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GENERALIZED BORWEIN CONJECTURE AND PARTITIONS OF NATURAL NUMBERS

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Abstract

In this paper, we have considered a generalization of the conjecture which was set up by P. Borwein [1]. We have proved some properties of generalized Borwein polynomials. An infinite versions of ordinary and generalized Borwein conjecture are also considered. We gave an explicit formula for the coefficients of infinite Borwein series and proved that all sequences of coefficients are particulary solutions of the same inhomogeneous difference equation. Finally, we found the one sequence of counterexamples for the considered generalization and gave modified conjecture for that sequence.

1 Introduction

Expansion of a polynomial over a set of positive polynomials became very interesting problem in mathematical considerations in the second half of twentieth century. A lot of such examples are noticed in theory of partitions. Between the others, we emphasize the results of G.E. Andrews [1, 2], D.M. Bressoud [3] and S.O. Warnaar [6, 7].

One of the notable examples is the famous Borwein conjecture. Denote by

$$J(n;q) = \prod_{i=0}^{n-1} (1-q^{3i+1})(1-q^{3i+2}). \tag{1.1}$$

Conjecture 1.1. (**P. Borwein**) *The polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ defined by*

$$J(n;q) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3), \tag{1.2}$$

have nonnegative coefficients.

Somewhere this conjecture is denoted like the property $+ \; - \; -$. We already made some considerations about this problem in our paper [5].

Now, our purpose is to signify on some modes in its generalization.

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2 A generalization of Borwein conjecture

Let us denote by

$$J_M(n;q) = \prod_{i=1}^{M-1} (q^i;q^M)_n = \prod_{i=0}^{n-1} (1-q^{Mi+1})(1-q^{Mi+2})\cdots(1-q^{Mi+M-1}).$$

where $M \geq 2$, $M \in \mathbb{N}$. Rewrite $J_M(n;q)$ in the following form:

$$J_M(n;q) = A_{n,0}^M(q^M) - qA_{n,1}^M(q^M) - q^2 A_{n,2}^M(q^M) - \ldots - q^{M-1}A_{n,M-1}^M(q^M)$$

Note that

$$\deg J_M(n;q) = \frac{n^2 M(M-1)}{2}\ , \quad \deg A_{n,0}^M(q) = \frac{M(M-1)(2n-1)}{2},$$

$$\deg J_M(n;q) = \frac{n^2 M(M-1)}{2}\ .$$

Theorem 2.1. *Let* $\vec{A}_n^M = [A_{n,0}^M(q),\ A_{n,1}^M(q),\ \ldots,\ A_{M-1,n}^M(q)]$. *There exists a polynomial matrix* $F_M(n,q)$ *such that*

$$\vec{A}_n^M = F_M(n,q)\vec{A}_{n-1}^M \qquad (n \in \mathbb{N}),$$

with initial value $\vec{A}_0^M(q) = [1,0,\ldots,0]$.

Proof. Denote by $d = M(M-1)(2n-1)/2$. We start from:

$$\begin{aligned} J_M(n,q) &= J_M(n-1,q)\prod_{i=1}^{M-1}(1-q^{M(n-1)+i}) = J_M(n-1,q)\left(\sum_{i=0}^{d} p_{n,i}^M q^i\right) \\ &= \left(A_{n-1,0}^M(q^M) - qA_{n-1,1}^M(q^M)\ldots - q^{M-1}A_{n-1,M-1}^M(q^M)\right)\left(\sum_{i=0}^{d} p_{n,i}^M q^i\right) \\ &= A_{n,0}^M(q) - qA_{n,1}^M(q^M) - \ldots - q^{M-1}A_{n,M-1}^M(q^M). \end{aligned}$$

By expanding the previous product and collecting factors with the same modulo of M, we establish the following formula:

$$A_{n,k}^M(q) = A_{n-1,0}^M(q)\sum_{j=0}^{[d-k/M]} p_{n,jM+k}^M q^j - \sum_{i=1}^{M-1} A_{n-1,i}^M(q)\sum_{j=1}^{[\frac{d-k+i}{M}]} p_{n,jM+k-i}^M q^j$$

