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Functional Analysis, Approximation and Computation **2:1** (2010), 19–32

HERMITIAN SUBSPACES AND FUGLEDE OPERATORS

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Abstract

We survey a little of the theory of “hermitian” and “normal” Banach space elements through the medium of a concept of “hermitian subspace”.

1 Hermitian subspaces

Suppose A is a complex linear algebra, with identity 1 : then a *Hermitian subspace* of A is a real-linear subspace $H \subseteq A$ for which

1.1 $$H_{\cap} iH = O \equiv \{0\}$$

with

1.2 $$1 \in H \ .$$

For example the real scalars $H = \mathbf{R} \subseteq A$ constitute a Hermitian subspace; the intersection of two Hermitian subspaces is Hermitian; an easy Zorn Lemma argument will show that every Hermitian subspace is contained in a maximal Hermitian subspace.

Hermitian subspaces $H \subseteq A$ give rise to complex “Palmer” subspaces $H + iH \subseteq A$ carrying involutions: we can define $\times : H + iH \to H + iH$ by setting

1.3 $$(h + ik)^\times = h - ik \ (h, k \in H) \ .$$

Evidently, for arbitrary $x, y \in H + iH$ and $\alpha, \beta \in \mathbf{C}$,

1.4 $$(\alpha x + \beta y)^\times = \overline{\alpha} x^\times + \overline{\beta} y^\times \ ; \ (x^\times)^\times = x \ ; \ 1^\times = 1.$$

Conversely a partially defined involution $\times : K \to K \subseteq A$ gives rise to the Hermitian subspace $H = \{a \in K : a^\times = a\}$. When A is a topological algebra then it is desireable that $H + iH$ is norm closed, and also that $\times$ is continuous:

2010 *Mathematics Subject Classifications.* 47A05.
Key words and Phrases. Hermitian subspaces, Fuglede operators.
Received: Februaru 20, 2010
Communicated by Dragan S. Djordjević

in a Banach algebra this asks for a certain mutual "orthogonality" between H and iH. We have looked [H3] at partially defined involutions before, but missed the clarity gained by focussing on the underlying Hermitian subspaces.

It is not clear that the Palmer subspace is closed under multiplication. Generally, if $a, b \in H + iH$ then

1.5 $$\{ab, b^\times a^\times\} \subseteq H + iH \Longleftrightarrow \{ab + b^\times a^\times, i(ab - b^\times a^\times)\} \subseteq H + iH \ ,$$

and

1.6 $$\{ab, b^\times a^\times\} \subseteq H \Longleftrightarrow \{ab + b^\times a^\times, i(ab - b^\times a^\times)\} \subseteq H \ .$$

We shall call the Hermitian subspace $H \subseteq A$ a *Lie* subspace if

1.7 $$i[H, H] \equiv \{i(ab - ba) : a, b \in H\} \subseteq H \ ,$$

and a *Jordan* subspace if instead

1.8 $$]H, H[\equiv \{ab + ba : a, b \in H\} \subseteq H \ .$$

When H is a Lie subspace then, for arbitrary $a, b \in H + iH$

1.9 $$[a, b]^\times = [b^\times, a^\times] \in H + iH \ ;$$

if instead H is a Jordan subspace then

1.10 $$]a, b[^\times =]b^\times, a^\times[\in H + iH \ .$$

If H is both a Lie and a Jordan subspace then $H + iH \subseteq A$ is a subalgebra, with for arbitrary $a, b \in H + iH$

1.11 $$(ab)^\times = b^\times a^\times \in H + iH \ .$$

2 Induced involutions

If $T : A \to B$ is a linear mapping, with $T(1) = 1$, then Hermitian subspaces of A are mapped into Hermitian subspaces of B and, if in addition $T^{-1}(0) = \{0\}$, the counterimages of Hermitian subspaces of B are Hermitian subspaces of A. In particular if

2.1 $$1 \in A \subseteq B$$

then a Hermitian subspace $H \subseteq A$ is also a Hermitian subspace of the larger algebra B, giving rise to the same involution. If instead $1 \in B \subseteq A$ then $H_\cap B$ is a Hermitian subspace of B. If

2.2 $$AJ + JA \subseteq J \subseteq A$$

is a two sided ideal then the image $(H+J)/J \subseteq A/J$ is a Hermitian subspace of the quotient algebra A/J, with

2.3 $$(a+J)^\times = a^\times + J \in (H+iH+J)/J \subseteq A/J \ .$$

If $H \subseteq A$ and $K \subseteq B$ are Hermitian subspaces then so also is

2.4 $$H \oplus K \subseteq A \oplus B \ , \ \textit{with} \ (a \oplus b)^\times = a^\times \oplus b^\times \ (a \in H+iH, b \in K+iK) \ ,$$

and more generally $\prod_j H_j \subseteq \prod_j A_j$. In particular, if Ω is a set and $H \subseteq A$ is Hermitian then also

