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A COMPARISON BETWEEN DIFFERENT CONCEPTS OF ALMOST ORTHOGONAL POLYNOMIALS

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Abstract

In this paper, we will discuss the notion of almost orthogonality in a functional sequence.Especially, we will define a few sequences of almost orthogonal polynomials which can be used successfully for modelling of electronic systems which generate orthonormal basis. We will include quasi-orthogonality and examine its influence on the behavior of these sequences.

1 Introduction

The first usage of the notion of *almost orthogonality* for operators is annotated in the M. Cotlar's paper [5]. Let E and F be the Hilbert spaces with their scalar products and norms. For a linear operator $S:\ E \to F$, the operator $S^\star:\ F \to E$ is his adjoined operator if it is satisfied

$$(Su, v)_F = (u, S^\star v)_E \qquad (\forall u \in E,\ \forall v \in F). \tag{1}$$

The operator norm is

$$\|S\| = \sup_{\|u\|_E=1} \|S(u)\|_F, \qquad \|S^\star\| = \sup_{\|v\|_F=1} \|S^\star(v)\|_E. \tag{2}$$

Definition 1.1. (Almost orthogonal operators). We will call a family of continuous operators $T_i:\ E \to F \quad (i \in \mathbb{Z})$, *almost orthogonal* if they satisfy the following conditions:

$$\|T_i^\star T_j\| \le a_{i,j}, \quad \|T_i T_j^\star\| \le b_{i,j}, \qquad (i,j \in \mathbb{Z}), \tag{3}$$

where $a_{i,j}$ and $b_{i,j}$ are non-negative symmetric functions on $\mathbb{Z} \times \mathbb{Z}$ which satisfy

$$\|a\|^\mu_{\infty,\mu} = \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} a^\mu_{i,j} < \infty \quad \|b\|^\nu_{\infty,\nu} = \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} b^\nu_{i,j} < \infty, \tag{4}$$

where $0 \le \mu, \nu \le 1, \quad \mu + \nu = 1$.

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Lemma 1.1. *(Cotlar-Stein Lemma). Let $\{T_i\}_{i\in\mathbb{Z}}$ be a family of almost orthogonal operators. Then the formal sum $\sum_i T_i$ converges in the strong operator topology to a continuous linear operator $T: E \to F$, which is bounded by*

$$\|T\| \leq \sqrt{\|a\|^{\mu}_{\infty,\mu}\ \|b\|^{\nu}_{\infty,\nu}}\ . \tag{5}$$

The concept of quasi-orthogonality was introduced in 1923. by M. Riesz [9] who considered the moment problem. It also appeared in Fejer's research of quadratures [8] in 1933. Later, a various aspects of this theory were considered by other mathematicians (T.S. Chihara [4], D.J. Dickinson [7], F. Marcelan,...).

Definition 1.2. (Quasi orthogonal functions). We say that a functional sequence $\{Q_n(x)\}$ is *quasi-orthogonal* of order ρ ($\rho \in \mathbb{N}_0$) with respect to the functional U if

$$U[Q_m Q_n] = 0 \qquad (m, n \in \mathbb{N}_0 : |m-n| > \rho). \tag{6}$$

In the special case $\rho = 0$, it becomes the regular orthogonality.

In our paper [6], we have introduced the next concept.

Definition 1.3. (Almost orthogonality by an error matrix) Let $\mathcal{E} = [\varepsilon_{i,j}]$ be a matrix whose elements are very small positive real numbers. If it exists, the sequence of the functions $\{P_n^{(\varepsilon)}(x)\}$ which satisfies the relation

$$\mathcal{L}\Big[P_n^{(\varepsilon)} \cdot P_i^{(\varepsilon)}\Big] = \varepsilon_{n,i} \quad (i = 0, 1, \ldots, n-1;\ n \in \mathbb{N}) \tag{7}$$

will be called *almost orthogonal* with respect to $\mathcal{L}$ and the error matrix $\mathcal{E}$.

