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GENERALIZED A-WEYL'S THEOREM AND PERTURBATIONS

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Abstract

In this paper we study the stability of generalized a-Weyl's theorem under perturbations by finite rank and nilpotent operators. Among other results, we prove that if T is a bounded linear operator acting on a Banach space X satisfies generalized a-Weyl's theorem and F is a finite rank operator commuting with T, then $T+F$ satisfies generalized a-Weyl's theorem if and only if $E_a(T+F)\cap\sigma_a(T)\subset E_a(T)$. Moreover we prove that if T is a bounded linear operator acting on a Banach space satisfies generalized a-Weyl's theorem and N is a nilpotent operator commuting with T, then $T+N$ satisfies generalized a-Weyl's theorem if and only if $\sigma_{SBF_+^-}(T+N)=\sigma_{SBF_+^-}(T)$.

1 Introduction

Throughout this paper $L(X)$ denote the Banach algebra of all bounded linear operators acting on a Banach space X. For $T\in L(X)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T)=\dim N(T)$ and $\beta(T)=\text{codim}R(T)$. If the range $R(T)$ of T is closed and $\alpha(T)<\infty$ (resp. $\beta(T)<\infty$), then T is called an upper (resp. a lower) semi-Fredholm operator. In the sequel $SF_+(X)$ denotes the class of all upper semi-Fredholm operators. If $T\in L(X)$ is either an upper or a lower semi-Fredholm operator, then T is called a semi-Fredholm operator , and the index of T is defined by $\text{ind}(T)=\alpha(T)-\beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator T is called a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum of T is defined by $\sigma_W(T)=\{\lambda\in\mathbb{C}:T-\lambda I \text{ is not Weyl }\}$. For $T\in L(X)$, let $SF_+^-(X)=\{T\in SF_+(X):\text{ind}(T)\leq 0\}$. Then the Weyl

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essential approximate spectrum of T is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T-\lambda I \notin SF_+^-(X)\}$.

Let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Following Coburn [11], we say that Weyl's theorem holds for $T \in L(X)$ if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T-\lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, iso$A$ denotes the set of all isolated points of A and acc$A$ denotes the set of all points of accumulation of A.

According to Rakočević [20], an operator $T \in L(X)$ is said to satisfy a-Weyl's theorem if $\Delta_a(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T-\lambda I) < \infty\}$. It is known [20] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but not conversely.

For $T \in L(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi-B-Fredholm operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [6]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator, see [7]. An operator $T \in L(X)$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not a B-Weyl operator}\}$.

Recall that the *ascent* of an operator $T \in L(X)$ is defined by $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and the *descent* of T, is defined by $\delta(T) = \inf\{\mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf\emptyset = \infty$. An operator T is called Drazin invertible if it has finite ascent and descent. The Drazin spectrum of T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not Drazin invertible }\}$. An operator $T \in L(X)$ is called an upper semi-Browder if it is an upper semi-Fredholm of finite ascent, and is called Browder if it is a Fredholm of finite ascent and descent. The upper semi-Browder spectrum of T is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not upper semi-Browder }\}$ and the Browder spectrum of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not Browder }\}$.
Define also the set $LD(X)$ as follows : $LD(X) = \{T \in L(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : \ T-\lambda I \notin LD(X)\}$. An operator $T \in L(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T-\lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T-\lambda I) < \infty$. Let $\Pi_a(T)$ denotes the set of all left poles of T and $\Pi_a^0(T)$ denotes the set of all left poles of T of finite rank. From [3, Theorem 2.8], it follows that if $T \in L(X)$ is left Drazin invertible, then T is an upper semi-B-Fredholm operator of index less or equal than zero.

Let $\Pi(T)$ be the set of all poles of the resolvent of T and let $\Pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T)\} : \alpha(T-\lambda I) < \infty\}$. According to [15], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\ (a(T-\lambda I), \delta(T-\lambda I)) < \infty$. Moreover, if this is true then $a(T-\lambda I) = \delta(T-\lambda I)$. According also to [15], the space $R((T-\lambda I)^{a(T-\lambda I)+1})$ is closed for each $\lambda \in \Pi(T)$. Hence we have always

$\Pi(T) \subset \Pi_a(T)$ and $\Pi^0(T) \subset \Pi_a^0(T)$.

Following [3], we say that generalized a-Browder's theorem holds for T if $\Delta_a^g(T) = \Pi_a(T)$ and that a-Browder's theorem holds for T if $\Delta_a(T) = \Pi_a^0(T)$. It is shown [2, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem.