Last equation has a form: $A_{n,k}^M(q) = \sum_{i=0}^{M-1} F_M(n,q)_{ki}A_{n-1,i}^M$, which completes the proof. □

Example 2.1. For $M = 2$ we have:

$$F_2(n,q) = \begin{bmatrix} 1 & q^n \\ q^{n-1} & 1 \end{bmatrix},$$

with the starting values $A^2_{0,0}(q) = 1$ and $A^2_{0,1}(q) = 0$. From the recurrence formula given in Theorem 2.1, we can easily conclude that all $A^M_{n,i}(q)$ $(i = 0, 1)$ have positive coefficients. Again, we consider the function

$$J_2(n;q) = \prod_{i=0}^{n-1} (1 - q^{2i+1}) \ .$$

Denote by Π^o_n (Π^e_n) the number of partitions where all parts are different, odd (even), and less than $2n+1$. Clearly, $J_2(n;q)$ is generating function of the sequence $\Pi^e_n - O^o_n$. From the positivity of coefficients of polynomials $A^2_{n,0}(q)$ and $A^2_{n,1}(q)$ we can conclude that if k is even, such difference is positive and otherwise it is negative.

In [2], it is proven that polynomials $A_n(q) = A^3_{n,0}(q)$ are reciprocal. This result can be easily extended to the general case:

Theorem 2.2. *The polynomial* $A^M_{1,n}(q)$ *is reciprocal itself and the polynomial* $A^M_{k+1,n}(q)$ *is reciprocal to the polynomial* $A^M_{M-k+1,n}(q)$*, i.e.,*

$$\begin{aligned} A^M_{1,n}(q) &= (-1)^{n(M-1)} q^{n^2(M-1)/2} A^M_{1,n}(1/q) \\ A^M_{k+1,n}(q) &= (-1)^{n(M-1)} q^{n^2(M-1)/2} A^M_{M-k+1,n}(1/q) \\ &(k = 1, 2, \ldots, M-1). \end{aligned}$$

For $M = 3$, we have that $J_3(n;q) = J(n;q)$ and polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ are equal to $A^3_{n,0}(q)$,$A^3_{n,1}(q)$ and $A^3_{n,2}(q)$ respectively. In [2], it is conjectured that all polynomials $A^5_{n,i}(q)$ have positive coefficients. Unfortunately, Conjecture 1.1 cannot be directly generalized in the case $M > 3$, since $A^4_{1,2}(q) = 1 - q$ and $A^7_{1,3}(q) = 1 + q - q^2$. Even more, for the polynomials $A^M_{n,0}(q)$, the generalization does not hold for all $M \in \mathbb{N}$ ($M = 14$ is counterexample).

Conjecture 2.1. *For every* $M \in \{2, 3, \ldots, 13\}$*, the polynomials* $A^M_{n,0}$ $(n \in \mathbb{N})$ *have all positive coefficients.*

Also, it seems that last conjecture is the best possible generalization in that direction.

Conjecture 2.2. *For every* $M > 13$ *and* $n \geq 3$*, polynomial* $A^M_{n,0}(q)$ *has at least one negative coefficient.*

3 Infinite products

Denote by

$$J(q) = (q;q^3)_\infty \ (q^2;q^3)_\infty = \prod_{i=0}^{\infty}(1-q^{3i+1})(1-q^{3i+2})$$

the infinite extension of the $J(n;q)$. It was proven (see [2]) that the decomposition of this product into the form:

$$J(q) = A(q) - qB(q) - q^2C(q),$$

where

$$A(q) = \sum_{n=0}^{+\infty} a_n q^n, \quad B(q) = \sum_{n=0}^{+\infty} b_n q^n, \quad C(q) = \sum_{n=0}^{+\infty} c_n q^n,$$

has only nonnegative coefficients.

Furthermore, we can conjecture the following fact.

Conjecture 3.1. *The coefficients in the infinite series $A(q)$, $B(q)$ and $C(q)$ are growing functions of n, i.e.,*

$$a_{n+1} > a_n, \quad b_{n+1} > b_n, \quad c_{n+1} > c_n \qquad (n = 0, 1, \ldots).$$

Let us notice that the first n coefficients of $A_n(q)$, $B_n(q)$ and $C_n(q)$ are equal to the first n coefficients of $A(q)$, $B(q)$ and $C(q)$ respectively.