2 Almost orthogonality by shifted zeros

Let $\lambda(x)$ be a positive Borel measure on an interval $(a, b) \subset \mathbb{R}$ with infinite support and such that all moments

$$\lambda_n = \mathcal{L}[x^n] = \int_a^b x^n d\lambda(x) \tag{8}$$

exist. In this manner, we define linear functional $\mathcal{L}$ in the linear space of real polynomials $\mathcal{P}$. Also, we can introduce an inner product as follows (see [10]):

$$(f,\ g) = \mathcal{L}\big[f \cdot g\big] \qquad (f, g \in \mathcal{P}), \tag{9}$$

which is positive-definite because of the property $\|f\|^2 = (f, f) \geq 0$. Hence it follows that monic polynomials $\{P_n(x)\}$ orthogonal with respect to this inner product exist and they satisfy the three-term recurrence relation

$$P_{k+1}(x) = (x - \alpha_k)P_k(x) - \beta_k P_{k-1}(x) \quad (k \geq 0), \qquad P_{-1} \equiv 0,\ P_0 \equiv 1. \tag{10}$$

The zeros of these polynomials are all contained in the interval $(a, b) = \operatorname{supp}\lambda(x)$ and they interlace each other. If we denote them by $\{x_{n,k}\}$, we can write

$$P_n(x) = \prod_{k=1}^{n} (x - x_{n,k}). \tag{11}$$

Let us denote by

$$\tilde{P}_n(x) = \sigma_n P_n(x), \quad \text{where} \quad \sigma_n = \frac{1}{\|P_n\|}. \tag{12}$$

Obviously, $\{\tilde{P}_n(x)\}$ is the corresponding orthonormal polynomial sequence.

$$\sqrt{\beta_{n+1}}\tilde{P}_{n+1}(x) = (x - \alpha_n)\tilde{P}_n(x) - \sqrt{\beta_n}\tilde{P}_{n-1}(x) \qquad (n \geq 0), \tag{13}$$

$$\tilde{P}_{-1} \equiv 0, \quad \tilde{P}_0 \equiv \frac{1}{\sqrt{\beta_0}}. \tag{14}$$

The next lemma, proven in [2], will be very useful

Lemma 2.1. *All leading principal minors of the matrix*

$$A = \begin{bmatrix} P_{n-1}(x_{n,1}) & P_{n-1}(x_{n,2}) & \cdots & P_{n-1}(x_{n,n}) \\ P_{n-2}(x_{n,1}) & P_{n-2}(x_{n,2}) & & P_{n-2}(x_{n,n}) \\ \vdots & & & \\ P_0(x_{n,1}) & P_0(x_{n,1}) & \cdots & P_0(x_{n,1}) \end{bmatrix} \tag{15}$$

are nonsingular.

Let us remind on notation

$$\alpha(x) = \mathcal{O}(\varepsilon^\beta) \quad \Leftrightarrow \quad \lim_{\varepsilon \to 0} \frac{\alpha(x)}{\varepsilon^\beta} = c \quad (0 < c < \infty). \tag{16}$$

The next lemma is slightly generalization of similar one from [11].

Lemma 2.2. *Let z_r be an isolated zero of a polynomial $f(z)$ and $g(z)$ a differentiable function in z_r. Then the function*

$$T(z) = f(z) + \varepsilon g(z) \quad (0 < \varepsilon \ll 1) \tag{17}$$

has a zero $z_r(\varepsilon)$ such that

$$z_r(\varepsilon) = z_r - \varepsilon \frac{g(z_r)}{f'(z_r)} + \mathcal{O}(\varepsilon^2). \tag{18}$$

Proof. Under assumptions, we have $f(z_r) = 0$, $f'(z_r) = \kappa \neq 0$ and $T(z_r(\varepsilon)) = 0$. According to mean valued theorem, we can write

$$\frac{f(z_r(\varepsilon)) - f(z_r)}{z_r(\varepsilon) - z_r} = f'(\eta_r(\varepsilon)) \quad (\eta_r(\varepsilon) \in (\min\{z_r(\varepsilon), z_r\},\ \max\{z_r(\varepsilon), z_r\})),$$

