GENERALIZED A-WEYL’S THEOREM
AND PERTURBATIONS

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Abstract

In this paper we study the stability of generalized a-Weyl’s theorem
under perturbations by finite rank and nilpotent operators. Among other
results, we prove that if $T$ is a bounded linear operator acting on a Banach
space $X$ satisfies generalized a-Weyl’s theorem and $F$ is a finite rank
operator commuting with $T$, then $T + F$ satisfies generalized a-Weyl’s
theorem if and only if $E_a(T + F) \cap \sigma_a(T) \subset E_a(T)$. Moreover we prove
that if $T$ is a bounded linear operator acting on a Banach space satisfies
generalized a-Weyl’s theorem and $N$ is a nilpotent operator commuting
with $T$, then $T + N$ satisfies generalized a-Weyl’s theorem if and only if
$\sigma_{SBF^+}(T + N) = \sigma_{SBF^+}(T)$.

1 Introduction

Throughout this paper $L(X)$ denote the Banach algebra of all bounded linear
operators acting on a Banach space $X$. For $T \in L(X)$, let $T^*$, $N(T)$, $R(T)$,
$\sigma(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the
spectrum and the approximate point spectrum of $T$. Let $\alpha(T)$ and $\beta(T)$ be
the nullity and the deficiency of $T$ defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \text{codim} R(T)$. If the range $R(T)$ of $T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$),
then $T$ is called an upper (resp. a lower) semi-Fredholm operator. In the sequel
$SF^+_+(X)$ denotes the class of all upper semi-Fredholm operators. If $T \in L(X)$
is either an upper or a lower semi-Fredholm operator, then $T$ is called a semi-
Fredholm operator , and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$.
If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called a Fredholm operator. An
operator $T$ is called a Weyl operator if it is a Fredholm operator of index zero.
The Weyl spectrum of $T$ is defined by $\sigma_W(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl } \}$. For $T \in L(X)$, let $SF^+_+(X) = \{ T \in SF^+_+(X) : \text{ind}(T) \leq 0 \}$. Then the Weyl

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essential approximate spectrum of $T$ is defined by $\sigma_{SF^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF^- (X)\}$.

Let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF^-}(T)$. Following Coburn [11], we say that Weyl's theorem holds for $T \in L(X)$ if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \text{iso}(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, $\text{iso}A$ denotes the set of all isolated points of $A$ and $\text{acc}A$ denotes the set of all points of accumulation of $A$.

According to Rakočević [20], an operator $T \in L(X)$ is said to satisfy a-Weyl’s theorem if $\Delta_a(T) = E^a_0(T)$, where $E^a_0(T) = \{\lambda \in \text{iso}_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$. It is known [20] that an operator satisfying a-Weyl’s theorem satisfies Weyl’s theorem, but not conversely.

For $T \in L(X)$ and a nonnegative integer $n$ define $T_{[n]}$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer $n$ the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then $T$ is called an upper (resp. a lower) semi-B-Fredholm operator. In this case the index of $T$ is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [6]. Moreover, if $T_{[n]}$ is a Fredholm operator, then $T$ is called a B-Fredholm operator, see [7]. An operator $T \in L(X)$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum of $T$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not a B-Weyl operator}. 

Recall that the ascent of an operator $T \in L(X)$ is defined by $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and the descent of $T$, is defined by $\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf\emptyset = \infty$. An operator $T$ is called Drazin invertible if it has finite ascent and descent. The Drazin spectrum of $T$ is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Drazin invertible $\}$. An operator $T \in L(X)$ is called an upper semi-Browder if it is an upper semi-Fredholm of finite ascent, and is called Browder if it is a Fredholm of finite ascent and descent. The upper semi-Browder spectrum of $T$ is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not upper semi-Browder $\}$ and the Browder spectrum of $T$ is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder $\}$.

Define also the set $LD(X)$ as follows : $LD(X) = \{T \in L(X) : a(T) < \infty$ and $R(T^{a(T)+1})$ is closed} and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$. An operator $T \in L(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of $T$ if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of $T$ of finite rank if $\lambda$ is a left pole of $T$ and $\alpha(T - \lambda I) < \infty$. Let $\Pi_a(T)$ denotes the set of all left poles of $T$ and $\Pi^0_a(T)$ denotes the set of all left poles of $T$ of finite rank. From [3, Theorem 2.8], it follows that if $T \in L(X)$ is left Drazin invertible, then $T$ is an upper semi-B-Fredholm operator of index less or equal than zero.

Let $\Pi(T)$ be the set of all poles of the resolvent of $T$ and let $\Pi^0(T)$ be the set of all poles of the resolvent of $T$ of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T) \setminus \Pi(T) : (a(T - \lambda I) = \infty\}$. According to [15], a complex number $\lambda$ is a pole of the resolvent of $T$ if and only if $0 < \max \{a(T - \lambda I), \delta(T - \lambda I)\} < \infty$. Moreover, if this is true then $a(T - \lambda I) = \delta(T - \lambda I)$. According also to [15], the space $R(T - \lambda I)^a(T - \lambda I) + 1$ is closed for each $\lambda \in \Pi(T)$.
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\( \Pi(T) \subset \Pi_a(T) \) and \( \Pi^0(T) \subset \Pi^0_a(T) \).