Let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$. We say that generalized Browder's theorem holds for T if $\Delta^g(T) = \Pi(T)$; where $\Pi(T)$ is the set of all poles of T and that Browder's theorem holds for T if $\Delta(T) = \Pi^0(T)$; where $\Pi^0(T)$ is the set of all poles of T of finite rank. It is proved in [2, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

Let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, $SBF_+^-(X) = \{T \in SBF_+(X) \ : \ \mathrm{ind}(T) \leq 0\}$. The upper B-Weyl spectrum of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SBF_+^-(X)\}$. Let $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. We say that T obeys generalized a-Weyl's theorem, if $\Delta_a^g(T) = E_a(T)$; where $E_a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that T obeys generalized Weyl's theorem if $\Delta^g(T) = E(T)$; where $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$ ([3, Definition 2.13]). Generalized a-Weyl's theorem has been studied in [3, 8]. In [3, Theorem 3.11], it is shown that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse is not true in general, and under the assumption $E_a(T) = \Pi_a(T)$, it is proved in [8, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem. It is also proved in [3, Theorem 3.7] that generalized a-Weyl's theorem implies generalized Weyl's which in turn implies from [3, Theorem 3.9] Weyl's theorem.

Definition 1.1. A bounded linear operator $T \in L(X)$ is called isoloid (resp. a-isoloid) if $\mathrm{iso}\sigma(T) = E(T)$ (resp. $\mathrm{iso}\sigma_a(T) = E_a(T)$). Moreover, if $\mathrm{iso}\sigma_a(T) = \Pi_a(T)$, then we will say that T is an a-polaroid operator.

We will say that $T \in L(X)$ has the single valued-extension property at λ_0, (SVEP for short) if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation: $(T - \lambda I)f(\lambda) = 0$, for all $\lambda \in U$ is the function $f = 0$. $T \in L(X)$ is said to have the SVEP if T has this property at every $\lambda \in \mathbb{C}$ (see [16]).

The aim of this paper is to study the stability of generalized a-Weyl's theorem under commuting nilpotent or finite rank perturbations. Thus, in the second section, we prove in Theorem 2.2 that if T is a bounded linear operator acting on a Banach space X satisfies generalized a-Weyl's theorem and F is a finite rank operator commuting with T, then $T + F$ satisfies generalized a-Weyl's theorem if and only if $E_a(T + F) \cap \sigma_a(T) \subset E_a(T)$. We obtain also similar results for a-Weyl's and Weyl's theorem in the case of compact perturbations. Moreover we prove also in Theorem 2.7 that if $T \in L(X)$ satisfies generalized Weyl's theorem and if $F \in L(X)$ is a finite rank operator commuting with T, then $T+F$ satisfies generalized Weyl's theorem if and only if $E(T + F) \cap \sigma(T) \subset E(T)$.

In the third section we consider in Theorem 3.2 an operator T satisfying generalized a-Weyl's theorem and a nilpotent operator N commuting with T, and we prove that $T+N$ satisfies generalized a-Weyl's theorem if and only if $\sigma_{SBF_+^-}(T+N)=\sigma_{SBF_+^-}(T)$. As a consequence, we show in Theorem 3.3 that if $T\in L(X)$ is an operator satisfying generalized a-Weyl's theorem, if $E_a(T)\subset iso\sigma(T)$ and if $N\in L(X)$ is a nilpotent operator commuting with T, then $T+N$ satisfies genralized a-Weyl's theorem. We conclude this paper by some open questions related to the ideas developed in this section.

2 Finite rank perturbations

The next theorem had been established in [3, Theorem 4.2] for Hilbert spaces operators. We show here that it holds also in the general case of Banach spaces.

For $T\in L(X)$, let $c'_n(T)=\dim\frac{N(T^{n+1})}{N(T^n)}$.

Theorem 2.1. *Let X be a Banach space and let $T\in L(X)$. Then*

$$\sigma_{LD}(T)=\bigcap_{F\in F(X),\ FT=TF}\sigma_{LD}(T+F)$$

where $F(X)$ denotes the ideal of finite rank operators in $L(X)$.

Proof. If $\lambda\notin\sigma_{LD}(T)$, then $\lambda\notin\sigma_{LD}(T+0)$. Since 0 is a finite rank operator, it follows that $\lambda\notin\bigcap\{\sigma_{LD}(T+F):F\in F(X),\ FT=TF\}$.