Denote by $\pi(m) = m(3m-1)/2$ and $S_n = \{m \in \mathbb{Z} \ : \ \pi(m) \le n\}$. In further consideration we use the following identity known as Euler's pentagonal number theorem:

$$\prod_{i=0}^{\infty}(1-q^i) = 1 + \sum_{m=1}^{\infty}\left(q^{\pi(m)} + q^{\pi(-m)}\right) = \sum_{m=-\infty}^{+\infty} q^{\pi(m)} \ .$$

Next theorem gives us an explicit formulas for coefficients a_n, b_n and c_n:

Theorem 3.1. *Denote by $p(n)$ the number of partitions of n for $n > 0$, $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Then it holds*

$$\begin{aligned}
a_n &= \sum_{m=0}^{n}(-1)^m p\left(n - \frac{m(9m-1)}{2}\right) + \sum_{m=1}^{n}(-1)^m p\left(n - \frac{m(9m+1)}{2}\right), \\
b_n &= \sum_{m=0}^{n}(-1)^{m+1} p\left(n - \frac{m(9m+5)}{2}\right) + \sum_{m=1}^{n}(-1)^{m+1} p\left(n - \frac{m(9m-5)}{2}\right), \\
c_n &= \sum_{m=0}^{n}(-1)^{m+1} p\left(n - \frac{m(9m-7)}{2}\right) + \sum_{m=1}^{n}(-1)^{m+1} p\left(n - \frac{m(9m+7)}{2}\right).
\end{aligned}$$

Proof. Let us write $J(q)$ in the form $J(q) = \sum_{n=0}^{\infty} j_n q^n$. Then $a_n = j_{3n}$. Starting from the relation

$$\prod_{i=0}^{\infty}(1-q^i) = \prod_{i=0}^{\infty}(1-q^{3i})\prod_{i=0}^{\infty}(1-q^{3i+1})\prod_{i=0}^{\infty}(1-q^{3i+2}) = \prod_{i=0}^{\infty}(1-q^{3i})J(q), \quad (3.1)$$

and using the Euler's pentagonal number theorem, we can establish

$$J(q) = \frac{\prod_{i=0}^{\infty}(1-q^i)}{\prod_{i=0}^{\infty}(1-q^{3i})} = \frac{1}{\prod_{i=0}^{\infty}(1-q^{3i})}\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{\pi(m)}\right). \quad (3.2)$$

It is known from the theory of partitions of the natural numbers that generating function of the sequence $\{p(n)\}_{n\in\mathbb{N}_0}$ is:

$$\frac{1}{\prod_{i=0}^{\infty}(1-x^i)} = \sum_{i=0}^{\infty} p(i)x^i.$$

Exchanging x with q^3 and replacing into (3.2), we obtain:

$$J(q) = \left(\sum_{k=0}^{\infty} p(k)q^{3k}\right)\left(-1 + \sum_{m=0}^{\infty}(-1)^m\left(q^{\pi(m)} + q^{\pi(-m)}\right)\right) \quad (3.3)$$

Let us notice that $3|\frac{m(3m\pm1)}{2}$ if and only if $3|m$. Hence we can conclude that $a_n = j_{3n}$ is the sum of all $(-1)^{m_1}p(k)$ such that $3n = 3k + \frac{m(3m\pm1)}{2}$ and $m = 3m_1$. It is equivalent to $k = n - \frac{m_1(9m_1\pm1)}{2}$. Finally, we have:

$$a_n = \sum_{m=0}^{n}(-1)^m p\left(n - \frac{m(9m-1)}{2}\right) + \sum_{m=1}^{n}(-1)^m p\left(n - \frac{m(9m+1)}{2}\right) \quad (3.4)$$