Following [3], we say that generalized a-Browder’s theorem holds for \( T \) if \( \Delta^g_a(T) = \Pi_a(T) \) and that a-Browder’s theorem holds for \( T \) if \( \Delta_a(T) = \Pi^0_a(T) \). It is shown [2, Theorem 2.2] that generalized a-Browder’s theorem is equivalent to a-Browder’s theorem.

Let \( \Delta^g_a(T) = \sigma(T) \setminus \sigma_{BW}(T) \). We say that generalized Browder’s theorem holds for \( T \) if \( \Delta^g(T) = \Pi(T) \); where \( \Pi(T) \) is the set of all poles of \( T \) and that Browder’s theorem holds for \( T \) if \( \Delta(T) = \Pi^0(T) \); where \( \Pi^0(T) \) is the set of all poles of \( T \) of finite rank. It is proved in [2, Theorem 2.1] that generalized Browder’s theorem is equivalent to Browder’s theorem.

Let \( SBF_+(X) \) be the class of all upper semi-B-Fredholm operators, \( SBF^-_+(X) = \{ T \in SBF_+(X) : \text{ind}(T) \leq 0 \} \). The upper B-Weyl spectrum of \( T \) is defined by \( \Delta^g(T) = \sigma_a(T) \setminus \sigma_{SBF^-_+(T)}(T) \). We say that \( T \) obeys generalized a-Weyl’s theorem, if \( \Delta^g_a(T) = \Pi_a(T) \); where \( \Pi_a(T) \) is the set of all eigenvalues of \( T \) which are isolated in \( \sigma_a(T) \) and that \( T \) obeys generalized Weyl’s theorem if \( \Delta^g(T) = \Pi(T) \); where \( \Pi(T) \) is the set of all eigenvalues of \( T \) which are isolated in \( \sigma(T) \) ([3, Definition 3.1]). Generalized a-Weyl’s theorem has been studied in [3, 8]. In [3, Theorem 3.11], it is shown that an operator satisfying generalized a-Weyl’s theorem satisfies a-Weyl’s theorem, but the converse is not true in general, and under the assumption \( E_a(T) = \Pi_a(T) \), it is proved in [8, Theorem 2.10] that generalized a-Weyl’s theorem is equivalent to a-Weyl’s theorem. It is also proved in [3, Theorem 3.7] that generalized a-Weyl’s theorem implies generalized Weyl’s which in turn implies from [3, Theorem 3.9] Weyl’s theorem.

**Definition 1.1.** A bounded linear operator \( T \in \mathcal{L}(X) \) is called isoloid (resp. a-isoloid) if \( \text{iso}(T) = E(T) \) (resp. \( \text{iso}_{a}(T) = E_a(T) \)). Moreover, if \( \text{iso}_{a}(T) = \Pi_a(T) \), then we will say that \( T \) is an a-polaroid operator.

We will say that \( T \in \mathcal{L}(X) \) has the single valued-extension property at \( \lambda_0 \), (SVEP for short) if for every open neighborhood \( U \) of \( \lambda_0 \), the only analytic function \( f : U \to X \) which satisfies the equation: \((T - \lambda I)f(\lambda) = 0\), for all \( \lambda \in U \) is the function \( f = 0 \). \( T \in \mathcal{L}(X) \) is said to have the SVEP if \( T \) has this property at every \( \lambda \in \mathbb{C} \) (see [16]).

The aim of this paper is to study the stability of generalized a-Weyl’s theorem under commuting nilpotent or finite rank perturbations. Thus, in the second section, we prove in Theorem 2.2 that if \( T \) is a bounded linear operator acting on a Banach space \( X \) satisfies generalized a-Weyl’s theorem and \( F \) is a finite rank operator commuting with \( T \), then \( T + F \) satisfies generalized a-Weyl’s theorem if and only if \( E_a(T + F) \cap \sigma_a(T) \subset E_a(T) \). We obtain also similar results for a-Weyl’s and Weyl’s theorem in the case of compact perturbations. Moreover we prove also in Theorem 2.7 that if \( T \in \mathcal{L}(X) \) satisfies generalized Weyl’s theorem and if \( F \in \mathcal{L}(X) \) is a finite rank operator commuting with \( T \), then \( T + F \) satisfies generalized Weyl’s theorem if and only if \( E(T + F) \cap \sigma(T) \subset E(T) \).

In the third section we consider in Theorem 3.2 an operator \( T \) satisfying generalized a-Weyl’s theorem and a nilpotent operator \( N \) commuting with \( T \),
and we prove that $T + N$ satisfies generalized a-Weyl's theorem if and only if $\sigma_{SBF^{-}}(T + N) = \sigma_{SBF^{-}}(T)$. As a consequence, we show in Theorem 3.3 that if $T \in L(X)$ is an operator satisfying generalized a-Weyl's theorem, if $E_a(T) \subset iso\sigma(T)$ and if $N \in L(X)$ is a nilpotent operator commuting with $T$, then $T + N$ satisfies generalized a-Weyl's theorem. We conclude this paper by some open questions related to the ideas developed in this section.

