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RIGHT INVERTIBILITY OF OPERATOR MATRICES

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Abstract

In this paper we consider the right invertibility of the operator matrix $M = \begin{bmatrix} A & C \\ T & S \end{bmatrix}$ on the Banach space $X \oplus Y$. Thus, some results of the paper (G. Hai and A. Chen, J. Operator Theory (to appear)) are generalized from Hilbert to Banach space settings.

1 Introduction

Let Z be a Banach space, such that $Z = X \oplus Y$ for some closed and complementary subspaces X and Y. This sum will be also denoted by $\begin{bmatrix} X \\ Y \end{bmatrix}$. If Z is a Hilbert space, then we always assume that X and Y are closed and mutually orthogonal subspaces of Z, so in this case $Z = X \oplus Y$ denotes the orthogonal sum.

Let $\mathcal{L}(X, Y)$ denote the set of all linear bounded operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X, X)$. The set of all finite rank operators from X to Y is denoted by $\mathcal{F}(X, Y)$. For $A \in \mathcal{L}(X, Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote the range and the null-space of A, respectively.

If W is a finite dimensional subspace of a Banach space, then $\dim W$ denotes the dimension of W. If W is infinite dimensional, then we simply write $\dim W = \infty$. However, if X is a Hilbert space and W is a closed subspace of X, then $\dim W$ is the orthogonal dimension of W.

If $Z = X \oplus Y$, then any $M \in \mathcal{L}(Z)$ can be decomposed as the following operator matrix

$$M = \begin{bmatrix} A & C \\ T & S \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix},$$

for some $A \in \mathcal{L}(X)$, $C \in \mathcal{L}(Y, X)$, $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$. On the other hand, any choice of A, C, T, S (linear and bounded operators on the corresponding subspaces), produces a linear and bounded operator M on the space Z.

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If A and C are fixed, then we use the notation $M_{(T,S)}$ to show that M depends on T and S. For given A and C, we are interested to find T and S, such that $M_{(T,S)}$ is right invertible. There are several papers that investigate invertibility of 2×2 operator matrices (see [1], [2], [3], [4]).

Among other things, for this purpose we need some properties of generalized inverses. Let $B \in \mathcal{L}(X,Y)$ be given. B is relatively regular (inner invertible) if there exists some $D \in \mathcal{L}(Y,X)$ such that $BDB = B$ holds. In this case D is an inner inverse of B. It is well-known that B is relatively regular, if and only if $\mathcal{R}(B)$ and $\mathcal{N}(B)$ are closed and complemented in Y and X, respectively. If $DBD = D$ holds and $D \neq 0$, then B is outer invertible, and D is an outer inverse of B. If $B \neq 0$, then it is a corollary of the Hahn-Banach theorem that there exists some non-zero outer inverse D of B. If D is both inner and outer inverse of B, then D is a reflexive inverse of B. Moreover, if D is an inner inverse of B, then DBD is a reflexive inverse of B.

If $D \in \mathcal{L}(Y,X)$ is a reflexive inverse of $B \in \mathcal{L}(X,Y)$, then BD is the projection from Y onto $\mathcal{R}(B)$ parallel to $\mathcal{N}(D)$, and DB is the projection from X onto $\mathcal{R}(D)$ parallel to $\mathcal{N}(B)$. On the other hand, if $X = U \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(B) \oplus V$ for closed subspaces: U of X and V of Y, then B have the matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} U \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ V \end{bmatrix},$$

and B_1 is invertible. It is easy to see that

$$D = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{N}(B) \end{bmatrix}$$

is the reflexive inverse of B satisfying $\mathcal{R}(B) = U$ and $\mathcal{N}(D) = V$.

This paper contains a generalization of one result from [2].

2 Results

In this section we investigate the right invertibility of the operator $M_{(T,S)}$. We use $\mathcal{G}_l(X,Y)$ and $\mathcal{G}_r(X,Y)$, respectively, to denote the set of all left and the set of all right invertible operators from $\mathcal{L}(X,Y)$. The abbreviations $\mathcal{G}_l(X)$ and $\mathcal{G}_r(X)$ are clear. Recall that $A \in \mathcal{G}_l(X,Y)$ if and only if $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A)$ is closed and complemented in Y. Also, $A \in \mathcal{G}_r(X)$ if and only if $\mathcal{N}(A)$ is complemented in X and $\mathcal{R}(A) = Y$. Notice that left or right invertible operators are always relatively regular. If A is left invertible and B is a left inverse of A, then B is a reflexive inverse of A. Similarly, if A is right invertible and B is a right inverse of A, then B is a reflexive inverse of A.

Two Hilbert spaces, among other things, can be compared by their orthogonal dimensions. In the case of Banach spaces it seems that the existence of left invertible operators is a useful substitution.

Definition 2.1. *If X and Y are Banach spaces, then X can be embedded in Y, if there exists a left invertible operator $W \in \mathcal{L}(X,Y)$. The notation is $X \preceq Y$.*

If X and Y are Hilbert spaces, then $X \preceq Y$ if and only if $\dim X \leq \dim Y$. Now we prove the result concerning the right invertibility of $M_{(T,S)}$.

