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SUBDIVISION IN POLYNOMIALS SPACES

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Abstract. For the Lagrange interpolation operator, a multi-subdivision scheme is established. The existence of the corresponding functional equation of Read-Bajraktarević type is proved and used in construction of this scheme. Associated algorithms are developed and illustrated through adequate examples.

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1. Introduction

The notion of subdivision is present in many areas of applied mathematics. If one speaks on polynomials, then subdivision is mainly relates to Bernstein polynomials and induced Bézier curves or surfaces [10, 11, 13]. As it is noted in [4], any polynomial undergoes some subdivision process, and this process is always linear. As it is shown in [5, 10], polynomial subdivision performs by an AIFS, a variant of IFS introduced in [5] and further developed in [6-9, 13] and [2]. So, this introduction begins with a reminder concerning IFS/AIFS stuff.

The concept of Iterated Function System (IFS), and its affine invariant counterpart AIFS appear to play a crucial role in constructive theory of fractal sets and in paving the way to have a good modeling tools for such sets. But, if the collection of objects to be modeled, besides fractals contains smooth objects as well (polynomials for ex.) then one needs to revisit classical algorithms for smooth objects generation and to introduce the new one that is capable to create both fractal and smooth forms. In this light, the purpose of this paper is to develop such algorithms for interpolating polynomials.

Let $\{w_i,\ i=1,2,\dots,n\}$, $n>1$, be a set of contractive affine mappings defined on the complete Euclidian metric space $\left(\mathrm{R}^m,\ d_E\right)$

$$w_i(\mathbf{x}) = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \quad \mathbf{x} \in \mathrm{R}^m, \quad i = 1, 2, \dots, n, \tag{1.1}$$

where $\mathbf{A}_i$ is an $m \times m$ real matrix and $\mathbf{b}_i$ is an m-dimensional real vector. Supposing that the Lipschitz factors $s_i = \mathrm{Lip}\{w_i\}$, satisfy condition $|s_i| < 1$, $i = 1, 2, ..., n$, the system $\left\{\mathbb{R}^m; w_1, w_2, \ldots, w_n\right\}$ is called (hyperbolic) Iterated Function System (IFS). Associated with given IFS, is so called *Hutchinson operator* $\mathcal{H}\left(\mathbb{R}^m\right) \to \mathcal{H}\left(\mathbb{R}^m\right)$, defined by

$$B \mapsto \bigcup_{i=1}^{n} w_i(B), \ \forall B \in \mathcal{H}\left(\mathbb{R}^m\right). \tag{1.2}$$

It turns to be a contractive mapping on the complete metric space $\left(\mathcal{H}\left(\mathbb{R}^m\right), h\right)$ with contractivity factor $s = \max\{s_i\}$. Here, $\mathcal{H}\left(\mathbb{R}^m\right)$ is the space of nonempty compact subsets of $\mathbb{R}^m$ and h stands for Hausdorff metric induced by d_E, i.e.

$$h(A, B) = \max\left\{\max_{a\in A} \min_{b\in B} d_E(a, b), \ \max_{b\in B} \min_{a\in A} d_E(b, a)\right\}, \ \forall A, B \in \mathcal{H}\left(\mathbb{R}^m\right).$$

Let $\mathsf{S}_{m+1} = \left[s_{ij}\right]_{i,j=1}^{m+1}$ be an $(m+1)\times(m+1)$ row-stochastic real matrix (its rows sum up to 1).

Definition 1. We refer to the linear mapping $\mathsf{L} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, such that $\mathsf{L}\,(\mathbf{r}) = \mathbf{S}^{\mathrm{T}}\mathbf{r}$ as the *linear mapping associated with* **S**. The corresponding Hutchinson operator is

$$W(B) = \bigcup_{i=1}^{n} \mathsf{L}_i(B), \ \forall B \in \mathcal{H}\left(\mathbb{R}^{m+1}\right) \tag{1.3}$$

According to the contraction mapping theorem, both Hutchinson operators (1.2) and (1.3) have the unique fixed point called the *attractor* of the IFS/AIFS. In the case of AIFS, the attractor $A \in \mathcal{H}\left(\mathbb{R}^{m+1}\right)$ satisfies $A = W(A)$.