2 Finite rank perturbations

The next theorem had been established in [3, Theorem 4.2] for Hilbert spaces operators. We show here that it holds also in the general case of Banach spaces.

For $T \in L(X)$, let $c_n(T) = \dim \frac{N(T^n+1)}{N(T^n)T}$.

Theorem 2.1. Let $X$ be a Banach space and let $T \in L(X)$. Then

$$\sigma_{LD}(T) = \bigcap_{F \in F(X), FT = TF} \sigma_{LD}(T + F)$$

where $F(X)$ denotes the ideal of finite rank operators in $L(X)$.

Proof. If $\lambda \notin \sigma_{LD}(T)$, then $\lambda \notin \sigma_{LD}(T + 0)$. Since 0 is a finite rank operator, it follows that $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F(X), FT = TF\}$. To show the opposite inclusion, let $\lambda \notin \bigcap \{\sigma_{LD}(T + F) : F \in F(X), FT = TF\}$. Then there exists a finite rank operator $F$ commuting with $T$ such that $T + F - \lambda I$ is left Drazin invertible. So $T + F - \lambda I$ is an upper semi-B-Fredholm. From [6, Theorem 2.7], $T - \lambda I$ is also an upper semi B-Fredholm operator. In particular the two operators $T - \lambda I$ and $T - \lambda I + F$ are operators of topological uniform descent [6]. By [14, Theorem 5.8], for $n$ large enough we have $c_n(T - \lambda I) = c_n(T - \lambda I + F)$. Since $T - \lambda I + F$ is left Drazin invertible, then for $n$ large enough we have $c_n(T - \lambda I + F) = 0$. So for $n$ large enough we have $c_n(T - \lambda I) = 0$ and $a(T - \lambda I) < \infty$. On the other hand, for $n$ large enough $R(T - \lambda I)^n$ is closed and by [19, Lemma 12], $R(T - \lambda I)^{n(T - \lambda I)}$ is also closed. Hence $T - \lambda I$ is left Drazin invertible.

From Theorem 2.1 we conclude that if $T \in L(X)$ and if $F \in L(X)$ is a finite operator commuting with $T$, then $\sigma_{LD}(T) = \sigma_{LD}(T + F)$. However, these result do not extend to commuting compact perturbations. To see this consider on the Hilbert space $\ell^2(\mathbb{N})$, the operators $T = 0$ and $Q$ defined by $Q(x_0, x_1, x_2, \ldots) = (x_0, x_1/2, x_2/3, \ldots)$. Then $Q$ is compact, $TQ = QT = 0$, $iso\sigma_a(T) = \Pi_a(T) = \{0\}$, $iso\sigma_a(T + Q) = \{0\}$ and $\Pi_a(T + Q) = \Pi_a(Q) = \emptyset$. So $\sigma_{LD}(T) = \emptyset$ but $\sigma_{LD}(T + Q) = \{0\}$.

Theorem 2.2. Let $X$ be a Banach space and let $T \in L(X)$ and $F \in L(X)$ be a finite rank operator commuting with $T$. If $T$ satisfies generalized a-Weyl's theorem, then the following assertions are equivalent.

(i) $T + F$ satisfies generalized a-Weyl's theorem;
(ii) \( E_a(T + F) = \Pi_a(T + F) \);
(iii) \( E_a(T + F) \cap \sigma_a(T) \subset E_a(T) \).

Proof. (i) \( \iff \) (ii) If \( T + F \) satisfies generalized a-Weyl’s theorem, then from [3, Corollary 3.2], we have \( E_a(T + F) = \Pi_a(T + F) \). Conversely, assume that \( E_a(T + F) = \Pi_a(T + F) \), since \( T \) satisfies generalized a-Weyl’s theorem, then \( \sigma_{SBF^c_+}(T) = \sigma_{LD}(T) \). Since \( F \) is a finite rank operator, from [5, Lemma 2.3] we have \( \sigma_{SBF^c_+}(T) = \sigma_{SBF^c_+}(T + F) \). As \( F \) commutes with \( T \), from Theorem 2.1 we have \( \sigma_{LD}(T) = \sigma_{LD}(T + F) \). So \( \sigma_{SBF^c_+}(T + F) = \sigma_{LD}(T + F) \). As \( E_a(T + F) = \Pi_a(T + F) \), then from [3, Corollary 3.2], \( T + F \) satisfies generalized a-Weyl’s theorem.

(iii) \( \implies \) (i) Let \( \lambda \in E_a(T + F) \). Then \( \lambda \in \text{iso}_a(T + F) \). If \( \lambda \notin \sigma_a(T) \), then \( \lambda \notin \text{iso}_a(T + F) \). As \( \lambda \in \sigma_{SBF^c_+}(T + F) \), it follows from [3, Theorem 2.8] that \( \lambda \in \Pi_a(T + F) \). If \( \lambda \notin \sigma_a(T) \), then \( \lambda \in E_a(T + F) \cap \sigma_a(T) \), and by assumption \( \lambda \in E_a(T) \). Since \( T \) satisfies generalized a-Weyl’s theorem, then \( \lambda \notin \sigma_{SBF^c_+}(T + F) \). Hence \( \lambda \in \Pi_a(T + F) \). In the two cases, we have \( E_a(T + F) \subset \Pi_a(T + F) \). As we have always \( E_a(T + F) \subset \Pi_a(T + F) \), then \( E_a(T + F) = \Pi_a(T + F) \).