2.5 $$H^\Omega \subseteq A^\Omega \ , \ \textit{with} \ a^\times(\xi) = a(\xi)^\times \ (\xi \in \Omega)$$

is a Hermitian subspace inducing pointwise involution. If $A \subseteq B(X)$ is among the (bounded) operators on a (Banach) space X then a Hermitian subspace $H \subseteq A$ has a sort of dual,

2.6 $$H^\ddagger = \{T^\dagger : T \in H\} \subseteq B(X^\dagger) \ , \ \textit{with} \ (T^\dagger)^\times = (T^\times)^\dagger \ (T \in H+iH) \ .$$

Analagous to (2.4), Hermitian subspaces are induced in tensor products: if $H \subseteq A$ and $K \subseteq B$ are Hermitian subspaces then so also is (the subspace generated by)

2.7 $$H \otimes K \subseteq A \otimes B \ , \ \textit{with} \ (a \otimes b)^\times = a^\times \otimes b^\times \ (a \in H+iH, b \in K+iK)$$

3 Vidav Hermitians

The archetypical Hermitians come ([BD] Lemma 5.7) from the *numerical range*: in a Banach algebra $a \in A$ is declared to be *Vidav Hermitian*, $a \in \mathrm{Re}(A)$, iff

3.1 $$V_A(a) \equiv \{\varphi(a) : \varphi \in \mathrm{State}(A)\} \subseteq \mathbf{R} \ ,$$

where

3.2 $$\mathrm{State}(A) = \{\varphi \in A^\dagger : \|\varphi\| = \varphi(1) = 1\} \ .$$

Equivalent to (3.1) are ([BD] Lemma 5.2; [P3] Theorem 2.6.7) each of the following two conditions:

3.3 $$\lim_{t \to 0} \frac{\|1+ita\| - 1}{t} = 0 \ ;$$

3.4 $$\forall t \in \mathbf{R} \ , \ \|e^{ita}\| = 1 \ .$$

We shall write the involution in this case as

3.5 $$* : \mathrm{Reim}(A) \to \mathrm{Reim}(A) = \mathrm{Re}(A) + i\mathrm{Re}(A) \ .$$

The Hermitian subspace $H = \mathrm{Re}(A)$ is ([BD] Lemma 5.4; [P3] Theorem 2.6.7) also a Lie subspace in the sense of (1.7). If $T : A \to B$ is isometric linear, with $T(1) = 1$, then

3.6 $$a \in A \Longrightarrow V_B(Ta) = V_A(a) \ .$$

Certainly if $\psi \in \mathrm{State}(B)$ then $\varphi = \psi \circ T \in \mathrm{State}(A)$; conversely if $\varphi \in \mathrm{State}(A)$ and we set

$$\psi_0(Ta) = \varphi(a) \ (a \in A)$$

then $\psi_0 : T(A) \to \mathbf{C}$ is well-defined, linear, with $\|\psi_0\| = \psi_0(1) = 1$: now Hahn-Banach extension gives $\psi \in \mathrm{State}(B)$. In particular when the Hermitian subspace $H \subseteq A$ generates the Vidav involution $\times = *$ on the Palmer subspace of a Banach algebra A then this remains the case in any closed subalgebra $B \subseteq A$ and any quotient A/J induced by a closed ideal $J \subseteq A$. The involution induced on the direct sum $A \oplus B$ by the Vidav involutions on A and B is again Vidav, as is the involution induced on $C(\Omega, A)$ when Ω is compact Hausdorff.

The dual of the Vidav Hermitians is again the Vidav Hermitians:

3.7 $$A = B(X) \Longrightarrow \{a^\dagger : a \in \mathrm{Re}(A)\} \subseteq \mathrm{Re}\ B(X^\dagger) \ :$$

this is because (3.6) the numerical range of the dual of an operator is the same as the numerical range of the operator. Alternatively the expression in (3.4) is unchanged when $a \in B(X)$ is replaced by $a^\dagger \in B(X^\dagger)$.

Generally in a Banach algebra an easy Hahn-Banach argument ([BD] Theorem 2.6; [H1] Theorem 9.10.2) gives

3.8 $$\sigma_A(a) \subseteq V_A(a) \ ;$$

it follows therefore that a non invertible Hermitian element must be a topological zero divisor:

3.9 $$a \in A \ Hermitian \Longrightarrow \sigma_A(a) = \partial\sigma_A(a) \subseteq \tau_A^{left}(a) \cap \tau_A^{left}(a) \ .$$

Conversely ([BD] Theorem 2.10; [P3] Theorem 2.6.7)

3.10 $$\mathrm{cvx}\ \sigma_A(a) = V_A(a) \ ,$$

which means that the only quasinilpotent Hermitian is zero.