Definition 2. A (non-degenerate) *m-dimensional simplex* $\hat{\mathbf{P}}_m$ (*m-simplex*) is the convex hull $\hat{\mathbf{P}}_m = \mathrm{conv}\{\mathbf{P}_m\}$ of a set $\mathbf{P}_m$ of $m+1$ affinely independent points/vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{m+1}$ in Euclidean space of dimension $\geq m$ that will be denoted in matrix form, $\mathbf{P}_m = \left[\mathbf{p}_1^{\mathrm{T}}\ \mathbf{p}_2^{\mathrm{T}}\ \ldots\ \mathbf{p}_{m+1}^{\mathrm{T}}\right]^{\mathrm{T}}$.

Definition 3. The *areal* (also *normalized barycentric*) *coordinates* of the point $\mathbf{x} \in {}^{m}$ w.r.t. the simplex $\hat{\mathbf{P}}_m$ are defined as $\rho_i = \mathrm{area}\,\hat{\mathbf{P}}_m^i / \mathrm{area}\,\hat{\mathbf{P}}_m$, $i = 1, 2, \ldots, m+1$, where $\mathrm{area}\,\hat{\mathbf{P}}_m$ is the signed m-dimensional volume of the simplex $\hat{\mathbf{P}}_m$ and $\hat{\mathbf{P}}_m^i$ is the simplex derived from $\hat{\mathbf{P}}_m$ by substituting the i-th vertex, $\mathbf{e}_i$ by the point $\mathbf{x}$. It follows by definition that areal coordinates obey the unity partition property $\sum_i \rho_i = 1$.

Definition 4. Let $\hat{\mathbf{P}}_m$ be a non-degenerate simplex and let $\{\mathbf{S}_i\}_{i=1}^n$ be a set of real square non-singular row-stochastic matrices of order $m+1$. The system $\Omega\left(\hat{\mathbf{P}}_m\right)=\left\{\hat{\mathbf{P}}_m;\,\mathbf{S}_1,\mathbf{S}_2,\ldots,\mathbf{S}_n\right\}$ is called (*hyperbolic*) *Affine invariant IFS* (*AIFS*), provided that the linear mappings associated with S_i are contractions in $\left(\mathsf{R}^m,\, d_E\right)$ ([5-7]).

2. Subdivision

Although the notion of subdivision is usually attributed to *m*-dimensional ($m \geq 1$) continuous parametric mapping $t \mapsto P_n(t)$, $t \in [a,\ b]$, $\left(a<b\right)$, so that $P_n(t) \in \mathsf{R}^{\mathsf{m}}$, for the purpose of study the basic properties, it is enough to consider one-dimensional case ($m=1$).

Let

$$P_n(t)=\sum_{k=0}^{n} A_k\, B_k^n(t), \quad t \in [a,\ b], \tag{2.1}$$

where A_k are real coefficients and

$$\mathsf{B}_n(t)=\{B_0^n(t),\, B_1^n(t),\, ...,\, B_n^n(t)\}, \quad t \in [a,\ b], \tag{2.2}$$

is some functional basis, it may happened that both A_k and $\mathsf{B}_n(t)$ depend on the interval of definition. To stress this fact, it is suitable to write $A_k[a,\ b]$ as well as $\mathsf{B}_n[a,\ b](t)$. Then, the subdivision is defined as follows.

Definition 5. The function P_n, defined by (2.1) is said to permit linear subdivision if and only if for each nonempty subinterval $[p,\ q] \subset [a,\ b]$, there exists a set of coefficients $\left\{A_k[p,\ q]\right\}_{k=0}^n$ such that

$$\sum_{k=0}^{n} A_k[p,\ q] B_k^n[p,\ q](t) = \sum_{k=0}^{n} A_k[a,\ b] B_k^n[a,\ b]\big(\varphi(t)\big), \tag{2.3}$$

for $t \in [a,\ b]$, where

$$\varphi(t)=\frac{1}{b-a}\big((q-p)t+b\,p-a\,q\big), \tag{2.4}$$

maps $[a,\ b]$ into $[p,\ q]$.