(ii) \( \implies \) (iii) Assume that \( E_a(T + F) = \Pi_a(T + F) \) and let \( \lambda \in E_a(T + F) \cap \sigma_a(T) \), then \( \lambda \in \Pi_a(T + F) \cap \sigma_a(T) \). So \( \lambda \notin \sigma_{LD}(T + F) \). As \( \sigma_{LD}(T) = \sigma_{LD}(T + F) \) and \( \lambda \in \sigma_a(T) \), then \( \lambda \in \Pi_a(T) \). Since the inclusion \( \Pi_a(T) \subset E_a(T) \) is always true, then \( \lambda \in E_a(T) \). Hence \( E_a(T + F) \cap \sigma_a(T) \subset E_a(T) \).

In the next result we prove a similar characterization for a-Weyl’s theorem, in the case of a compact perturbation.

**Theorem 2.3.** Let \( X \) be a Banach space and let \( T \in L(X) \) and \( K \in L(X) \) be a compact operator commuting with \( T \). If \( T \) satisfies a-Weyl’s theorem, then the following properties are equivalent.
(i) \( T + K \) satisfies a-Weyl’s theorem;
(ii) \( E^0_a(T + K) = \Pi^0_a(T + K) \);
(iii) \( E^0_a(T + K) \cap \sigma_a(T) \subset E^0_a(T) \).

Proof. (i) \( \iff \) (ii) If \( T \) satisfies a-Weyl’s theorem, then from [3, Theorem 3.4] we have \( E^0_a(T + K) = \Pi^0_a(T + K) \). Conversely, if \( E^0_a(T + K) = \Pi^0_a(T + K) \), since \( T \) satisfies a-Weyl’s theorem, then from [3, Theorem 3.4] we have \( E^0_a(T) = \Pi^0_a(T) \). Since \( K \) is a compact operator, then we also have \( \sigma_{SBF^c_+}(T + K) = \sigma_{SBF^c_+}(T) = \sigma_a(T) \setminus E^0_a(T) = \sigma_a(T) \setminus \Pi^0_a(T) = \sigma_{ab}(T) \). Since \( K \) commutes with \( T \), then from [1, Corollary 3.45], we have \( \sigma_{ab}(T + K) = \sigma_{ab}(T + K) \setminus \Pi^0_a(T + K) = \sigma_a(T + K) \setminus E^0_a(T + K) \). Therefore \( \sigma_{SBF^c_+}(T + K) = \sigma_a(T + K) \setminus E^0_a(T + K) \) and \( T + K \) satisfies a-Weyl’s theorem.

(ii) \( \implies \) (iii) Suppose that \( E^0_a(T + K) = \Pi^0_a(T + K) \). If \( \lambda \in E^0_a(T + K) \cap \sigma_a(T) \), then \( \lambda \in \Pi^0_a(T + K) \cap \sigma_a(T) \). So \( \lambda \notin \sigma_{ab}(T + K) \). As \( \sigma_{ab}(T + K) = \sigma_{ab}(T + K) \) and \( \lambda \in \sigma_a(T) \), then \( \lambda \in \Pi^0_a(T + K) \). Hence \( E^0_a(T + K) \cap \sigma_a(T) \subset E^0_a(T) \).

(iii) \( \implies \) (ii) Suppose that \( E^0_a(T + K) \cap \sigma_a(T) \subset E^0_a(T) \). Since \( \Pi^0_a(T + K) \subset E^0_a(T + K) \) is always true, we only have to show that \( \Pi^0_a(T + K) \supset E^0_a(T + K) \). Let
Theorem 2.2 extends [17, Theorem 2.4] which establishes if \( \lambda \notin \sigma_a(T) \), then \( \lambda \notin \sigma_{ab}(T) \). As \( \sigma_{ab}(T) = \sigma_{ub}(T+K) \) and \( \lambda \in \sigma_a(T+K) \), then \( \lambda \in \Pi^0_a(T+K) \). If \( \lambda \in \sigma_a(T) \), then \( \lambda \in \Pi^0_a(T+K) \cap \sigma_a(T) \), and by hypothesis \( \lambda \in \Pi^0_a(T) = \Pi^0_a(T+K) \). So \( \lambda \notin \sigma_{ab}(T) \). As \( \sigma_{ab}(T) = \sigma_{ub}(T+K) \), then \( \lambda \in \Pi^0_a(T+K) \). In the two cases, we have \( \Pi^0_a(T+K) \supset E^0_a(T+K) \).

**Remark 2.4.** (1)– Theorem 2.2 extends [17, Theorem 2.4] which establishes that \( T + F \) satisfies generalized a-Weyl’s theorem when \( T \) is an a-isoloid operator satisfying generalized a-Weyl’s theorem and \( F \) is a finite rank operator commuting with \( T \). Since \( acc \sigma_a(T) = acc \sigma_a(T + F) \) (see [13, Theorem 3.2]), we observe that if \( T \) is an a-isoloid operator, then \( E_a(T + F) \cap \sigma_a(T) \subset E_a(T) \).