To show the opposite inclusion, let $\lambda\notin\bigcap\{\sigma_{LD}(T+F):F\in F(X),\ FT=TF\}$. Then there exists a finite rank operator F commuting with T such that $T+F-\lambda I$ is left Drazin invertible. So $T+F-\lambda I$ is an upper semi-B-Fredholm. From [6, Theorem 2.7], $T-\lambda I$ is also an upper semi B-Fredholm operator. In particular the two operators $T-\lambda I$ and $T-\lambda I+F$ are operators of topological uniform descent [6]. By [14, Theorem 5.8], for n large enough we have $c'_n(T-\lambda I)=c'_n(T-\lambda I+F)$. Since $T-\lambda I+F$ is left Drazin invertible, then for n large enough we have $c'_n(T-\lambda I+F)=0$. So for n large enough we have $c'_n(T-\lambda I)=0$ and $a(T-\lambda I)<\infty$. On the other hand, for n large enough $R(T-\lambda I)^n$ is closed and by [19, Lemma 12], $R(T-\lambda I)^{a(T-\lambda I)+1}$ is also closed. Hence $T-\lambda I$ is left Drazin invertible. □

From Theorem 2.1 we conclude that if $T\in L(X)$ and if $F\in L(X)$ is a finite operator commuting with T, then $\sigma_{LD}(T)=\sigma_{LD}(T+F)$. However, these result do not extend to commuting compact perturbations. To see this,consider on the Hilbert space $\ell^2(\mathbb{N})$, the operators $T=0$ and Q defined by $Q(x_0,x_1,x_2,...)=(x_0,x_1/2,x_2/3,...)$. Then Q is compact, $TQ=QT=0$, $\text{iso}\sigma_a(T)=\Pi_a(T)=\{0\}$, $\text{iso}\sigma_a(T+Q)=\{0\}$ and $\Pi_a(T+Q)=\Pi_a(Q)=\emptyset$. So $\sigma_{LD}(T)=\emptyset$ but $\sigma_{LD}(T+Q)=\{0\}$.

Theorem 2.2. *Let X be a Banach space and let $T\in L(X)$ and $F\in L(X)$ be a finite rank operator commuting with T. If T satisfies generalized a-Weyl's theorem, then the following assertions are equivalent.*
(i) $T+F$ satisfies generalized a-Weyl's theorem;

(ii) $E_a(T+F) = \Pi_a(T+F)$;
(iii) $E_a(T+F) \cap \sigma_a(T) \subset E_a(T)$.

Proof. $(i) \Longleftrightarrow (ii)$ If $T+F$ satisfies generalized a-Weyl's theorem, then from [3, Corollary 3.2], we have $E_a(T+F) = \Pi_a(T+F)$. Conversely, assume that $E_a(T+F) = \Pi_a(T+F)$, since T satisfies generalized a-Weyl's theorem, then $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$. Since F is a finite rank operator, from [5, Lemma 2.3] we have $\sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+F)$. As F commutes with T, from Theorem 2.1 we have $\sigma_{LD}(T) = \sigma_{LD}(T+F)$. So $\sigma_{SBF_+^-}(T+F) = \sigma_{LD}(T+F)$. As $E_a(T+F) = \Pi_a(T+F)$, then from [3, Corollary 3.2], $T+F$ satisfies generalized a-Weyl's theorem.
$(iii) \Longrightarrow (ii)$ Let $\lambda \in E_a(T+F)$. Then $\lambda \in \text{iso}\sigma_a(T+F)$. If $\lambda \notin \sigma_a(T)$, then $\lambda \notin \sigma_{SBF_+^-}(T)$, then $\lambda \notin \sigma_{SBF_+^-}(T+F)$. As $\lambda \in \text{iso}\sigma_a(T+F)$, it follows from [3, Theorem 2.8] that $\lambda \in \Pi_a(T+F)$. If $\lambda \in \sigma_a(T)$, then $\lambda \in E_a(T+F) \cap \sigma_a(T)$, and by assumption $\lambda \in E_a(T)$. Since T satisfies generalized a-Weyl's theorem, then $\lambda \notin \sigma_{SBF_+^-}(T+F)$. Hence $\lambda \in \Pi_a(T+F)$. In the two cases, we have $E_a(T+F) \subset \Pi_a(T+F)$. As we have always $E_a(T+F) \supset \Pi_a(T+F)$, then $E_a(T+F) = \Pi_a(T+F)$.
$(ii) \Longrightarrow (iii)$ Assume that $E_a(T+F) = \Pi_a(T+F)$ and let $\lambda \in E_a(T+F) \cap \sigma_a(T)$, then $\lambda \in \Pi_a(T+F) \cap \sigma_a(T)$. So $\lambda \notin \sigma_{LD}(T+F)$. As $\sigma_{LD}(T) = \sigma_{LD}(T+F)$ and $\lambda \in \sigma_a(T)$, then $\lambda \in \Pi_a(T)$. Since the inclusion $\Pi_a(T) \subset E_a(T)$ is always true, then $\lambda \in E_a(T)$. Hence $E_a(T+F) \cap \sigma_a(T) \subset E_a(T)$. □

In the next result we prove a similar characterization for a-Weyl's theorem, in the case of a compact pertubation.