Moreover, restriction on $P_n(t)$ to belong to the set P_n of algebraic polynomials of dg≤ n, allows the subdivision to be linear and only linear.

Theorem 2 (Goldman and Heath [4]). The function P_n, defined by (2.1) admits linear subdivision if and only if $B_n(t)$ is a polynomial basis.

The classic, and best known subdivision phenomena is connected with Bernstein polynomial basis, but subdivision is also possible for monomial, Lagrange, Newton or any other polynomial basis ([1], [10]). Here, the focus will be set on Lagrange interpolation polynomials.

3. Lagrange subdivision

Let the interval $[t_0, t_n]$ be subdivided by a nonuniformly spaced knots

$$\boldsymbol{\tau} = \left[t_0\ t_1\ \ldots\ t_n\right]^{\mathrm{T}}\ t_0 < t_1 < \ldots < t_n, \quad (n \ge 2), \tag{3.1}$$

and let the vector $\mathbf{l}_n(\boldsymbol{\tau},\ t) = \left[\ \ell_0^n(t) \ldots \ell_n^n(t)\right]^{\mathrm{T}}$ contains the Lagrange basis functions over

$$\ell_k^n(\boldsymbol{\tau},\ t) = \prod_{\substack{i=0\\ i\ne k}}^{n} \frac{t - t_i}{t_k - t_i}, \quad t \in [t_0,\ t_n], \quad k = 0, 1, \ldots n. \tag{3.2}$$

Then, the vector of ordinates $\mathbf{y} = [\,y_0\ y_1\ \ldots\ y_n]^{\mathrm{T}}$ defines the matrix of the interpolating nodes

$$\mathbf{P}^{\mathrm{T}} = \left[[t_0\ \ y_0]^{\mathrm{T}} \ldots [t_n\ \ y_n]^{\mathrm{T}}\right], \tag{3.3}$$

that uniquely defines $L_n(\boldsymbol{\tau}, \mathbf{P}, \cdot) \in \mathsf{P}_n$, the interpolation polynomial which Lagrange form is

$$L_n(\boldsymbol{\tau}, \mathbf{P}, t) = \sum_{k=0}^{n} y_k \ell_k^n(t) = \mathbf{l}_n(t)^{\mathrm{T}} \cdot \mathbf{y}, \quad t_0 \le t \le t_n. \tag{3.4}$$

Lemma 1. Let $\boldsymbol{\theta} = \left[\theta_0\ \theta_1\ \ldots\ \theta_n\right]^{\mathrm{T}}$ be any set of knots satisfying $-\infty < \theta_0 < \theta_1 < \ldots < \theta_n < +\infty$, $(n \ge 2)$. Then, the following expansion holds

$$\ell_\nu^n(\boldsymbol{\tau},\ t) = \sum_{k=0}^{n} \ell_\nu^n(\boldsymbol{\tau},\ \theta_k)\ell_k^n(\boldsymbol{\theta},\ t). \tag{3.5}$$

Proof. The right side of (3.5) is a polynomial from P_n, $p(t) = \sum_{k=0}^{n} \lambda_k^\nu \ell_k^n(\boldsymbol{\theta},\ t)$, where for fixed $0 \le \nu \le n$, $\lambda_k^\nu = \ell_\nu^n(\boldsymbol{\tau},\ \theta_k)$. By the cardinal property of Lagrange basis, $\ell_k^n(\boldsymbol{\theta},\ \theta_\nu) = \delta_{\nu k}$ (Kronecker "delta"), it follows

$p(\theta_\nu) = \sum_{k=0}^{n} \lambda_k^\nu \ell_k^n(\boldsymbol{\theta},\ \theta_\nu) = \lambda_\nu^\nu = \ell_\nu^n(\boldsymbol{\tau},\ \theta_\nu)$ for $0 \le \nu \le n$. Thus, $p(t)$ equals to $\ell_\nu^n(\boldsymbol{\tau},\ t)$ in $(n+1)$ non-coincident points, which means that $p(t) \equiv \ell_\nu^n(\boldsymbol{\tau},\ t)$. □