(2)– There exists an operator \( T \) which is not a-isoloid, satisfying generalized a-Weyl’s theorem and a finite rank operator commuting with \( T \) such that \( E_a(T + F) \cap \sigma_a(T) \subset E_a(T) \). To see this, consider the operator \( T \) defined on the Hilbert space \( \ell^2(\mathbb{N}) \) by \( T(x_1, x_2, x_3, ...) = (x_1/2, x_2/3, ...) \) and let \( F = 0 \). Then \( \sigma_a(T) = \{0\}, E_a(T) = \emptyset \) and \( \sigma_{SBF}^-(T) = \{0\} \). So \( T \) satisfies generalized a-Weyl’s theorem, \( E_a(T + F) \cap \sigma_a(T) = E_a(T) \), but \( T \) is not a-isoloid.

(3)– Theorem 2.3 extends [13, Theorem 3.4] which establishes that if \( T \) is an a-isoloid operator satisfying a-Weyl’s theorem and if \( F \) is a finite rank operator commuting with \( T \), then \( T + F \) satisfies a-Weyl’s theorem. To see this, we know that \( \alpha(T) < \infty \) if and only if \( \alpha(T + F) < \infty \) (see [18, Lemma 2.1]), so it follows that if \( T \) is a-isoloid then \( E^0_a(T + F) \cap \sigma_a(T) \subset E^0_a(T) \).

There exists quasinilpotent operators which do not satisfy generalized a-Weyl’s theorem. For example, if we consider the operator \( T \) defined on \( \ell^2(\mathbb{N}) \) by \( T(x_1, x_2, x_3, ...) = (0, x_2/2, x_3/3, ...) \), then \( T \) is quasinilpotent but generalized a-Weyl’s theorem fails for \( T \), since \( \sigma_a(T) = \sigma_{SBF}^-(T) = \{0\} \) and \( E_a(T) = \{0\} \). But if a quasinilpotent operator satisfies generalized a-Weyl’s theorem, then the following perturbation result holds.

**Corollary 2.5.** Let \( T \in L(X) \) be a quasinilpotent operator and let \( F \in L(X) \) be a finite rank operator commuting with \( T \). If \( T \) satisfies generalized a-Weyl’s theorem, then \( T + F \) satisfies generalized a-Weyl’s theorem.

**Proof.** If \( T \) is injective, as \( TF \) is a finite rank quasinilpotent operator, then \( TF \) is a nilpotent operator. Since \( T \) is injective, then \( F \) is nilpotent. Therefore \( \sigma_a(T + F) = \sigma_a(T) \) and \( E_a(T + F) = E_a(T) \) (see Lemma 3.1). Moreover, since \( F \) is of finite rank, it follows that \( \sigma_{SBF}^-(T + F) = \sigma_{SBF}^-(T) \). As \( T \) satisfies generalized a-Weyl’s theorem then \( \Delta_a^g(T) = E_a(T) \). So \( \Delta_a^g(T + F) = E_a(T + F) \) and \( T + F \) satisfies generalized a-Weyl’s theorem.

If \( T \) is not injective, then \( \text{iso} \sigma_a(T) = E_a(T) = \{0\} \) and \( T \) is an a-isoloid operator. Therefore by Theorem 2.2, we conclude that \( T + F \) satisfies generalized a-Weyl’s theorem.

**Remark 2.6.** The hypothesis of commutativity in Corollary 2.5 is crucial. Indeed, if we consider the Hilbert space \( H = \ell^2(\mathbb{N}) \), and the operators \( T \) and \( F \) defined on \( H \) by:

\[
T(x_1, x_2, x_3, ...) = (0, x_1/2, x_2/3, ...), \quad F(x_1, x_2, x_3, ...) = (0, -x_1/2, 0, 0, ...).
\]
Then \( T \) is quasi-nilpotent, \( F \) is a finite rank operator which do not commutes with \( T \). Moreover, we have \( \sigma_a(T) = \sigma_{SBF^-}(T) = \{0\} \) and \( E_a(T) = \emptyset \). Hence \( T \) satisfies generalized a-Weyl's theorem. But \( T + N \) does not satisfy generalized a-Weyl's theorem because \( \sigma_a(T + N) = \sigma_{SBF^-}(T + F) = \{0\} \) and \( E_a(T + N) = \{0\} \).

**Theorem 2.7.** Let \( X \) be a Banach space and let \( T \in L(X) \) and \( F \in L(X) \) be a finite rank operator commuting with \( T \). If \( T \) satisfies generalized Weyl's theorem, then the following properties are equivalent.

(i) \( T + F \) satisfies generalized Weyl's theorem;

(ii) \( E(T + F) = \Pi(T + F) \);

(iii) \( E(T + F) \cap \sigma(T) \subset E(T) \).