Including it into (17), we have

$$T(z_r(\varepsilon)) = f'(\eta_r(\varepsilon))(z_r(\varepsilon) - z_r) + \varepsilon\ g(z_r(\varepsilon)) = 0,$$

wherefrom

$$z_r(\varepsilon) - z_r = -\varepsilon\ \frac{g(z_r(\varepsilon))}{f'(\eta_r(\varepsilon))}. \tag{19}$$

Since f' and g are continuous functions in the point z_r, the real constants k_1 and k_2 exist such that

$$\varphi(\varepsilon) = \frac{g(z_r(\varepsilon))}{f'(\eta_r(\varepsilon))} = \frac{g(z_r) + k_1\varepsilon}{f'(z_r) + k_2\varepsilon}.$$

By using Taylor series of the function $\varphi(\varepsilon)$, we obtain

$$\varphi(\varepsilon) = \frac{g(z_r)}{f'(z_r)} + \varepsilon \frac{k_1 f'(z_r) - k_2 g(z_r)}{[f'(z_r)]^2} + \mathcal{O}(\varepsilon^2).$$

Hence we finish the proof of the formula (18). □

For the next two theorems we find inspiration in R. Brent's paper [2]. There, discussion about almost orthogonality was motivated by iterative methods for zero-finding, but we find that echo of this paper could be large in the theory of orthogonality itself. Our purpose is to improve conclusions in that way.

Let

$$0 < \varepsilon \ll 1, \quad s \in \{1, \dots, n-1\}, \quad |\gamma_{n,k} - x_{n,k}| < \varepsilon \qquad (k = 1, \dots, s), \tag{20}$$

where $x_{n,k}$ are the zeros of $P_n(x)$ given by (11).

Theorem 2.3. *Under the condition (20), the polynomial*

$$Q_n(x) = \sigma_n \prod_{i=1}^{s} (x - \gamma_{n,i}) \prod_{i=s+1}^{n} (x - x_{n,i}), \tag{21}$$

is almost orthogonal with respect to $\{\tilde{P}_k(x)\}_{k=0}^{n}$, *i.e.*

$$f_i = \mathcal{L}\big[\tilde{P}_i\ Q_n\big] = \begin{cases} \varepsilon\ \omega_i\ , & 0 \le i \le n-1\ , \\ \\ 1\ , & i = n\ , \end{cases} \qquad (\omega_i \in \mathbb{R},\ 1 \le i \le n). \tag{22}$$

Proof. Let us denote by

$$R^{(\ell)}_{n;k_1,k_2,\dots,k_\ell}(x) = \frac{\tilde{P}_n(x)}{\displaystyle\prod_{i=1}^{\ell} (x - x_{n,k_i})} \quad (1 \le k_1 < \ldots < k_\ell \le n,\ 1 \le \ell \le n,\ n \in \mathbb{N}).$$

According to (20), we can write

$$\gamma_{n,k} = x_{n,k} + \varepsilon_{n,k}, \quad \text{where} \quad |\varepsilon_{n,k}| < \varepsilon \quad (k = 1, \dots, s). \tag{23}$$

Then

$$Q_n(x) = \tilde{P}_n(x) + \sum_{m=1}^{s} (-1)^m \sum_{1\le i_1<\cdots<i_m\le s} \prod_{k=1}^{m} \varepsilon_{n,i_k}\ R^{(m)}_{n;i_1,i_2,\ldots,i_m}(x). \tag{24}$$

Hence

$$Q_n(x) = \tilde{P}_n(x) + R_{n-1}(x)\mathcal{O}(\varepsilon) \qquad (R_{n-1}\in\mathcal{P}), \tag{25}$$

wherefrom the conclusion follows. □.