Now, let the set of n contractive affine mappings $\{\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_n\}$ be given so that $\varphi_k:\ [t_0,\ t_n] \to [t_{k-1},\ t_k]$. In other words,

$$\varphi_k(t) = \frac{t_k - t_{k-1}}{t_n - t_0}\, t \ + \ \frac{t_{k-1}\, t_n - t_k\, t_0}{t_n - t_0},\quad t \in [t_0,\ t_n]. \tag{3.6}$$

The image $\left[\varphi_k(t_0)\ \varphi_k(t_1)\ \ldots\ \varphi_k(t_n)\right]^{\mathrm{T}}$ of the knot vector (3.1) upon the mapping φ_k will be denoted simply by $\varphi_k(\boldsymbol{\tau})$.

Theorem 1. Let $L_n(\boldsymbol{\tau},\ \mathbf{P},\ \cdot)$ be polynomial from P_n, interpolating the points (3.3). Let $\mathbf{Q}^{[k]} = \left[\mathbf{Q}_0^k\ \mathbf{Q}_1^k\ \ldots\ \mathbf{Q}_n^k\right]^{\mathrm{T}}$, where

$$\mathbf{Q}_i^k = \left(\varphi_k(t_i),\ L_n(\boldsymbol{\tau},\ \mathbf{P},\ \varphi_k(t_i))\right),\quad i = 0,\ \ldots,\ n. \tag{3.7}$$

Then, the operator $L_n\left(\varphi_k(\boldsymbol{\tau}),\ \mathbf{Q}^{[k]},\ \cdot\right)$ coincides with $L_n(\boldsymbol{\tau},\ \mathbf{P},\ \cdot)$.

Proof. Let $\boldsymbol{\theta}$ be any set of knots satisfying conditions of Lemma 1. The Lagrange basis over $\boldsymbol{\theta}$ will be $\left\{\ell_k^n(\boldsymbol{\theta},\ t)\right\}_{k=0,n}$. From the graph of the interpolating polynomial $L_n(\boldsymbol{\tau},\ \mathbf{P},\ \cdot)$, let the points $\mathbf{Q}_k = \left(\theta_k,\ L_n(\boldsymbol{\tau},\ \mathbf{P},\ \theta_k)\right) = (\theta_k,\ q_k)$ be selected with abscisae at the knots $\boldsymbol{\theta}$. The interpolating polynomial to the points $\mathbf{Q} = \left[\mathbf{Q}_0\ \ldots\ \mathbf{Q}_n\right]^{\mathrm{T}}$ will be

$$\begin{aligned}
L_n(\boldsymbol{\theta},\ \mathbf{Q},\ t) &= \sum_{k=0}^{n} \ell_k^n(\boldsymbol{\theta},\ t)\, q_k = \sum_{k=0}^{n} \ell_k^n(\boldsymbol{\theta},\ t)\, L_n(\boldsymbol{\tau},\ \mathbf{P},\ \theta_k) \\
&= \sum_{k=0}^{n} \ell_k^n(\boldsymbol{\theta},\ t) \sum_{\nu=0}^{n} \ell_\nu^n(\boldsymbol{\tau},\ \theta_k)\, y_\nu = \sum_{k=0}^{n} \sum_{\nu=0}^{n} \ell_k^n(\boldsymbol{\theta},\ t)\, \ell_\nu^n(\boldsymbol{\tau},\ \theta_k)\, y_\nu \\
&= \sum_{\nu=0}^{n} \left(\sum_{k=0}^{n} \ell_k^n(\boldsymbol{\theta},\ t)\, \ell_\nu^n(\boldsymbol{\tau},\ \theta_k) \right) y_\nu .
\end{aligned}$$

In virtue of Lemma 1, $\sum_{k=0}^{n} \ell_k^n(\boldsymbol{\theta},\ t)\,\ell_\nu^n(\boldsymbol{\tau},\ \theta_k) = \ell_\nu^n(\boldsymbol{\tau},\ t)$, so $L_n(\boldsymbol{\theta}, \mathbf{Q}, t) = \sum_{\nu=0}^{n} \ell_\nu^n(\boldsymbol{\tau},\ t)\, y_\nu = L_n(\boldsymbol{\tau}, \mathbf{P}, t)$, for all finite *t*. The proof completes by setting $\boldsymbol{\theta} = \varphi_k(\boldsymbol{\tau})$, $k = 1, \ldots, n$. □