**Proof.** The equivalence of the two first properties is well known in [9, Theorem 3.2]. Let us show that (ii) is equivalent to (iii). Assume that \( E(T + F) \cap \sigma(T) \subset E(T) \). Let \( \lambda \in E(T + F) \), then \( \lambda \in \text{iso}(T + F) \). If \( \lambda \not\in \sigma(T) \), then \( \lambda \not\in \sigma_D(T) \). Since \( F \) commutes with \( T \), from [10, Theorem 2.7] we have \( \sigma_D(T) = \sigma_D(T + F) \). As \( \lambda \in \sigma(T + F) \), then \( \lambda \in \Pi(T + F) \). If \( \lambda \in \sigma(T) \), then \( \lambda \in E(T + F) \cap \sigma(T) \) and by hypothesis we have \( \lambda \in E(T) \). As \( T \) satisfies generalized Weyl's theorem, it follows that \( \lambda \in \Pi(T) \). As \( \sigma_D(T) = \sigma_D(T + F) \) and \( \lambda \in \sigma(T + F) \), then \( \lambda \in \Pi(T + F) \). Finally we have \( E(T + F) \subset \Pi(T + F) \). As we have always \( E(T + F) \cap \Pi(T + F) \), then \( E(T + F) = \Pi(T + F) \).

Conversely, suppose that \( E(T + F) = \Pi(T + F) \). If \( \lambda \in E(T + F) \cap \sigma(T) \), then \( \lambda \in \Pi(T + F) \cap \sigma(T) \). Therefore \( \lambda \not\in \sigma_D(T + F) \). As \( \sigma_D(T) = \sigma_D(T + F) \) and \( \lambda \in \sigma(T) \), then \( \lambda \in \Pi(T) = E(T) \). Hence \( E(T + F) \cap \sigma(T) \subset E(T) \). \( \square \)

Similarly to Theorem 2.7, we have the following characterization in the case of Weyl's theorem. We give this result without proof.

**Theorem 2.8.** Let \( X \) be a Banach and let \( T \in L(X) \) and \( K \in L(X) \) be a compact operator commuting with \( T \). If \( T \) satisfies Weyl's theorem, then the following properties are equivalent.

(i) \( T + K \) satisfies Weyl's theorem;

(ii) \( E^0(T + K) = \Pi^0(T + K) \);

(iii) \( E^0(T + K) \cap \sigma(T) \subset E^0(T) \).

**Remark 2.9.** (1) It is proved in [5, Theorem 2.6] that generalized Weyl's theorem for isoloid operators is preserved under perturbations by commuting finite rank operators. This result becomes as an immediate consequence of Theorem 2.7. As \( \text{acc} \sigma(T) = \text{acc} \sigma(T + F) \) (see [18, Lemma 2.1]), we observe that if \( T \) is isoloid, then \( E(T + F) \cap \sigma(T) \subset E(T) \).

(2) Since \( \alpha(T) < \infty \) if and only if \( \alpha(T + F) < \infty \), we observe that if \( T \) is isoloid then \( E^0(T + F) \cap \sigma(T) \subset E^0(T) \). Therefore Theorem 2.8 extends a result of W. Y. Lee and S. H. Lee in [18], where Weyl's theorem was proved for \( T + F \) when \( T \) is an isoloid operator satisfying Weyl's theorem, and \( F \) is a finite rank operator commuting with \( T \).
Examples 2.10. (a)– In general generalized a-Weyl’s theorem, a-Weyl’s theorem, generalized Weyl’s theorem and Weyl’s theorem are not transmitted from an operator to a commuting finite rank perturbation as the following example shows.

Let \( S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) be an injective quasinilpotent operator which is not nilpotent. We define \( T \) on the Banach space \( X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \) by \( T = I \oplus S \) where \( I \) is the identity operator on \( \ell^2(\mathbb{N}) \). Then \( \sigma(T) = \sigma_a(T) = \{0,1\} \) and \( E_a(T) = \{1\} \). It follows from [9, Example 2] that \( \sigma_{BW}(T) = \{0\} \) and \( \sigma_{SBF}(T) = \{0\} \). Hence \( \sigma_a(T) \setminus \sigma_{SBF}(T) = E_a(T) = \{1\} \) and \( T \) satisfies generalized a-Weyl’s theorem, so it satisfies a-Weyl’s theorem, generalized Weyl’s theorem and Weyl’s theorem.

We define the operator \( U \) on \( \ell^2(\mathbb{N}) \) by \( U(\xi_1, \xi_2, ..., \xi_n, ...) = (-\xi_1, 0, 0, ...) \) and \( F = U \oplus 0 \) on the Banach space \( X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \). Then \( F \) is a finite rank operator commuting with \( T \). On the other hand, \( \sigma(T + F) = \sigma(T) = \{0,1\} \) and \( E_a(T + F) = \{0,1\} \). As \( \sigma_{SBF}(T + F) = \sigma_{SBF}(T) = \{0\} \), then \( \sigma_a(T + F) \setminus \sigma_{SBF}(T + F) = \{1\} \neq \sigma_{a}(T + F) \) and \( T + F \) does not satisfy generalized a-Weyl’s theorem. Not that \( E_a(T + F) \cap \sigma_a(T) \not\subset E_a(T) \). Moreover, \( E(T + F) = \{0,1\} \), and as by [4, Theorem 4.3] we have \( \sigma_{BW}(T + F) = \sigma_{BW}(T) = \{0\} \), then \( T + F \) does not satisfy generalized Weyl’s theorem. Observe that \( E(T + F) \cap \sigma(T) \not\subset E(T) = \{1\} \).