Notice that the second sum starts from $m = 1$ to avoid duplicating of the addend $p(n)$. In the sums, the upper bound of index m should be $\lfloor\frac{1+\sqrt{1+72n}}{18}\rfloor$ and $\lfloor\frac{-1+\sqrt{1+72n}}{18}\rfloor$ respectively, but for the sake of simplicity they are extended to n. The function $p(n)$ vanishes for negative argument. The expressions for b_n and c_n can be similarly proven. □

The numbers $\{m(9m\pm1)/2,\ m\in\mathbb{N}\}$ satisfy the following relation

$$\frac{m(9m-1)}{2} < \frac{m(9m+1)}{2} < \frac{(m+1)(9(m+1)-1)}{2}.$$

The first few members of this sequence are $0, 4, 5, 17, 19, 39, 42, 70, \ldots$.

The relation (3.4) can be interpreted in the following way:

$$a_n = p(n) - p(n-4) - p(n-5) + p(n-17) + p(n-19) - p(n-39) - p(n-42) + \ldots.$$

Previous expansion is very similar to the famous Euler relation for $p(n)$:

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \ldots = 0, \quad (3.5)$$

where the sum goes over all generalized pentagonal numbers $m(3m \pm 1)/2$. Also, it is very interesting that the sequence

$$p(n) - p(n-4) - p(n-5) + p(n-17)$$

is growing for $n \leq 69$ and for $n \geq 77$ it is negative. This property holds if we take a few more members from (3.4).

Now, let us consider the sequence:

$$\alpha_n = \sum_{m=0}^{n} (-1)^m P\left(n - \frac{m(9m-1)}{2}\right) + \sum_{m=1}^{n} (-1)^m P\left(n - \frac{m(9m+1)}{2}\right),$$

where

$$P(n) = \begin{cases} \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}, & n > 0, \\ 1, & n = 0, \\ 0, & n < 0, \end{cases}$$

is an asymptotic formula for $p(n)$. Numerical computation suggest that the following conjecture seems to be true:

Conjecture 3.2. *For every $n \in \mathbb{N}$, it holds*

$$\alpha_{n+1} > \alpha_n > 0, \qquad \lim_{n \to +\infty} \frac{\alpha_n}{a_n} = 1,$$

i.e., α_n is an asymptotic formula for a_n.

In the same way we can define sequences β_n and γ_n (the corresponding to b_n and c_n) and similar conjectures can be made.

4 Generalized infinite product

We extend the generalized Borwein conjecture to infinite series. Let us define

$$J_M(q) = \prod_{i=0}^{+\infty} \prod_{j=1}^{M-1} \left(1 - q^{Mi+j}\right).$$

Let $A_i^M(q)$ $(i = 0, \ldots M-1)$ be power series such that

$$J_M(q) = A_0^M(q^M) - qA_1^M(q^M) - \ldots - q^{M-1} A_{M-1}^M(q^M).$$

Denote by $A_j^M(q) = \sum_{k=0}^{+\infty} a_{k,j}^M q^k \quad (j = 0, 1, \ldots, M-1)$ and $J_M(q) = \sum_{n=0}^{+\infty} j_n^M q^n$. Now we construct the recurrence relation for the sequence $\{j_n^M\}_{n \in \mathbb{N}}$. Like in the previous section, we consider the product

$$\prod_{i=0}^{+\infty} \left(1 - q^i\right) = J_M(q) \cdot \prod_{i=0}^{+\infty} \left(1 - q^{Mi}\right).$$

By using Euler's pentagonal number theorem twice, we obtain

$$\sum_{m=-\infty}^{+\infty} (-1)^m q^{\pi(m)} = \left(\sum_{n=0}^{+\infty} j_n^M q^n\right)\left(\sum_{m=-\infty}^{+\infty} (-1)^m q^{M\pi(m)}\right).$$

By expanding the second product and collecting addends with the same degree of q, we establish the following recurrence relation

$$\sum_{M\pi(m)\le n} (-1)^m j^M_{n-M\pi(m)} = f(n) = \begin{cases} (-1)^l, & n = \pi(l),\ l \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

