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A STUDY ON STATISTICAL CONVERGENCE

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Abstract

A characterization of statistical convergence of sequences in topological groups is obtained and extensions of a decomposition theorem, a completeness theorem and a Tauberian theorem to the topological group setting are proved.

1 Introduction

The notion of statistical convergence for real or complex number sequences was first given by Fast in [6] and has been investigated in a number of papers [7], [8],[9] ,[10], [13], and [14]. Most of the existing work on statistical convergence appears to have been restricted to real or complex sequences except the works of Kolk, Maddox, Bulut and Çakallı. This notion was used by Kolk in [11] to extend statistical convergence to normed spaces; by Maddox [12] to extend to locally convex Hausdorff topological linear spaces; and used by Çakallı [3] to extend to topological Hausdorff groups (see also [1] and [2]).

The purpose of this paper is to give certain characterizations of statistical convergent sequences in topological groups and to obtain extensions of a decompositon theorem, a completeness theorem and a Tauberian theorem to the topological groups.

2 Statistical Convergence in topological groups

In this paper X will always denote a topological Hausdorff group, written additively, which satisfies the first axiom of countability.

In [2], a sequence (x_k) of points in X is called to be statistically convergent to an element ℓ of X if for each neighborhood U of 0

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n : x_k-\ell\notin U\}|=0,$$

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and this is denoted by $st - \lim_{n\to\infty} x_n = \ell$; and a sequence (x_k) of points in X is called statistically Cauchy in X if for each neighborhood U of 0, there is an integer $M(U)$ such that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k - x_{M(U)} \notin U\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. We see that these two concepts coincide on complete topological groups, i.e. a sequence (x_k) is statistically Cauchy if and only if it is statistically convergent when the topological group considered is complete. For a counterexample consider the topological group $\mathbf{Q}$ with its subspace topology induced by the usual topology of $\mathbf{R}$. Let (x_n) be a rational sequence converging (in $\mathbf{R}$) to $\sqrt{2}$. Then (x_n) is statistically Cauchy in $\mathbf{Q}$ but not statistically convergent in $\mathbf{Q}$.

Any convergent sequence is statistically convergent, but the converse is not generally true. For example, for any element x of X, different from 0, the sequence defined by $x_k = x$ if k is a square and $x_k = 0$ otherwise is statistically convergent to 0, but not convergent; and the sequence defined by $x_k = x$ if k is odd and $x_k = 0$ if k is even is not statistically convergent at all.

We note that every statistically convergent sequence has only one limit, that is, if a sequence statistically convergent to ℓ_1 and ℓ_2, then $\ell_1 = \ell_2$, any nonthin subsequence of a statistically convergent sequence is statistically convergent to the same statistical limit, sum of two statistically convergent sequences is statistically convergent.

The following definition is a special application to the work given in [4].

Definition 1. Let $F \subset X$ and $\ell \in X$. Then ℓ is in the statistically-sequential closure of F if there is a sequence $\mathbf{x} = (x_n)$ of points in F such that $st{-}\lim_{n\to\infty} x_n = \ell$. We denote statistically-sequential closure of a set F by $\overline{F}^{st}$. We say that a set is statistically-sequentially closed if it contains all of the points in its statistically-sequential closure.

Now we give three definitions which can be obtained by taking $G := st - \lim$ in [4].

Definition 2. A point ℓ is called a statistically-sequential accumulation point of F (or is in the statistically-sequential derived set) if there is a sequence $\mathbf{x} = (x_n)$ of points in $F \backslash \{\ell\}$ such that $st - \lim_{n\to\infty} x_n = \ell$.

Definition 3. A subset F of X is called statistically-sequentially countably compact if any infinite subset of F has at least one statistically-sequentially accumulation point in F.

Definition 4. A subset F of X is called statistically-sequentially compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in F there is a subsequence $\mathbf{y} = (y_{n_k})$ of $\mathbf{x}$ with $st - \lim_{k\to\infty} y_{n_k} = \ell \in F$.

Now we give a characterization of statistical convergence in the following:

Theorem 1. A sequence (x_n) is statistically convergent if and only if the following condition is satisfied.