Theorem 2.3. *Let X be a Banach space and let $T \in L(X)$ and $K \in L(X)$ be a compact operator commuting with T. If T satisfies a-Weyl's theorem, then the following properties are equivalent.*
(i) $T+K$ satisfies a-Weyl's theorem;
(ii) $E_a^0(T+K) = \Pi_a^0(T+K)$;
(iii) $E_a^0(T+K) \cap \sigma_a(T) \subset E_a^0(T)$.

Proof. $(i) \Longleftrightarrow (ii)$ If T satisfies a-Weyl's theorem, then from [3, Theorem 3.4] we have $E_a^0(T+K) = \Pi_a^0(T+K)$. Conversely, if $E_a^0(T+K = \Pi_a^0(T+K)$, since T satisfies a-Weyl's theorem, then from [3, Theorem 3.4] we have $E_a^0(T) = \Pi_a^0(T)$. Since K is a compact operator, then we also have $\sigma_{SF_+^-}(T+K) = \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_a^0(T) = \sigma_a(T) \setminus \Pi_a^0(T) = \sigma_{ub}(T)$. Since K commutes with T, then from [1, Corollary 3.45], we have $\sigma_{ub}(T) = \sigma_{ub}(T+K) = \sigma_a(T+K) \setminus \Pi_a^0(T+K) = \sigma_a(T+K) \setminus E_a^0(T+K)$. Therefore $\sigma_{SF_+^-}(T+K) = \sigma_a(T+K) \setminus E_a^0(T+K)$ and $T+K$ satisfies a-Weyl's theorem.
$(ii) \Longrightarrow (iii)$ Suppose that $E_a^0(T+K) = \Pi_a^0(T+K)$. If $\lambda \in E_a^0(T+K) \cap \sigma_a(T)$, then $\lambda \in \Pi_a^0(T+K) \cap \sigma_a(T)$. So $\lambda \notin \sigma_{ub}(T+K)$. As $\sigma_{ub}(T) = \sigma_{ub}(T+K)$ and $\lambda \in \sigma_a(T)$, then $\lambda \in \Pi_a^0(T) = E_a^0(T)$. Hence $E_a^0(T+K) \cap \sigma_a(T) \subset E_a^0(T)$.
$(iii) \Longrightarrow (ii)$ Suppose that $E_a^0(T+K) \cap \sigma_a(T) \subset E_a^0(T)$. Since $\Pi_a^0(T+K) \subset E_a^0(T+K)$ is always true, we only have to show that $\Pi_a^0(T+K) \supset E_a^0(T+K)$. Let

$\lambda \in E^0_a(T+K)$. If $\lambda \notin \sigma_a(T)$, then $\lambda \notin \sigma_{ub}(T)$. As $\sigma_{ub}(T) = \sigma_{ub}(T+K)$ and $\lambda \in \sigma_a(T+K)$, then $\lambda \in \Pi^0_a(T+K)$. If $\lambda \in \sigma_a(T)$, then $\lambda \in E^0_a(T+K) \cap \sigma_a(T)$, and by hypothesis $\lambda \in E^0_a(T) = \Pi^0_a(T)$. So $\lambda \notin \sigma_{ub}(T)$. As $\sigma_{ub}(T) = \sigma_{ub}(T+K)$, then $\lambda \in \Pi^0_a(T+K)$. In the two cases, we have $\Pi^0_a(T+K) \supset E^0_a(T+K)$. □

Remark 2.4. (1)– Theorem 2.2 extends [17, Theorem 2.4] which establishes that $T+F$ satisfies generalized a-Weyl's theorem when T is an a-isoloid operator satisfying generalized a-Weyl's theorem and F is a finite rank operator commuting with T. Since $\operatorname{acc}\sigma_a(T) = \operatorname{acc}\sigma_a(T+F)$ (see [13, Theorem 3.2]), we observe that if T is an a-isoloid operator, then $E_a(T+F) \cap \sigma_a(T) \subset E_a(T)$.