Furthermore, by replacing $a^M_{n,0} = j^M_{nM}$ and $a^M_{n,l} = -j^M_{nM+l}$ for $0 < l < M$, the equation (4.1) becomes the recurrence relation for the sequence $\{a^M_{n,l}\}_{n\in\mathbb{N}}$:

$$\sum_{m\in S_n} (-1)^m a^M_{n-\pi(m),l} = f^M_l(n) = \begin{cases} -f(Mn+l), & l > 0 \\ f(Mn), & l = 0 \end{cases} \tag{4.2}$$

Hence, we proved the following theorem:

Theorem 4.1. *The sequences* $\{a^M_{n,l}\}_{n\in\mathbb{N}}$ $(0 \le l < M)$ *are solutions of the following inhomogeneous difference equation:*

$$\sum_{m\in S_n} (-1)^m x\left(n - \pi(m)\right) = g(n) \quad \forall (n \in \mathbb{N}_0), \tag{4.3}$$

for the right-hand side equal to $g(n) = f^M_l(n)$.

If $g(n) = 0$ (homogeneous case), solution of (4.3) with starting condition $x(0) = 1$ is partition function $p(n)$. Let us define function

$$g_i(n) = \begin{cases} 1, & n = i \\ 0, & n \ne i. \end{cases}$$

In the case $g(n) = g_0(n)$, the solution of (4.3) is $x(n) = p(n)$. It is obviously from the relation (3.5). The case $g(n) = g_i(n)$ is similar to the previous one and the corresponding solution is $x(n) = p(n-i)$ (let us remember that $p(z) = 0$ for $z < 0$). Now, let us consider an arbitrary integer function $g(n)$ written in the form $g(n) = \sum_{i=0}^{+\infty} g_i(n)g(i)$. Then, the solution of inhomogeneous equation (4.1) with the starting condition $x(0) = g(0)$ is given by

$$x(n) = \sum_{i=0}^{n} p(n-i)g(i). \tag{4.4}$$

The previous relation holds directly from the linearity of equation (4.3). It also implies that, if we have two functions $g'(n)$ and $g''(n)$, such that $g'(n) \le g''(n)$, then $x'(n) \le x''(n)$, where $x'(n)$ and $x''(n)$ are corresponding solutions of (4.3).

Example 4.1. In the case $g(n) = f_l^3(n)$ $(l = 0, 1, 2)$, we can derive expressions for a_n, b_n and c_n.

Remark 4.1. For the computation of sequence $\{a_{n,l}^M\}_{n\in\mathbb{N}}$, the recurrence relation (4.4), written in the form

$$a_{n,l}^M = -\sum_{m\in S_n\setminus\{0\}} (-1)^m a_{n-\pi(m)}^M + f_l^M(n),$$

establishes an $\mathcal{O}(n^{3/2})$–algorithm (i.e., it requires working time proportional to $n^{3/2}$).

Our computer evaluation shows that the following conjecture seems to be true be true:

Conjecture 4.1. *All coefficients in the series $A_l^M(q)$ for $2 \le M \le 13$ and $0 \le l < M$ are positive. Also holds $a_{k+1,l}^M \ge a_{k,l}^M$ for all $k \in \mathbb{N}$.*

Moreover, increasing M we noticed that the following conjecture seems to be true:

Conjecture 4.2. *All coefficients in the series $A_0^M(q)$ are positive for all $2 \le M \le 34$. Further more, it holds: $a_{k+1,0}^M \ge a_{k,0}^M$.*

Example 4.2. In the case $M = 35$, which is a pentagonal number, we have that $f_0^{35}(1) = f_0^{35}(2) = -1$. So, the first few members of the sequence $\{a_{n,0}^{35}\}_{n\in\mathbb{N}}$ are $(1, 0, 0, 0, -1, 0, -2, \ldots)$, and after that all other members are negative.

Also, our computer evaluation shows that the coefficients of the series $A_0^M(q)$ are positive for all positive integers M except $M = 35, 115, 260, 330, 518, 658, 805, 910, 969, \ldots$.