(sC) For each neighborhood U of 0 there exists a subsequence of $(x_{k'(r)})$ of (x_n) such that $\lim_{r\to\infty} x_{k'(r)} = \ell$ and

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : x_k - x_{k'(r)} \notin U\}| = 0.$$

Proof. Take any sequence with $st - \lim_{n\to\infty} x_n = \ell$. Let (U_n) be a sequence of nested base of neighborhoods of 0. Write $K^{(j)} = \{k \in \mathbf{N} : k \le n \textit{ and } x_k - \ell \in U_j\}$ for any positive integer j. Thus for each j, $K(j+1) \subset K(j)$ and $\lim_{n\to\infty} \frac{1}{n}|K^j| = 1$. Choose $m(1)$ such that $n > m(1)$ implies that $\frac{1}{n}|K^{(1)}| > 0$, i.e. $K^{(1)} \ne \emptyset$. Then for each positive integer r such that $m(1) \le r < m(2)$, choose $k(r) \in K^p$, i.e. $x_{k'(r)} - \ell \in U_1$. In general, choose $m(p+1) > m(p)$ such that $r > m(p+1)$ implies that $K^{p+1} \ne \emptyset$. Then for all r satisfying $m(p) \le r < m(p+1)$, choose $k'(r) \in K^p$, i.e. $x_{k'(r)} - \ell \in U_p$. Furthermore, we have for every neighborhood U of 0, a symmetric neighborhood W of 0 such that $W + W \subset U$. Thus we get $\frac{1}{n}|\{k \le n : x_k - x_{k'(r)} \notin U\}| \le \frac{1}{n}|\{k \le n : x_k - \ell \notin W\}| + \frac{1}{n}|\{k \le n : \ell - x_{k'(r)} \notin W\}|$. Since $st - \lim_{n\to\infty} x_n = \ell$, $lim_{n\to\infty}\frac{1}{n}|\{k \le n : x_k - \ell \notin W\}| = 0$ and $\lim_{r\to\infty} x_{k'(r)} = \ell$ we have $lim_{n\to\infty}\frac{1}{n}|\{k \le n : x_{k'(r)} - \ell \notin W\}| = 0$. Thus we we have $lim_{n\to\infty}\frac{1}{n}|\{k \le n : x_k - x_{k'(r)} \notin U\}| = 0$, hence the condition (sC) is satisfied.

Now assume that condition (sC) is satisfied. Then given any neighborhood U of 0 we may choose a neighborhood W of 0 such that $W + W \subset U$. Then we have

$$\frac{1}{n}|\{k \le n : x_k - \ell \notin U\}| \le \frac{1}{n}|\{k \le n : x_k - x_{k'(r)} \notin W\}| + \frac{1}{n}|\{k \le n : x_{k'(r)} - \ell \notin W\}|.$$

Thus the condition (sC) is satisfied from which it follows that the sequence (x_n) is statistically convergent. □

Theorem 2. If $\lim_{k\to\infty} x_k = \ell$ and $st - \lim_{k\to\infty} y_k = 0$, then

$$st - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k.$$

Proof. Let U be any neighborhood of 0. Then we may choose a symetric neighbourhood W of 0 such that $W + W \subset U$. Since $\lim_{k\to\infty} x_k = \ell$, there exists an integer k_0 such that $k \ge k_0$ implies that $x_k - \ell \in W$. Hence $lim_{n\to\infty}\frac{1}{n}|\{k \le n : x_k - \ell \notin W\}| \le lim_{n\to\infty}\frac{k_0}{n} = 0$ and by the assumption that $st - \lim_{k\to\infty} y_k = 0$ we have $lim_{n\to\infty}\frac{1}{n}|\{k \le n : y_k \notin W\}| = 0$. Now we have $\{k \le n : (x_k - \ell) + y_k \notin U\} \subset \{k \le n : x_k - \ell \notin W\} \bigcup \{k \le n : y_k \notin W\}$. Hence

$$\frac{1}{n}|\{k \le n : (x_k - \ell) + y_k \notin U\}| \le \frac{1}{n}|\{k \le n : x_k - \ell \notin W\}| + \frac{1}{n}|\{k \le n : y_k \notin W\}|.$$

It follows from the above inequality that $lim_{n\to\infty}\frac{1}{n}|\{k \le n : (x_k - \ell) + y_k \notin U\}| = 0$. Thus $st - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k$. □

Theorem 3. If a sequence (x_k) is statistically convergent to ℓ, then there are sequences (y_k) and (z_k) such that $\lim_{k\to\infty} y_k = \ell$, $x = y + z$ and $\lim_{n\to\infty} \frac{1}{n}|\{k \le n : x_k \ne y_k\}| = 0$ and z is a statistically null sequence.