Moreover we have \( \sigma_a(T + F) = \{0,1\} \) and \( \sigma^0(T + F) = \{0\} \). As \( \sigma(T + F) = \{0,1\} \) then \( \Delta(T + F) \neq \sigma^0(T + F) \) and \( T + F \) does not satisfy Weyl’s theorem. So \( T + F \) does not satisfy a-Weyl’s theorem. Note that \( \sigma_a(T) \not\subset \sigma^0(T) \), \( E_a(T + F) \cap \sigma_a(T) = \{0\} \cap \{0,1\} \not\subset E^0_a(T) = \emptyset \).

(b)– Theorem 2.2 and Theorem 2.7 do not extend to a commuting compact perturbation. Indeed, if we consider on the Hilbert space \( \ell^2(\mathbb{N}) \) the operators \( T = 0 \) and \( Q \) defined by \( Q(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, x_4/4, ...) \). Then \( Q \) is a compact operator commuting with \( T \). Moreover, we have \( \sigma_a(T) = \{0\} \), \( \sigma_{SBF}(T) = \emptyset \), \( E_a(T) = \{0\} \). Hence \( T \) satisfies generalized a-Weyl’s theorem. So it satisfies generalized Weyl’s theorem. But generalized a-Weyl’s theorem and generalized Weyl’s fail for \( T + Q = Q \). Indeed \( \sigma_{SBF}(T + Q) = \emptyset \), \( E_a(T + Q) = \{0\} \) and \( \sigma(T + Q) = \{0\} \), \( \sigma_{BW}(T + Q) = \{0\} \), \( E(T + Q) = E(T) = \{0\} \). Thought we have \( E_a(T + Q) \cap \sigma_a(T) \subset E_a(T) \) and \( E(T + Q) \cap \sigma(T) \subset E(T) \).

3 Nilpotent perturbations

Let \( T \in L(X) \) and let \( N \) be a nilpotent operator commuting with \( T \). In a first step we prove that \( T \) and \( T + N \) have the same isolated eigenvalues in the approximate spectrum.

Lemma 3.1. Let \( X \) be a Banach space and let \( T \in L(X) \). If \( N \in L(X) \) is a nilpotent operator commuting with \( T \), then \( E_a(T + N) = E_a(T) \).

Proof. Let \( \lambda \in E_a(T) \) be arbitrary. There is no loss of generality if we assume
that $\lambda = 0$. As $N$ is nilpotent we know that $\sigma_a(T + N) = \sigma_a(T)$, thus $0 \in \text{iso}_a(T + N)$. Let $m \in \mathbb{N}$ be such that $N^m = 0$. If $x \in N(T)$, then $(T + N)^m(x) = \sum_{k=0}^{m} c_k T^k N^{m-k}(x) = 0$. So $N(T) \subset N(T + N)^m$. As $\alpha(T) > 0$, it follows that $\alpha((T + N)^m) > 0$ and this implies that $\alpha(T + N) > 0$. Hence $0 \in E_a(T + N)$. So $E_a(T) \subset E_a(T + N)$. By symmetry we have $E_a(T) = E_a(T + N)$.

In the next theorem, we consider an operator $T \in L(X)$ satisfying generalized a-Weyl’s theorem, a nilpotent operator commuting with $T$, and we give necessary and sufficient conditions for $T + N$ to satisfy generalized a-Weyl’s theorem.

**Theorem 3.2.** Let $X$ be a Banach space and $T \in L(X)$ and $N \in L(X)$ be a nilpotent operator commuting with $T$. If $T$ satisfies generalized a-Weyl’s theorem, then the following statements are equivalent.

(i) $T + N$ satisfies generalized a-Weyl’s theorem;
(ii) $\sigma_{SBF^-}(T + N) = \sigma_{SBF^-}(T)$;
(iii) $E_a(T) = \Pi_a(T + N)$.

**Proof.** (i) $\iff$ (ii) Assume that $T + N$ satisfies generalized a-Weyl’s theorem, then

$$\sigma_a(T + N) \setminus \sigma_{SBF^-}(T + N) = E_a(T + N).$$

As $\sigma_a(T + N) = \sigma_a(T)$ and $E_a(T + N) = E_a(T)$ then $\sigma_a(T) \setminus \sigma_{SBF^-}(T + N) = E_a(T)$. Since $T$ satisfies generalized a-Weyl’s theorem, then

$$\sigma_a(T) \setminus \sigma_{SBF^-}(T) = E_a(T).$$

Hence $\sigma_{SBF^-}(T + N) = \sigma_{SBF^-}(T)$. Conversely, assume that $\sigma_{SBF^-}(T + N) = \sigma_{SBF^-}(T)$, then $T$ satisfies generalized a-Weyl’s theorem it follows that $T + N$ satisfies also generalized a-Weyl’s theorem.