(2)– There exists an operator T which is not a-isoloid, satisfying generalized a-Weyl's theorem and a finite rank operator commuting with T such that $E_a(T+F) \cap \sigma_a(T) \subset E_a(T)$. To see this, consider the operator T defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, ...) = (x_1/2, x_2/3, ...)$ and let $F = 0$. Then $\sigma_a(T) = \{0\}, E_a(T) = \emptyset$ and $\sigma_{SBF^-_+}(T) = \{0\}$. So T satisfies generalized a-Weyl's theorem, $E_a(T+F) \cap \sigma_a(T) = E_a(T)$, but T is not a-isoloid.

(3)– Theorem 2.3 extends [13, Theorem 3.4] which establishes that if T is an a-isoloid operator satisfying a-Weyl's theorem and if F is a finite rank operator commuting with T, then $T+F$ satisfies a-Weyl's theorem. To see this, we know that $\alpha(T) < \infty$ if and only if $\alpha(T+F) < \infty$ (see [18, Lemma 2.1]), so it follows that if T is a-isoloid then $E^0_a(T+F) \cap \sigma_a(T) \subset E^0_a(T)$.

There exists quasinilpotent operators which do not satisfy generalized a-Weyl's theorem. For example, if we consider the operator T defined on $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, ...) = (0, x_2/2, x_3/3, ...)$, then T is quasinilpotent but generalized a-Weyl's theorem fails for T, since $\sigma_a(T) = \sigma_{SBF^-_+}(T) = \{0\}$ and $E_a(T) = \{0\}$. But if a quasinilpotent operator satisfies generalized a-Weyl's theorem, then the following perturbation result holds.

Corollary 2.5. *Let $T \in L(X)$ be a quasinilpotent operator and let $F \in L(X)$ be a finite rank operator commuting with T. If T satisfies generalized a-Weyl's theorem, then $T+F$ satisfies generalized a-Weyl's theorem.*

Proof. If T is injective, as TF is a finite rank quasinilpotent operator, then TF is a nilpotent operator. Since T is injective, then F is nilpotent . Therefore $\sigma_a(T+F) = \sigma_a(T)$ and $E_a(T+F) = E_a(T)$ (see Lemma 3.1). Moreover, since F is of finite rank, it follows that $\sigma_{SBF^-_+}(T+F) = \sigma_{SBF^-_+}(T)$. As T satisfies generalized a-Weyl's theorem then $\Delta^g_a(T) = E_a(T)$. So $\Delta^g_a(T+F) = E_a(T+F)$ and $T+F$ satisfies generalized a-Weyl's theorem.

If T is not injective, then $\operatorname{iso}\sigma_a(T) = E_a(T) = \{0\}$ and T is an a-isoloid operator. Therefore by Theorem 2.2, we conclude that $T+F$ satisfies generalized a-Weyl's theorem. □

Remark 2.6. The hypothesis of commutativity in Corollary 2.5 is crucial. Indeed, if we consider the Hilbert space $H = \ell^2(\mathbb{N})$, and the operators T and F defined on H by:

$$T(x_1, x_2, x_3, ...) = (0, x_1/2, x_2/3, ...) \ , \ F(x_1, x_2, x_3, ...) = (0, -x_1/2, 0, 0, ...).$$

Then T is quasi-nilpotent, F is a finite rank operator which do not commutes with T. Moreover, we have $\sigma_a(T) = \sigma_{SBF_+^-}(T) = \{0\}$ and $E_a(T) = \emptyset$. Hence T satisfies generalized a-Weyl's theorem. But $T+N$ does not satisfy generalized a-Weyl's theorem because $\sigma_a(T+N) = \sigma_{SBF_+^-}(T+F) = \{0\}$ and $E_a(T+N) = \{0\}$.

Theorem 2.7. *Let X be a Banach space and let $T \in L(X)$ and $F \in L(X)$ be a finite rank operator commuting with T. If T satisfies generalized Weyl's theorem, then the following properties are equivalent.*
(i) $T+F$ satisfies generalized Weyl's theorem;
(ii) $E(T+F) = \Pi(T+F)$;
(iii) $E(T+F) \cap \sigma(T) \subset E(T)$.