Corollary 1. Let ϕ_k be the polynomial mapping, $\phi_k : (t,\ y) \mapsto L_n\left(\varphi_k(\boldsymbol{\tau}), \mathbf{Q}^{[k]}, \varphi_k(t)\right)$. Then, the Lagrange interpolating polynomial $L_n(\boldsymbol{\tau}, \mathbf{P}, t)$ satisfies the Read-Bajraktarević type functional equation

$$L_n(\boldsymbol{\tau}, \mathbf{P}, t) = \phi_k\left[\varphi_k^{-1}(t),\ L_n\left(\boldsymbol{\tau}, \mathbf{P}, \varphi_k^{-1}(t)\right)\right], \quad t \in [t_{k-1}, t_k].$$

For the next theorem the following lemma is needed.

Lemma 2. Let **P** and **Q** are two *m*-simplexes and let $\mathbf{Q} = \mathcal{L}(\mathbf{P})$, $\mathcal{L}(\mathbf{r}) = \mathbf{S}^{\mathrm{T}}\mathbf{r}$ in the sense of Def. 1. Let $\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x}) = \left[\rho_1^{\mathbf{P}}\ \ldots\ \rho_{m+1}^{\mathbf{P}}\right]^{\mathrm{T}}$ be areal coordinates of the point **x** w.r.t. **P**. Then, if $\mathbf{x}' = \mathcal{L}(\mathbf{x})$, it holds

$$\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x}') = \mathbf{S}^T \boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x}). \tag{3.8}$$

Proof. The relation $\mathbf{Q} = \mathcal{L}(\mathbf{P})$, means that **P** and **Q** are linearly equivalent, which implies equality of areal coordinates of the corresponding point i.e. $\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x}) = \boldsymbol{\rho}^{\mathbf{Q}}(\mathbf{x}')$. Also, $\mathbf{Q} = \mathbf{S} \cdot \mathbf{P}$, so

$$\mathbf{x}' = \boldsymbol{\rho}^{\mathbf{Q}}(\mathbf{x}')^{\mathrm{T}} \cdot \mathbf{Q} = \boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x})^{\mathrm{T}} \cdot (\mathbf{S} \cdot \mathbf{P}) = \left(\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x})^{\mathrm{T}} \cdot \mathbf{S}\right) \cdot \mathbf{P} = \boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x}')^{\mathrm{T}} \cdot \mathbf{P},$$

wherefrom, $\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x}') = \left(\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{x})^{\mathrm{T}} \cdot \mathbf{S}\right)^{\mathrm{T}}$ which is (3.8). □

Theorem 2. The points $\mathbf{Q}^{[k]} = \left[\mathbf{Q}_0^k\ \mathbf{Q}_1^k\ \ldots\ \mathbf{Q}_n^k\right]^{\mathrm{T}}$, $k = 1, \ldots, n$ $(n \geq 2)$, given by (3.7) are images of the interpolating nodes **P** given by (3.3), upon the linear mapping $\mathcal{L}_k(\mathbf{r}) = \mathbf{S}_k^{\mathrm{T}}\mathbf{r}$, where

$$\mathbf{S}_k = \left[\ell_j^n\left(\boldsymbol{\tau},\ \varphi_k(t_i)\right)\right]_{0 \leq i, j \leq n}, \quad k = 1, \ldots, n. \tag{3.9}$$

and φ_k is given by (3.6). More precisely, $\mathbf{Q}^{[k]} = \mathbf{S}_k \cdot \mathbf{P}$.