Especially, let be

$$R_{n,k}(x) \equiv R^{(1)}_{n,k}(x) = \frac{\tilde{P}_n(x)}{x-x_{n,k}}, \quad \tau_{i,n,k} = \mathcal{L}\big[\tilde{P}_i\ R_{n,k}\big] \quad (1\le k\le n). \tag{26}$$

Because of orthogonality, we can write

$$0 = \mathcal{L}\big[\tilde{P}_i\ \tilde{P}_n\big] = \mathcal{L}\big[\tilde{P}_i\ (x-x_{n,k})R_{n,k}\big] = \mathcal{L}\big[x\tilde{P}_i\ R_{n,k}\big] - x_{n,k}\mathcal{L}\big[\tilde{P}_i\ R_{n,k}\big].$$

From three-term recurrence relation (13), we have

$$x\tilde{P}_i(x) = \sqrt{\beta_{i+1}}\tilde{P}_{i+1}(x) + \alpha_i\tilde{P}_i(x) + \sqrt{\beta_i}\tilde{P}_{i-1}(x). \tag{27}$$

Hence

$$\sqrt{\beta_{i+1}}\ \tau_{i+1,n,k} = (x_{n,k}-\alpha_i)\ \tau_{i,n,k} - \sqrt{\beta_i}\ \tau_{i-1,n,k} \quad (0\le i<n; 1\le k\le n). \tag{28}$$

Lemma 2.4. *Let*

$$h = \min_{0\le i\le n}\sqrt{\beta_i}, \quad R = \max_{0\le i\le n}\sqrt{\beta_i}, \quad C = \max_{\substack{0\le i\le n\\ 1\le k\le n}} |x_{n,k}-\alpha_i|. \tag{29}$$

Then

$$|\tau_{i,n,k}| \le |\tau_{0,n,k}|\Big(\frac{C}{h}\Big)^i \sum_{j=0}^{[i/2]} \binom{i-j}{j}\Big(\frac{Rh}{C^2}\Big)^j. \tag{30}$$

Proof. By mathematical induction.□

By using the form (24) of the polynomial $Q_n(x)$, we can write

$$f_i = \mathcal{L}\big[\tilde{P}_i\ Q_n\big] = \mathcal{L}\big[\tilde{P}_i\ \tilde{P}_n(x)\big] - \sum_{k=1}^{s}\varepsilon_{n,k}\mathcal{L}\big[\tilde{P}_i\ R_{n,k}\big]$$
$$+ \sum_{\substack{k_1,k_2=1\\ k_1<k_2}}^{s} \varepsilon_{n,k_1}\varepsilon_{n,k_2}\mathcal{L}\big[\tilde{P}_i\ R^{(2)}_{n;k_1,k_2}\big] \ + \ \cdots + (-1)^s\mathcal{L}\big[\tilde{P}_i\ R^{(s)}_{n;1,2,\ldots,s}\big]\prod_{i=1}^{s}\varepsilon_{n,i}\ .$$

Hence

$$f_i = -\sum_{k=1}^{s}\varepsilon_{n,k}\ \tau_{i,n,k} + \mathcal{O}(\varepsilon^2) \quad (1\le i\le n-1), \qquad f_n = 1.$$

According to (23) and Lemma 2.4, the following estimate is valid:

$$|f_i| \le \varepsilon s |\tau_{i,n,k}| + \mathcal{O}(\varepsilon^2) \ .$$

So, we can say that

$$\omega_i \le s|\tau_{0,n,k}|\Big(\frac{C}{h}\Big)^i \sum_{j=0}^{[i/2]} \binom{i-j}{j} \Big(\frac{Rh}{C^2}\Big)^j + \mathcal{O}(\varepsilon^2) \qquad (i=0,1,\dots,n-1) \ .$$

Notice that

$$Q_n(x) = \sum_{i=0}^{n} f_i \tilde{P}_i(x).$$

Theorem 2.5. *Under the condition (20), the real numbers* $\gamma_{n,s+1},\dots,\gamma_{n,n}$ *exist such that*

$$\gamma_{n,k} = x_{n,k} + \mathcal{O}(\varepsilon) \qquad (k = s+1,\dots,n),$$

and the polynomial

$$P_n^{(\varepsilon)}(x) = \sigma_n \prod_{k=1}^{n} (x - \gamma_{n,k})$$

is quasi almost orthogonal with respect to $\{P_k(x)\}_{k=0}^n$, *i.e.*

$$\mathcal{L}\big[\tilde{P}_k P_n^{(\varepsilon)}\big] = \begin{cases} 0\ , & 0 \le k \le n-s-1\ , \\ \mathcal{O}(\varepsilon)\ , & n-s \le k \le n-1\ , \\ 1\ , & k = n\ . \end{cases}$$