Proof. The equivalence of the two first properties is well known in [9, Theorem 3.2]. Let us show that (ii) is equivalent to (iii). Assume that $E(T+F) \cap \sigma(T) \subset E(T)$. Let $\lambda \in E(T+F)$, then $\lambda \in \text{iso}\sigma(T+F)$. If $\lambda \notin \sigma(T)$, then $\lambda \notin \sigma_D(T)$. Since F commutes with T, from [10, Theorem 2.7] we have $\sigma_D(T) = \sigma_D(T+F)$. As $\lambda \in \sigma(T+F)$, then $\lambda \in \Pi(T+F)$. If $\lambda \in \sigma(T)$, then $\lambda \in E(T+F) \cap \sigma(T)$ and by hypothesis we have $\lambda \in E(T)$. As T satisfies generalized Weyl's theorem, it follows that $\lambda \in \Pi(T)$. As $\sigma_D(T) = \sigma_D(T+F)$ and $\lambda \in \sigma(T+F)$ then $\lambda \in \Pi(T+F)$. Finally we have $E(T+F) \subset \Pi(T+F)$. As we have always $E(T+F) \supset \Pi(T+F)$, then $E(T+F) = \Pi(T+F)$.

Conversely, suppose that $E(T+F) = \Pi(T+F)$. If $\lambda \in E(T+F) \cap \sigma(T)$, then $\lambda \in \Pi(T+F) \cap \sigma(T)$. Therefore $\lambda \notin \sigma_D(T+F)$. As $\sigma_D(T) = \sigma_D(T+F)$ and $\lambda \in \sigma(T)$, then $\lambda \in \Pi(T) = E(T)$. Hence $E(T+F) \cap \sigma(T) \subset E(T)$. □

Similarly to Theorem 2.7, we have the following characterization in the case of Weyl's theorem. We give this result without proof.

Theorem 2.8. *Let X be a Banach and let $T \in L(X)$ and $K \in L(X)$ be a compact operator commuting with T. If T satisfies Weyl's theorem, then the following properties are equivalent.*
(i) $T+K$ satisfies Weyl's theorem;
(ii) $E^0(T+K) = \Pi^0(T+K)$;
(iii) $E^0(T+K) \cap \sigma(T) \subset E^0(T)$.

Remark 2.9. (1) It is proved in [5, Theorem 2.6] that generalized Weyl's theorem for isoloid operators is preserved under perturbations by commuting finite rank operators. This result becomes as an immediate consequence of Theorem 2.7. As $\text{acc}\,\sigma(T) = \text{acc}\,\sigma(T+F)$ (see [18, Lemma 2.1]), we observe that if T is isoloid, then $E(T+F) \cap \sigma(T) \subset E(T)$.
(2) Since $\alpha(T) < \infty$ if and only if $\alpha(T+F) < \infty$, we observe that if T is isoloid then
$E^0(T+F) \cap \sigma(T) \subset E^0(T)$. Therefore Theorem 2.8 extends a result of W. Y. Lee and S. H. Lee in [18], where Weyl's theorem was proved for $T+F$ when T is an isoloid operator satisfying Weyl's theorem, and F is a finite rank operator commuting with T.

Examples 2.10. (a)– In general generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and Weyl's theorem are not transmitted from an operator to a commuting finite rank perturbation as the following example shows.
Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be an injective quasinilpotent operator which is not nilpotent. We define T on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = I \oplus S$ where I is the identity operator on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \sigma_a(T) = \{0, 1\}$ and $E_a(T) = \{1\}$. It follows from [9, Example 2] that $\sigma_{BW}(T) = \{0\}$. This implies that $\sigma_{SBF_+^-}(T) = \{0\}$. Hence $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T) = \{1\}$ and T satisfies generalized a-Weyl's theorem, so it satisfies a-Weyl's theorem, generalized Weyl's theorem and Weyl's theorem.

We define the operator U on $\ell^2(\mathbb{N})$ by $U(\xi_1, \xi_2, \xi_3, ...) = (-\xi_1, 0, 0, ...)$ and $F = U \oplus 0$ on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Then F is a finite rank operator commuting with T. On the other hand, $\sigma(T+F) = \sigma_a(T+F) = \{0, 1\}$ and $E_a(T+F) = \{0, 1\}$. As $\sigma_{SBF_+^-}(T+F) = \sigma_{SBF_+^-}(T) = \{0\}$, then $\sigma_a(T+F) \setminus \sigma_{SBF_+^-}(T+F) = \{1\} \neq E_a(T+F)$ and $T+F$ does not satisfy generalized a-Weyl's theorem. Not that $E_a(T+F) \cap \sigma_a(T) \not\subset E_a(T)$. Moreover, $E(T+F) = \{0, 1\}$, and as by [4, Theorem 4.3] we have $\sigma_{BW}(T+F) = \sigma_{BW}(T) = \{0\}$, then $T+F$ does not satisfy generalized Weyl's theorem. Observe that $E(T+F) \cap \sigma(T) \not\subset E(T) = \{1\}$.
Moreover we have $\sigma_W(T+F) = \{0, 1\}$ and $E^0(T+F) = \{0\}$. As $\sigma(T+F) = \{0, 1\}$ then $\Delta(T+F) \neq E^0(T+F)$ and $T+F$ does not satisfy Weyl's theorem. So $T+F$ does not satisfy a-Weyl's theorem. Note that $E^0(T+F) \cap \sigma(T) \not\subset E^0(T) = \emptyset$, and $E^0_a(T+F) \cap \sigma_a(T) = \{0\} \cap \{0, 1\} \not\subset E^0_a(T) = \emptyset$.