Theorem 2.1. *Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y,X)$ be given operators. Then the following statements are equivalent:*

(a) *There exist some $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y)$ such that $M_{(T,S)}$ is right invertible;*

(b) $[A \quad C] \in \mathcal{L}(X \oplus Y, Y)$ *is right invertible and* $Y \preceq \mathcal{N}([A \quad C])$.

Proof. (a) $\Longrightarrow$ (b): Suppose that $M_{(T,S)}$ is right invertible for some $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y)$. Then there exists a bounded linear operator

$$\begin{bmatrix} E & G \\ H & F \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}$$

such that

$$\begin{bmatrix} A & C \\ T & S \end{bmatrix} \begin{bmatrix} E & G \\ H & F \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix}.$$

It follows that $[A \quad C] \begin{bmatrix} E \\ H \end{bmatrix} = I_X$, so $[A \quad C]$ is right invertible. On the other hand, we have $[T \quad S] \begin{bmatrix} G \\ F \end{bmatrix} = I_Y$ and $[A \quad C] \begin{bmatrix} G \\ F \end{bmatrix} = 0$, so there exists a left invertible operator $\begin{bmatrix} G \\ F \end{bmatrix} : Y \to \begin{bmatrix} X \\ Y \end{bmatrix}$ such that $\mathcal{R}\left(\begin{bmatrix} G \\ F \end{bmatrix}\right) \subseteq \mathcal{N}([A \quad C])$. Hence $Y \preceq \mathcal{N}([A \quad C])$.

(b) $\Longrightarrow$ (a): Suppose that $[A \quad C] : \begin{bmatrix} X \\ Y \end{bmatrix} \to X$ is right invertible, and suppose that $Y \preceq \mathcal{N}([A \quad C])$ holds. Let $K = \begin{bmatrix} E \\ H \end{bmatrix} : X \to \begin{bmatrix} X \\ Y \end{bmatrix}$ be a bounded right inverse of $[A \quad C]$. Then $\begin{bmatrix} X \\ Y \end{bmatrix} = \mathcal{R}(K) \oplus (\mathcal{N}([A \quad C]))$ and

$$[A \quad C] \begin{bmatrix} E \\ H \end{bmatrix} = I_X. \tag{1}$$

Let $L : Y \to \begin{bmatrix} X \\ Y \end{bmatrix}$ be a left invertible operator such that $\mathcal{R}(L) \subset \mathcal{N}([A \quad C])$. Then $L = \begin{bmatrix} G \\ F \end{bmatrix} : Y \to \begin{bmatrix} X \\ Y \end{bmatrix}$. Since $\mathcal{R}(L) \subset \mathcal{N}([A \quad C])$, we get that

$$[A \quad C] \begin{bmatrix} G \\ F \end{bmatrix} = 0. \tag{2}$$

$\mathcal{N}([A \quad C])$ is complemented in $X \oplus Y$ and $\mathcal{R}(L)$ is complemented in $X \oplus Y$. From $\mathcal{R}(L) \subset \mathcal{N}([A \quad C])$ it follows that $\mathcal{R}(L)$ is complemented in $\mathcal{N}([A \quad C])$. It follows that there exists a closed subspace W such that $\mathcal{N}([A \quad C]) = \mathcal{R}(L) \oplus W$. Now we have $X \oplus Y = \mathcal{R}(K) \oplus \mathcal{N}([A \quad C]) = \mathcal{R}(K) \oplus W \oplus \mathcal{R}(L)$. There exists the bounded left inverse N of L, such that $\mathcal{N}(N) = \mathcal{R}(K) \oplus W$. Such N has the matrix form $N = [T \quad S] : \begin{bmatrix} X \\ Y \end{bmatrix} \to Y$. Then

$$[T \quad S] \begin{bmatrix} G \\ F \end{bmatrix} = I_Y. \tag{3}$$

From $\mathcal{R}(K) \subset \mathcal{N}(N)$ we have

$$[T \quad S] \begin{bmatrix} E \\ H \end{bmatrix} = 0. \tag{4}$$

Finally, from (1), (2), (3) and (4) it follows that

$$\begin{bmatrix} A & C \\ T & S \end{bmatrix} \begin{bmatrix} E & G \\ H & F \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix}.$$

Thus, the proof is completed. □

As a corollary, we obtain the following result for Hilbert space operators, which is proved in ([2], Theorem 1.1).

Corollary 2.1. *Let $Z = X \oplus Y$ be a Hilbert space, where X, Y are closed and mutually orthogonal. Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y, X)$ be given operators. Then the following statements are equivalent:*

(a) *There exist some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$ such that $M_{(T,S)}$ is right invertible;*

(b) $\mathcal{R}(A) + \mathcal{R}(C) = Y$ *and* $\dim Y \leq \mathcal{N}([A \quad C])$*;*

(c) $[A \quad C]$ *is right invertible operator and* $\dim Y \leq \mathcal{N}([A \quad C])$.

Notice that (b) is equivalent to (c) from the following reason: $[A \quad C]$ is right invertible if and only if $\mathcal{R}([A \quad C]) = Y$; on the other hand, it is easy to see that $\mathcal{R}([A \quad C]) = \mathcal{R}(A) + \mathcal{R}(C)$.

Finally, we mention that it seems more difficult to prove the analogous result considering the left invertibility of $M_{(T,S)}$ on the Banach space $X \oplus Y$.

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