(i) $\iff$ (iii) Assume that $T + N$ satisfies generalized a-Weyl’s theorem, then from [3, Corollary 3.2], we have $E_a(T + N) = \Pi_a(T + N)$. Therefore $E_a(T) = \Pi_a(T + N)$. Conversely, assume that $E_a(T) = \Pi_a(T + N)$. Since $T$ satisfies generalized a-Weyl’s theorem, then $T$ satisfies generalized a-Browder’s theorem. As we know from [2, Theorem 2.2] that a-Browder’s theorem is equivalent to generalized a-Browder’s theorem, then $T$ satisfies a-Browder’s theorem. So $\sigma_{SBF^-}(T) = \sigma_{ab}(T)$. From [12, Theorem 2.13], we know that $\sigma_{SF^-}(T) = \sigma_{SF^-}(T + N)$. By [1, Theorem 3.65], we know that $\sigma_{ab}(T) = \sigma_{SF^-}(T) \cup \text{aco}_a(T)$. Hence $\sigma_{ab}(T) = \sigma_{ab}(T + N)$. Therefore $\sigma_{SF^-}(T + N) = \sigma_{ab}(T + N)$ and $T + N$ satisfies a-Browder’s theorem. So it satisfies generalized a-Browder’s that is $\sigma_a(T + N) \setminus \sigma_{SBF^-}(T + N) = \Pi_a(T + N)$. As by assumption $E_a(T) = \Pi_a(T + N)$, it follows that $\sigma_a(T + N) \setminus \sigma_{SBF^-}(T + N) = E_a(T + N)$ and so $T + N$ satisfies generalized a-Weyl’s theorem.

**Theorem 3.3.** Let $X$ be a Banach space and let $T \in L(X)$ be an operator satisfying generalized a-Weyl’s theorem. If $E_a(T) \subset \text{iso}(T)$ and if $N \in L(X)$ is a nilpotent operator commuting with $T$, then $T + N$ satisfies generalized a-Weyl’s theorem.
Proof. Let $\lambda \in E_a(T)$, since $E_a(T) \subset iso\sigma(T)$, then $\lambda \in E(T)$.

As $T$ satisfies generalized a-Weyl’s theorem, from [3, Theorem 3.7] $T$ satisfies generalized Weyl’s theorem. Hence $\lambda \in \Pi(T)$. As we know that $\sigma_D(T) = \sigma_D(T + N)$, then $\lambda \in \Pi(T + N)$. Hence $\lambda \in \Pi_a(T + N)$. Consequently we have $E_a(T) = \Pi_a(T + N)$. Conversely if $\lambda \in \Pi_a(T + N)$, then $\lambda \in E_a(T + N)$. As we know that $E_a(T) = E_a(T + N)$, (see Lemma 3.1), then $\lambda \in E_a(T + N)$. So $E_a(T) = \Pi_a(T + N)$. From Theorem 3.2, $T + N$ satisfies generalized a-Weyl’s theorem. $\square$

Remark 3.4. 1- The hypothesis of commutativity in the Theorem 3.2 corollary is crucial. The following example shows that if we do not assume that $N$ commutes with $T$, then the result may fails. Let $H = \ell^2(\mathbb{N})$, and let $T$ and $N$ defined by:

$$T(x_1, x_2, x_3, \ldots) = (0, x_1/2, x_2/3, \ldots), \quad N(x_1, x_2, x_3, \ldots) = (0, -x_1/2, 0, 0, \ldots).$$

Clearly $N$ is a nilpotent operator which does not commute with $T$. Moreover, we have $\sigma_a(T) = \sigma_{SBF^{-}}(T) = \{0\}$ and $E_a(T) = \emptyset$. Therefore $T$ satisfies generalized a-Weyl’s theorem. But $T + N$ does not satisfy generalized a-Weyl’s theorem because $\sigma_a(T + N) = \sigma_{SBF^{+}}(T + N) = \{0\}$ and $E_a(T + N) = \{0\}$.

(2) Generally, generalized a-Weyl’s theorem does not extend to a quasinilpotent perturbation: Define on the Banach space $\ell^2(\mathbb{N})$ the operator $T = 0$ and the quasinilpotent operator $Q$ defined by $Q(x_1, x_2, x_3, ...) = (x_2/2, x_3/3, x_4/4, ...)$. Then $\sigma_a(T) = \{0\}$ and $\sigma_{SBF^{-}}(T) = \emptyset$. Moreover we have $E_a(T) = \{0\}$. Hence $T$ satisfies generalized a-Weyl’s theorem. But generalized a-Weyl’s theorem does not hold for $T + Q = Q$, since $\sigma_{SBF^{-}}(T + Q) = \sigma_a(T + Q) = \{0\}$ and $E_a(T + Q) = \{0\}$.

Open questions: The proof of Theorem 3.2 suggests the following questions:

1. let $T \in L(X)$ and let $N \in L(X)$ be a nilpotent operator commuting with $T$. Under which conditions $a(T + N)$ is finite if $a(T)$ is finite?

2. let $T \in L(X)$ and let $N \in L(X)$ be a nilpotent operator commuting with $T$. Under which conditions $R((T + N)^m)$ is closed for $m$ large enough if $R(T^m)$ is closed for $m$ large enough?

3. let $T \in L(X)$ and let $N \in L(X)$ be a nilpotent operator commuting with $T$. Under which conditions $\Pi_a(T + N) = \Pi_a(T)$?

References


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