Proof. Choose an arbitrary value $t_A \in [t_0,\ t_n]$ and consider the point $\mathbf{A} = \left(t_A,\ L_n(\boldsymbol{\tau},\ \mathbf{P},\ t_A)\right)$. The areal coordinates of the point **A** w.r.t. simplex **P**, $\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{A})$ are given by the Lagrange basic functions for $t = t_A$, i.e. $\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{A}) = \left[\ell_0^n\left(\boldsymbol{\tau},\ t_A\right) \ldots \ell_n^n\left(\boldsymbol{\tau},\ t_A\right)\right]^{\mathrm{T}}$. It is the consequence of the fact that $\mathbf{A} = \sum_{k=0}^{n} \mathbf{P}_k \ell_k^n(\boldsymbol{\tau},\ t_A)$ and the unity partition property $\sum_{k=0}^{n} \ell_k^n(\boldsymbol{\tau},\ t) = 1$. On the other hand, the corresponding point to **A**, for the transformed node mesh $\varphi_k(\boldsymbol{\tau})$ is the point $\mathbf{B} = \left(\varphi_k(t_A),\ L_n(\boldsymbol{\tau},\ \mathbf{P},\ \varphi_k(t_A))\right) = \sum_{i=0}^{n} \mathbf{Q}_i^k \ell_i^n(\boldsymbol{\tau},\ \varphi_k(t_A))$. Consequently, $\boldsymbol{\rho}^{\mathbf{Q}^{[k]}}(\mathbf{B}) = \left[\ell_0^n(\boldsymbol{\tau},\ \varphi_k(t_A) \ldots \ell_n^n(\boldsymbol{\tau},\ \varphi_k(t_A)\right]^{\mathrm{T}}$. By Lemma 2, then $\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{B}) = \mathbf{S}_k^{\mathrm{T}} \boldsymbol{\rho}^{\mathbf{P}}(\mathbf{A})$, with $\mathbf{S}_k$ as it is given in (3.9). By setting $t_A = t_i \quad (0 \le i \le n)$, the last equality turns into $\boldsymbol{\rho}^{\mathbf{P}}(\mathbf{Q}_i) = \mathbf{S}_k^{\mathrm{T}} \boldsymbol{\rho}^{\mathbf{P}}(\mathbf{P}_i)$ which yields $\mathbf{Q}^{[k]} = \mathbf{S}_k \cdot \mathbf{P}$. □

Finally, one has

Theorem 3. The graph G_p of the polynomial $P_n(t) \in \mathsf{P}_n$, defined by the Lagrange interpolation data (3.3), is the attractor of the AIFS {**P**; $\mathbf{S}_1$,..., $\mathbf{S}_n$}, where $\mathbf{S}_k$ is given by (3.9), and φ_k is given by (3.6).

Proof. First, the hyperbolicity of the AIFS will be shown. The spectrum of the matrix $\mathbf{S}_k$, as it is known from [5-7] is of the form

$\mathrm{sp}\left(\mathbf{S}_k\right) = \left\{1,\ \lambda_1^k,\ \lambda_2^k, \ldots,\ \lambda_{n-1}^k\right\}$, where $\lambda_j^k = \left(\dfrac{t_k - t_{k-1}}{t_n - t_0}\right)^j$. Since $n \ge 2$, $\lambda_1^k = \dfrac{t_k - t_{k-1}}{t_n - t_0} < 1$ for all k, making λ_1^k the second greatest eigenvalue of $\mathbf{S}_k$. So, there exists matrix norm such that $\left\|\mathbf{S}_k\right\| = \lambda_1^k < 1$. So, the AIFS is hyperbolic, ensuring existence of the unique attractor. Denote it by G_p. Since matrix S_k, that defines the mapping $\mathsf{L}_k(\mathbf{r}) = \mathbf{S}_k^{\mathrm{T}} \mathbf{r}$, maps $\mathbf{P}_0$ and $\mathbf{P}_n$ into $\mathbf{P}_{k1}$ and $\mathbf{P}_k$, the image of **P** upon Hutchinson operator W contains as a subset. The same is for any power W^m of W. So, $\mathbf{P} \in W^{\infty} = G_p$, i.e., the interpolating points lie on the attractor. Further, by Theorem 2, and Theorem 1, the mapping L_k maps the graph of the polynomial $L_n(t)$, interpolating the points **P**, into the segment of the graph of $L_n(t)$ on the subinterval $[t_{k-1},\ t_k]$. The proof is then follows by the Collage theorem [3]. □