Theorem 4.2. *The infinitely many numbers $M \in \mathbb{N}$ exist such that the first six elements of the sequence $\{a_{n,0}^M\}_{n\in\mathbb{N}_0}$ are respectively equal to $1, 0, 0, 0, 0, -1$.*

Proof. The statement of the theorem is fulfilled if and only if the first six values of sequence $\{f_0^M(n)\}_{n\in\mathbb{N}_0}$ are respectively $1, -1, -1, 0, 0, 0$. This is satisfied if numbers M and $2M$ are pentagonal, but $3M$, $4M$ and $5M$ are not. Hence there should exist odd numbers: $m', m'' \in \mathbb{Z}$ such that $\pi(m') = M$ and $\pi(m'') = 2M$. Obviously:

$$m' = \frac{1 \pm \sqrt{1+24M}}{6}, \quad m'' = \frac{1 \pm \sqrt{1+48M}}{6}\ . \tag{4.5}$$

Hence there should exist numbers $x, y \in \mathbb{Z}$ such that $1+24M = y^2$ and $1+48M = x^2$. Eliminating M, we obtain Pell's equation $x^2 - 2y^2 = -1$, which has infinitely many solutions of the form:

$$x_k = \frac{(1+\sqrt{2})^{2k+1} + (1-\sqrt{2})^{2k+1}}{2}, \quad y_k = \frac{(1+\sqrt{2})^{2k+1} - (1-\sqrt{2})^{2k+1}}{2\sqrt{2}}$$

where $k \in \mathbb{N}_0$ is arbitrary number. Numbers x_k and y_k are both solutions of following linear difference equation:

$$e_{n+2} - 6e_{n+1} + e_n = 0,$$

with the different starting values. By induction, from that recurrence relation it can be easily shown that $x_k \equiv_3 1$ for $k \equiv_4 0, 1$ and $x_k \equiv_3 1$ for $k \equiv_4 2, 3$. So, we conclude that $3|x_k^2 - 1$. By similar consideration, it holds $16|x_k^2 - 1$ and $M_k = \frac{x_k^2-1}{48} \in \mathbb{N}$, for every k.

If $k \equiv_4 2, 3$, then if we choose in (4.5) sign $+$ for m' and sign $-$ for m'', both m' and m'' are even integers. For $k \equiv_4 0, 1$, we will choose opposite signs, again to obtain $m', m'' \in \mathbb{Z}$. In that case, it is not hard to prove that m' and m'' are both odd, so $f_0^{M_k}(1) = -1$ and $f_0^{M_k}(2) = -1$. If $k \equiv_4 2, 3$, then $f_0^{M_k}(1) = 1$ and $f_0^{M_k}(2) = 1$. Hence we only need to ensure that numbers $3M_k$, $4M_k$ and $5M_k$ are not pentagonal, or equivalently, that $1 + 72M_k$, $1 + 96M_k$ and $1 + 120M_k$ are not squares. That's why we consider the reminders of $1 + 72M_k$ to modulo 5. For $k \equiv_3 0$ we have that $1 + 72M_k \equiv_5 3$ and $1 + 96M_k \equiv_5 2$. None of them is a quadratic reminder. Similarly, considering the reminders to modulo 13, we have that for $k \equiv_7 2$ holds $1 + 120M_k \equiv_{13} 2$. Also, they are not the quadratic reminders. Finally, if k satisfies the following three congruences:

$$((k \equiv_4 2) \vee (k \equiv_4 3)) \wedge (k \equiv_3 0) \wedge (k \equiv_7 2), \tag{4.6}$$

then the numbers M_k, $2M_k$ are pentagonal, but $3M_k$, $4M_k$ and $5M_k$ are not. So, the condition of the theorem is satisfied. From the Chinese reminder theorem [8], there exist infinitely many numbers k such that (4.6) holds. □

Our numerical evaluation also shows that the theorem still holds if the second and third congruence in (4.6) are dropped.

Corollary 4.1. *There exists infinitely many $M \in \mathbb{N}$ such that the coefficients of the series $A_0^M(q)$ are not all positive.*

Conjecture 4.3. *Let the first six elements of the sequence $\{a_{n,0}^M\}_{n\in\mathbb{N}_0}$ are respectively equal to $1, 0, 0, 0, 0, -1$. Then all other elements of the sequence are negative.*

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