Proof. Using the same notation like in the previous lemmas, we can define

$$T_n(x) = Q_n(x) + \varepsilon\left\{-\sum_{i=0}^{n-s-1} \omega_i \tilde{P}_i(x) + \sum_{i=n-s}^{n-1} \mu_i \tilde{P}_i(x)\right\}, \tag{31}$$

where constants μ_i $(i = n-s,\dots,n-1)$ will be determined. Also, it can be written in the form

$$T_n(x) = \tilde{P}_n(x) + \varepsilon \sum_{i=n-s}^{n-1} (\omega_i + \mu_i)\tilde{P}_i(x). \tag{32}$$

Then we find

$$g_j = \mathcal{L}\big[\tilde{P}_j\ T_n\big] = \begin{cases} 0\ , & 0 \le j \le n-s-1\ , \\ \varepsilon(\omega_j + \mu_j)\ , & n-s \le j \le n-1\ , \\ 1\ , & j = n\ . \end{cases} \tag{33}$$

If $\{t_{n,k}\}$ are the zeros of $T_n(x)$, we can write

$$T_n(x) = \sigma_n \prod_{k=1}^{n} (x - t_{n,k}). \tag{34}$$

By applying Lemma 2.2 onto (32), for $k = 1, \dots, s$, we can write

$$t_{n,k} = \gamma_{n,k} + \varepsilon \left\{ \sum_{i=0}^{n-s-1} \omega_i \, \frac{\tilde{P}_i(\gamma_{n,k})}{Q'_n(\gamma_{n,k})} - \sum_{i=n-s}^{n-1} \mu_i \frac{\tilde{P}_i(\gamma_{n,k})}{Q'_n(\gamma_{n,k})} \right\} + \mathcal{O}(\varepsilon^2). \tag{35}$$

It can be written in the matrix form

$$A_s(\varepsilon)\vec{\mu} = \vec{b}(\varepsilon), \tag{36}$$

where

$$A_s(\varepsilon) = \begin{bmatrix} \tilde{P}_{n-s}(\gamma_{n,1}) & \cdots & \tilde{P}_{n-1}(\gamma_{n,1}) \\ \vdots & & \\ \tilde{P}_{n-s}(\gamma_{n,s}) & & \tilde{P}_{n-1}(\gamma_{n,s}) \end{bmatrix}, \ \vec{\mu} = \begin{bmatrix} \mu_{n-s} \\ \vdots \\ \mu_{n-1} \end{bmatrix}, \ \vec{b}(\varepsilon) = \begin{bmatrix} b_1(\varepsilon) \\ \vdots \\ b_s(\varepsilon) \end{bmatrix}, \tag{37}$$

with

$$b_k(\varepsilon) = Q'_n(\gamma_{n,k}) \frac{\gamma_{n,k} - t_{n,k}}{\varepsilon} + \sum_{j=0}^{n-s-1} \omega_j \; \tilde{P}_j(\gamma_{n,k}) + \mathcal{O}(\varepsilon) \qquad (k = 1, \dots, s). \tag{38}$$

Let us consider the system

$$A_s(\varepsilon)\vec{\mu} = \vec{b}'(\varepsilon), \quad \text{where} \quad b'_k(\varepsilon) = \sum_{j=0}^{n-s-1} \omega_j \tilde{P}_j(\gamma_{n,k}) + \mathcal{O}(\varepsilon). \tag{39}$$

According to Lemma 2.1, all leading principal minors of the matrix A, defined by (15), are nonsingular. Hence, for sufficiently small ε, the matrix $A_s(\varepsilon)$ is nonsingular too. Therefore exists the solution

$$\vec{\mu} = A_s^{-1}(\varepsilon) \; \vec{b}'(\varepsilon).$$

of the system (39). In that case, it is valid $t_{n,k} = \gamma_{n,k}$ $(k = 1, \dots, s)$.