Example 1. Let **P** = $\mathrm{diag}\left(\boldsymbol{\tau} \times \mathbf{y}\right)$ be the interpolating data in R^2, where

$\boldsymbol{\tau} = \left[-1 \;\; -\tfrac{1}{5} \;\; \tfrac{3}{5} \;\; \tfrac{7}{5} \;\; \tfrac{11}{5} \;\; 3\right]^{\mathrm{T}}$ and $\mathbf{y} = \left[-1.2 \;\; 2.5 \;\; 3.25 \;\; 2.38 \;\; -2.2 \;\; 0.4\right]^{\mathrm{T}}$.

The graph of the polynomial interpolating these data

$$L_5(t) = 2.69951 + 0.934912\, t - 0.0703125\, t^2 + 0.878092\, t^3 - 1.59912\, t^4 + 0.417074\, t^5$$

on the interval [1, 3], is given in Fig.1.

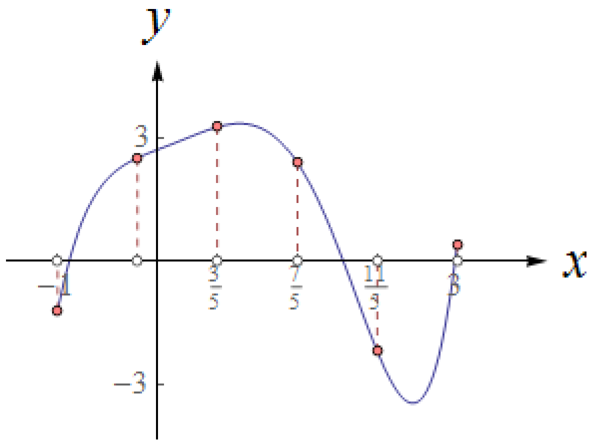

Figure 1.

There are five subintervals, so the AIFS will be $\{\mathbf{P}; \mathbf{S}_1, ..., \mathbf{S}_5\}$, where the matrices are given by (3.9), which makes

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{9576}{15625} & \frac{2394}{3125} & -\frac{2128}{3125} & \frac{1368}{3125} & -\frac{504}{3125} & \frac{399}{15625} \\ \frac{5382}{15625} & \frac{3588}{3125} & -\frac{2691}{3125} & \frac{1656}{3125} & -\frac{598}{3125} & \frac{468}{15625} \\ \frac{2618}{15625} & \frac{3927}{3125} & -\frac{2244}{3125} & \frac{1309}{3125} & -\frac{462}{3125} & \frac{357}{15625} \\ \frac{924}{15625} & \frac{3696}{3125} & -\frac{1232}{3125} & \frac{672}{3125} & -\frac{231}{3125} & \frac{176}{15625} \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{399}{15625} & \frac{2394}{3125} & \frac{1197}{3125} & -\frac{532}{3125} & \frac{171}{3125} & -\frac{126}{15625} \\ -\frac{468}{15625} & \frac{1638}{3125} & \frac{2184}{3125} & -\frac{819}{3125} & \frac{252}{3125} & -\frac{182}{15625} \\ -\frac{357}{15625} & \frac{952}{3125} & \frac{2856}{3125} & -\frac{816}{3125} & \frac{238}{3125} & -\frac{168}{15625} \\ -\frac{176}{15625} & \frac{396}{3125} & \frac{3168}{3125} & -\frac{528}{3125} & \frac{144}{3125} & -\frac{99}{15625} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix};$$

$$S_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{126}{15625} & -\frac{231}{3125} & \frac{2772}{3125} & \frac{693}{3125} & -\frac{154}{3125} & \frac{99}{15625} \\ \frac{182}{15625} & -\frac{312}{3125} & \frac{2184}{3125} & \frac{1456}{3125} & -\frac{273}{3125} & \frac{168}{15625} \\ \frac{168}{15625} & -\frac{273}{3125} & \frac{1456}{3125} & \frac{2184}{3125} & -\frac{312}{3125} & \frac{182}{15625} \\ \frac{99}{15625} & -\frac{154}{3125} & \frac{693}{3125} & \frac{2772}{3125} & -\frac{231}{3125} & \frac{126}{15625} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix};$$