(b)– Theorem 2.2 and Theorem 2.7 do not extend to a commuting compact perturbation. Indeed, if we consider on the Hilbert space $\ell^2(\mathbb{N})$ the operators $T = 0$ and Q defined by $Q(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, x_4/4, ...)$. Then Q is a compact operator commuting with T. Moreover,we have

$\sigma_a(T) = \{0\}$, $\sigma_{SBF_+^-}(T) = \emptyset, E_a(T) = \{0\}$. Hence T satisfies generalized a-Weyl's theorem. So it satisfies generalized Weyl's theorem. But generalized a-Weyl's theorem and generalized Weyl's fails for $T+Q = Q$. Indeed $\sigma_{SBF_+^-}(T+Q) = \sigma_a(T+Q) = \{0\}$, $E_a(T+Q) = \{0\}$ and $\sigma(T+Q) = \{0\}, \sigma_{BW}(T+Q) = \{0\}, E(T+Q) = E(T) = \{0\}$. Thought we have $E_a(T+Q) \cap \sigma_a(T) \subset E_a(T)$ and $E(T+Q) \cap \sigma(T) \subset E(T)$.

3 Nilpotent perturbations

Let $T \in L(X)$ and let N be a nilpotent operator commuting with T. In a first step we prove that T and $T+N$ have the same isolated eigenvalues in the approximate spectrum.

Lemma 3.1. Let X be a Banach space and let $T \in L(X)$. If $N \in L(X)$ is a nilpotent operator commuting with T, then $E_a(T+N) = E_a(T)$.

Proof. Let $\lambda \in E_a(T)$ be arbitrary. There is no loss of generality if we assume that $\lambda = 0$. As N is nilpotent we know that $\sigma_a(T+N) = \sigma_a(T)$, thus $0 \in \text{iso}\sigma_a(T+N)$. Let $m \in \mathbb{N}$ be such that $N^m = 0$. If $x \in N(T)$, then $(T+N)^m(x) = \sum_{k=0}^{m} C_m^k T^k N^{m-k}(x) = 0$. So $N(T) \subset N(T+N)^m$. As $\alpha(T) > 0$, it follows that $\alpha((T+N)^m) > 0$ and this implies that $\alpha(T+N) > 0$. Hence $0 \in E_a(T+N)$. So $E_a(T) \subset E_a(T+N)$. By symmetry we have $E_a(T) = E_a(T+N)$. □

In the next theorem, we consider an operator $T \in L(X)$ satisfying generalized a-Weyl's theorem, a nilpotent operator commuting with T, and we give necessary and sufficient conditions for $T+N$ to satisfy generalized a-Weyl's theorem.

Theorem 3.2. *Let X be a Banach space and $T \in L(X)$ and $N \in L(X)$ be a nilpotent operator commuting with T. If T satisfies generalized a-Weyl's theorem, then the following statements are equivalent.*
(i) $T+N$ satisfies generalized a-Weyl's theorem;
(ii) $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$;
(iii) $E_a(T) = \Pi_a(T+N)$.

Proof. $(i) \iff (ii)$ Assume that $T+N$ satisfies generalized a-Weyl's theorem, then

$\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = E_a(T+N)$. As $\sigma_a(T+N) = \sigma_a(T)$ and $E_a(T+N) = E_a(T)$ then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T+N) = E_a(T)$. Since T satisfies generalized a-Weyl's theorem, then

