a(A)$ (or, $\lambda\in\rho(A)$). Since A^* has SVEP at λ, $\lambda\in p_0(A)$, so that $\beta(A-\lambda)<\infty$. Arguing as before, it is seen that $\lambda\in\ p_0^a(B)$ (or, $\lambda\in\rho(B)$). Applying Proposition 2.10 we conclude that $\lambda\in p_0^a(T)$. $\square$

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