$$S_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{99}{15625} & \frac{144}{3125} & -\frac{528}{3125} & \frac{3168}{3125} & \frac{396}{3125} & -\frac{176}{15625} \\ -\frac{168}{15625} & \frac{238}{3125} & -\frac{816}{3125} & \frac{2856}{3125} & \frac{952}{3125} & -\frac{357}{15625} \\ -\frac{182}{15625} & \frac{252}{3125} & -\frac{819}{3125} & \frac{2184}{3125} & \frac{1638}{3125} & -\frac{468}{15625} \\ -\frac{126}{15625} & \frac{171}{3125} & -\frac{532}{3125} & \frac{1197}{3125} & \frac{2394}{3125} & -\frac{399}{15625} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

$$S_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{176}{15625} & -\frac{231}{3125} & \frac{672}{3125} & -\frac{1232}{3125} & \frac{3696}{3125} & \frac{924}{15625} \\ \frac{357}{15625} & -\frac{462}{3125} & \frac{1309}{3125} & -\frac{2244}{3125} & \frac{3927}{3125} & \frac{2618}{15625} \\ \frac{468}{15625} & -\frac{598}{3125} & \frac{1656}{3125} & -\frac{2691}{3125} & \frac{3588}{3125} & \frac{5382}{15625} \\ \frac{399}{15625} & -\frac{504}{3125} & \frac{1368}{3125} & -\frac{2128}{3125} & \frac{2394}{3125} & \frac{9576}{15625} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

The Figure 2. shows results of repeated application of Hutchinson operator on the interpolation data **P**.

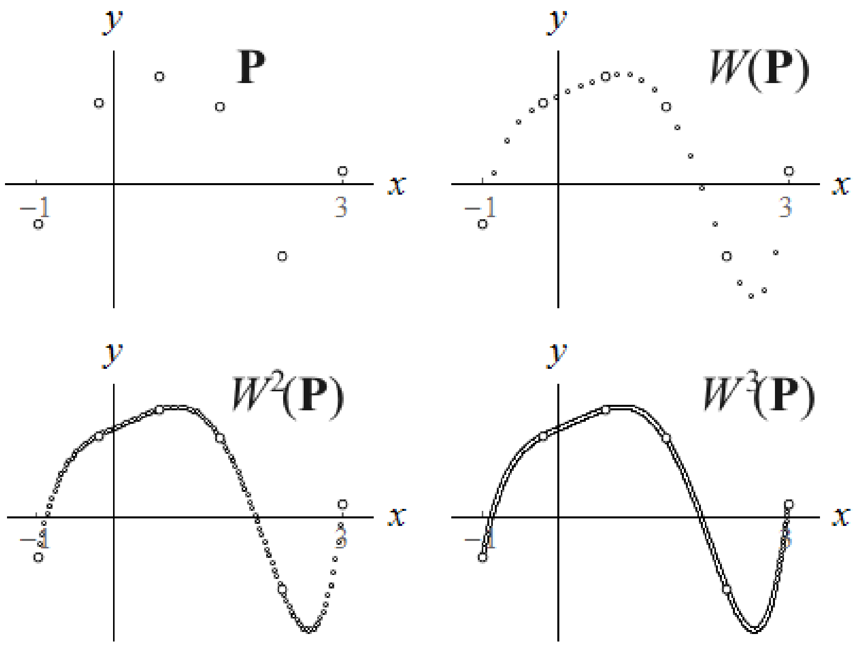

Figure 2.

4. Conclusion

The subdivision scheme, developed in this paper allows fast evaluation of interpolating polynomial, and it is suitable for graphical visualization. The scheme is given in the form of iterated function system, and therefore compatible with other schemes and algorithms that are in use for evaluation and visualization of fractal sets. As a consequence, it makes possible to combine fractal shapes with smooth, polynomial shapes, with possibility of continuous altering from one to other. The problems that are still open are comparing the speed of the algorithm that rises from Theorem 3 and other algorithms for polynomial evaluation.

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