Taking $\gamma_{n,k} = t_{n,k}$ $(k = s+1, \dots, n)$, we have

$$T_n(x) = \sigma_n \prod_{k=1}^{n} (x - \gamma_{n,k}), \tag{40}$$

and

$$\gamma_{n,k} = x_{n,k} + \mathcal{O}(\varepsilon) \qquad (k = 1, 2, \dots n).$$

Choosing $P_n^{(\varepsilon)}(x) = T_n(x)$, we prove its existence. $\square$

Remark 2.1. Because of its quasi orthogonality, the sequence $\{P_n^{(\varepsilon)}(x)\}$ satisfies $(s+2)$-term recurrence relation of the form

$$x\ P_n^{(\varepsilon)}(x) = \sum_{k=n-s}^{n+1} d_{n,k}\ P_k^{(\varepsilon)}(x). \tag{41}$$

Remark 2.2. Writing $P_n^{(\varepsilon)}(x)$ in the form

$$P_n^{(\varepsilon)}(x) = \sigma_n \prod_{k=1}^{s}(x-\gamma_{n,k})\ V_{n-s}(x), \quad \text{where} \quad V_{n-s}(x) = x^{n-s} + \sum_{i=0}^{n-s-1} v_{n-s,i}x^i,$$

we can evaluate numerically $v_{n-s,i}$ $(0 \le i \le n-s-1)$ from linear algebraic system obtained from the fact

$$\mathcal{L}\big[P_j\ P_n^{(\varepsilon)}\big] = 0 \quad (0 \le j \le n-s-1).$$

2.1 Examples

In the examples we will take upper limit ε and choose $\varepsilon_{n,k}$ by the function `Random` from package *Mathematica* in the interval $(-\varepsilon, \varepsilon)$. We will repeat the whole procedure 20 times. Let us consider $P_4(x)$ and $P_4^{(\varepsilon)}(x)$ provided by $s=2$. In the tables, the notation $a(n)$ means $a \cdot 10^n$. In the first column is ε. In the second column is the maximal distance between the zeros of orthogonal and almost orthogonal polynomials of the same degree. In the third column, it is given maximal absolute value of inner products $\mathcal{L}\big[P_j\ P_n^{(\varepsilon)}\big]$ $(j = 0, 1, \ldots, n-1)$, some kind of almost orthogonality between the members.

Example 1. Let us consider Legendre polynomials $\{P_n(x)\}$ which are orthonormal with respect to the functional

$$\mathcal{L}[f \cdot g] = \int_{-1}^{1} f(x)g(x)\,dx\ .$$

Weak	orthogonality
ε	inner products
0.1(-1)	0.377056 (-1)
0.1(-2)	0.446765 (-2)
0.1(-3)	0.158524 (-2)
0.1(-4)	0.191905 (-3)

Quasi	almost	orthogonality
ε	zero distance	inner products
0.1(-1)	0.500665(-1)	0.186743(0)
0.1(-2)	0.584278(-2)	0.181221(-1)
0.1(-3)	0.493928(-3)	0.158524(-2)
0.1(-4)	0.602947(-4)	0.191905(-3)

Example 2. The Laguerre polynomials $\{L_n(x)\}$ are orthonormal with respect to the functional

$$\mathcal{L}[f \cdot g] = \int_{0}^{+\infty} f(x)g(x)\mathrm{e}^{-x}\,dx\ .$$

Weak	orthogonality
ε	inner products
0.1(-1)	0.166269(-1)
0.1(-2)	0.145173(-2)
0.1(-3)	0.174978(-3)
0.1(-4)	0.177084(-4)

Quasi	almost	orthogonality
ε	zero distance	inner products
0.1(-1)	0.231992(1)	0.647862(0)
0.1(-2)	0.167538(0)	0.499177(-1)
0.1(-3)	0.178939(-1)	0.530434(-2)
0.1(-4)	0.167632(-2)	0.497250(-3)

We can notice from the tables that insisting to have quasi orthogonality included, has the consequence weakness of almost orthogonality of the members with high degrees, i.e. increasing of the values of inner products $\mathcal{L}\big[P_j\ P_n^{(\varepsilon)}\big]$ for high j's.

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