Proof. Let (U_j) be a nested base of neighborhoods of 0. Take $n_o = 0$ and choose an increasing sequence (n_j) of positive integers such that $\frac{1}{n}|\{k \leq n : x_k - \ell \notin U_j\}| < \frac{1}{j}$ for $n > n_j$. Let us define sequences $y = (y_k)$ and $z = (z_k)$ in the following way. Write $z_k = 0$ and $y_k = x_k$ if $n_o < k \leq n_1$ and suppose that $n_j < n_{j+1}$ for $j \geq 1$. $z_k = 0$ and $y_k = x_k$ if $x_k - \ell \in U_j$, $y_k = \ell$ and $z_k = x_k - \ell$ if $x_k - \ell \notin U_j$. Firstly, we prove that $\lim_{k\to\infty} y_k = \ell$. Let U be any neighborhood of 0. We may choose a positive integer j such that $U_j \subset U$. Then $y_k - \ell = x_k - \ell \in U_j$ and so $y_k - \ell \in U$ for $k > n_j$. If $x_k - \ell \notin U_j$, then $y_k - \ell = \ell - \ell = 0 \in U$. Hence $\lim_{k\to\infty} y_k = \ell$. Finally we show that $z = (z_k)$ is a statistically null sequence. It is enough to show that $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : z_k \neq 0\}| = 0$. For any $n \in \mathbf{N}$ any neighborhood U of 0, we have $|\{k \leq n : z_k \notin U\}| \leq |\{k \leq n : z_k \neq 0\}|$. If $j \in \mathbf{N}$ such that $U_j \subset U$ and $\varepsilon > 0$, we are going to show that $\frac{1}{n}|\{k \leq n : z_k \neq 0\}| < \varepsilon$. If $n_p < n \leq n_{p+1}$, then $\{k \leq n : z_k \neq 0\} \subset \{k \leq n : x_k - \ell \notin U_p\}$. If $p > j$ and $n_p < n \leq n_{p+1}$, then $\frac{1}{n}|\{k \leq n : z_k \neq 0\}| \leq \frac{1}{n}|\{k \leq n : x_k - \ell \notin U_p\}| < \frac{1}{p} < \frac{1}{j} < \varepsilon$. This completes the proof. □

Corollary 4. Any statistically convergent sequence has a convergent subsequence.

Proof. The proof of this result follows from the preceding theorem. □

Theorem 5. If X is statistically-sequentially compact, then X is complete.

Proof. Take any Cauchy sequence $\mathbf{x} = (x_k)$ of points in X. As X is statistically sequentially compact, there exists a statistically convergent subsequence $\mathbf{y} = (y_{n_k})$ of the sequence $\mathbf{x}$ to an element ℓ of X. Theorem 1 ensures the existence of a convergent subsequence $\mathbf{z} = (z_{n_k})$ of the sequence $\mathbf{y}$ to ℓ. Hence $\mathbf{x}$ converges. This completes the proof of the theorem. □

Definition 5. A sequence (x_n) in X is called slowly oscillating if, for each neighborhood U of 0, there exists a positive integer n_0 and $\delta > 0$ such that if $n_0 \leq n \leq k \leq (1+\delta)n$, then $(x_k - x_n) \in U$.

Sum of two slowly oscillating sequences is again slowly oscillating. A statistically Cauchy sequence need not be slowly oscillating. Neither the converse holds. For example, for any element x of X, different from 0, the sequence defined by $x_k = \sqrt{k}x$ if k is a square and $x_k = 0$ otherwise is not slowly oscillating but it is statistically Cauchy; and when X is a torsion free topological Hausdorff group which satisfies the first axiom of countability, the sequence (s_n) defined by $s_n = \sum_{k=1}^{n} \frac{1}{n}x$ is slowly oscillating but it is not statistically Cauchy .

We now give a Tauberian theorem.

Theorem 6. If (x_n) is statistically convergent and slowly oscillating, then it is convergent.

Proof. Let $st - \lim_{n\to\infty} x_n = \ell$. By Corollary 4, we have a subsequence (j_m) with $1 \leq j_1 < j_2 < ... < j_m < ...$ of those indices k for which $y_k = x_k$. Since $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k \neq y_k\}| = 0$, $\lim_{m\to\infty} \frac{1}{j_m}|\{k \leq j_m : x_k = y_k\}| = \lim_{m\to\infty} \frac{m}{j_m} = 1$. Consequently, it follows that

(iii) $\lim_{m\to\infty} \frac{j_{m+1}}{j_m} = \lim_{m\to\infty} \frac{j_{m+1}}{m+1} \cdot \frac{m+1}{m} \cdot \frac{m}{j_m} = 1.$

By the definition of (j_m), we get

(iv) $\lim_{m\to\infty} x_{j_m} = \lim_{m\to\infty} y_{j_m} = \ell.$

By (iii) and (iv) we conclude that for each closed neighborhood U of 0, there exists an n_0 such that if $m > n_0$ then $(x_k - x_{j_m}) \in U$ whenever $j_m < k < j_{m+1}$. Since U is arbitrary, it follows that $\lim_{m\to\infty}(x_m - x_{j_m}) = 0$. By (iv), (x_m) is convergent to ℓ. Thus the proof is completed. □

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