$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$. So $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$. Conversely, assume that $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$, then as T satisfies generalized a-Weyl's theorem it follows that $T+N$ satisfies also generalized a-Weyl's theorem.
$(i) \iff (iii)$ Assume that $T+N$ satisfies generalized a-Weyl's theorem, then from [3, Corollary 3.2], we have $E_a(T+N) = \Pi_a(T+N)$. Therefore $E_a(T) = \Pi_a(T+N)$. Conversely, assume that $E_a(T) = \Pi_a(T+N)$. Since T satisfies generalized a-Weyl's theorem, then T satisfies generalized a-Browder's theorem. As we know from [2, Theorem 2.2] that a-Browder's theorem is equivalent to generalized a-Browder's theorem, then T satisfies a-Browder's theorem. So $\sigma_{SF_+^-}(T) = \sigma_{ub}(T)$. From [12, Theorem 2.13], we know that $\sigma_{SF_+^-}(T) = \sigma_{SF_+^-}(T+N)$. By [1, Theorem 3.65], we know that $\sigma_{ub}(T) = \sigma_{SF_+^-}(T) \cup \text{acc}\sigma_a(T)$. Hence $\sigma_{ub}(T) = \sigma_{ub}(T+N)$. Therefore $\sigma_{SF_+^-}(T+N) = \sigma_{ub}(T+N)$ and $T+N$ satisfies a-Browder's theorem. So it satisfies generalized a-Browder's that is $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = \Pi_a(T+N)$. As by assumption $E_a(T) = \Pi_a(T+N)$, it follows that $\sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = E_a(T+N)$ and so $T+N$ satisfies generalized a-Weyl's theorem. □

Theorem 3.3. *Let X be a Banach space and let $T \in L(X)$ be an operator satisfying generalized a-Weyl's theorem. If $E_a(T) \subset \text{iso}\sigma(T)$ and if $N \in L(X)$ is a nilpotent operator commuting with T, then $T+N$ satisfies generalized a-Weyl's theorem.*

Proof. Let $\lambda \in E_a(T)$, since $E_a(T) \subset iso\sigma(T)$, then $\lambda \in E(T)$.

As T satisfies generalized a-Weyl's theorem, from [3, Theorem3.7] T satisfies generalized Weyl's theorem. Hence $\lambda \in \Pi(T)$. As we know that $\sigma_D(T) = \sigma_D(T+N)$, then $\lambda \in \Pi(T+N)$. Hence $\lambda \in \Pi_a(T+N)$. Consequently we have $E_a(T) = \Pi_a(T+N)$. Conversely if $\lambda \in \Pi_a(T+N)$, then $\lambda \in E_a(T+N)$. As we know that $E_a(T) = E_a(T+N)$, (see Lemmma 3.1), then $\lambda \in E_a(T+N)$. So $E_a(T) = \Pi_a(T+N)$. From Theorem 3.2, $T+N$ satisfies generalized a-Weyl's theorem. □

Remark 3.4. 1- The hypothesis of commutativity in the Theorem 3.2 corollary is crucial. The following example shows that if we do not assume that N commutes with T, then the result may fails. Let $H = \ell^2(\mathbb{N})$, and let T and N defined by:

$$T(x_1, x_2, x_3, ...) = (0, x_1/2, x_2/3, ...) \ , \ N(x_1, x_2, x_3, ...) = (0, -x_1/2, 0, 0, ...).$$

Clearly N is a nilpotent operator which does not commute with T. Moreover, we have $\sigma_a(T) = \sigma_{SBF_+^-}(T) = \{0\}$ and $E_a(T) = \emptyset$. Therefore T satisfies generalized a-Weyl's theorem. But $T+N$ does not satisfy generalized a-Weyl's theorem because $\sigma_a(T+N) = \sigma_{SBF_+^-}(T+N) = \{0\}$ and $E_a(T+N) = \{0\}$.

(2) Generally, generalized a-Weyl's theorem does not extend to a quasinilpotent perturbation: Define on the Banach space $\ell^2(\mathbb{N})$ the operator $T = 0$ and the quasinilpotent operator Q defined by $Q(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, x_4/4, ...)$. Then $\sigma_a(T) = \{0\}$ and $\sigma_{SBF_+^-}(T) = \emptyset$. Moreover we have $E_a(T) = \{0\}$. Hence T satisfies generalized a-Weyl's theorem. But generalized a-Weyl's theorem does not hold for $T+Q = Q$, since $\sigma_{SBF_+^-}(T+Q) = \sigma_a(T+Q) = \{0\}$ and $E_a(T+Q) = \{0\}$.

Open questions: The proof of Theorem 3.2 suggests the following questions:

1. let $T \in L(X)$ and let $N \in L(X)$ be a nilpotent operator commuting with T. Under which conditions $a(T+N)$ is finite if $a(T)$ is finite?

2. let $T \in L(X)$ and let $N \in L(X)$ be a nilpotent operator commuting with T. Under which conditions $R((T+N)^m)$ is closed for m large enough i if $R(T^m)$ is closed for m large enough?

3. let $T \in L(X)$ and let $N \in L(X)$ be a nilpotent operator commuting with T. Under which conditions $\Pi_a(T+N) = \Pi_a(T)$?

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