4 Elementary operators

In any (Banach) algebra each element gives rise to two (bounded) operators acting on the underlying linear space. More generally if A and B are Banach algebras, and M is a *Banach (A, B) bimodule*, then elements $a \in A$ and $b \in B$ induce (bounded) operators

4.1 $$L_a : m \mapsto am \ ; \ R_b : m \mapsto mb \ .$$

The "mixed associative law" says that each L_a commutes with each R_b. We shall say that the module M is (A,B) *prime* if [M],[HH] there is implication, for arbitrary $a \in A$ and $b \in B$,

4.2 $$aMb = O \Longrightarrow \big(a = 0 \in A \textit{ or } b = 0 \in B\big) \ ,$$

and (A,B) *ultra prime* if for each $a \in A$ and $b \in B$

4.3 $$\|L_a R_b\| = \|a\| \ \|b\| \ .$$

Obviously ultra prime modules are prime; for example (4.3) holds when $A = B(X)$, $B = B(Y)$ and $Y^\dagger \odot X \subseteq M \subseteq BL(Y,X)$. The *elementary operators* on a bimodule M are the operators

4.4 $$L_A \circ R_B = \{L_a \circ R_b : n \in \mathbf{N}, a \in A^n, b \in B^n\} \subseteq B(M) \ ,$$

where

4.5 $$(L_a \circ R_b)(m) = \sum_{j=1}^n a_j m b_j \ (m \in M, a \in A^n, b \in B^n) \ .$$

Now if the module M is prime then the elementary operators are linearly isomorphic to the tensor product $A \otimes B$, and if M is ultra prime the operator norm of an elementary operator on M induces a uniform crossnorm on the tensor product $A \otimes B$. Thus if M is prime then Hermitian subspaces $H \subseteq A$ and $K \subseteq B$ combine to give a Hermitian subspace of $L_A \circ R_B$ supporting an involution $\sim$ defined by setting

4.6 $$(L_a \circ R_b)^\sim = L_{a^\times} \circ R_{b^\times} \ (a \in A^n, b \in B^n) \ ,$$

where of course

4.7 $$(a_1, a_2, \ldots, a_n)^\times = (a_1^\times, a_2^\times, \ldots, a_n^\times) \ , \ (b_1, b_2, \ldots, b_n)^\times = (b_1^\times, b_2^\times, \ldots, b_n^\times) \ .$$

5 Anderson/Foias

Possibly the simplest non trivial elementary operators are the "generalized inner derivations" induced by $a \in A$ and $b \in B$:

5.1 $$(L_a - R_b)(m) = am - mb \ (m \in M) \ .$$

If in particular the involutions $\times = *$ on the algebras A and B are the Vidav involutions then for arbitrary $a \in A$, $b \in B$,

5.2 $$(L_a - R_b)^\sim \equiv L_{a^*} - R_{b^*} = (L_a - R_b)^* \ :$$

if we start with Vidav Hermitians on A and B the construction of (4.6) gives back Vidav Hermitians for (generalized inner) derivations. This is clear by comparing numerical ranges: if M is ultra prime

5.3 $$V_{B(M)}(L_a - R_b) \subseteq V_A(a) - V_B(b) \ ,$$

since $\Phi \in B(M)^\dagger$ generates $\varphi \in A^\dagger$ and $\psi \in B^\dagger$ by setting, for each $a \in A$, $b \in B$,

5.4 $$\varphi(a) = \Phi(L_a)\ ,\ \psi(b) = \Phi(R_b)\ .$$

Generally if M is a Banach (A, B) bimodule then its dual $M^\dagger$ is a Banach (B, A) bimodule, with

5.5 $$L_b = (R_b)^\dagger\ ,\ R_a = (L_a)^\dagger\ ,\ (a \in A, b \in B)\ .$$

It would have been nice to report that (5.2) extended to more general elementary operators; unfortunately an old example of Anderson and Foias says different: **Example** ([AF] Theorem 5.8): *if* $0 \neq p^* = p = p^2 \neq 1$ *in* $A = B(X)$ *for Hilbert space* X *then* L_pR_p *is not Hermitian.*

To see this suppose $\dim(X) = 2 = 2 \dim p(X)$: with

5.6 $$a \sim \begin{pmatrix} i/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} : \begin{pmatrix} p(X) \\ (1-p)(X) \end{pmatrix} \to \begin{pmatrix} p(X) \\ (1-p)(X) \end{pmatrix}$$

observe $\exp(itL_pR_p) = I + (e^{it} - 1)L_pR_p$ and hence

5.7 $$t = 3\pi/2 \Longrightarrow q = \exp(iL_pR_p)(a) \sim \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\ .$$

Evidently $q^* = q = q^2$ is another projection: but now we have

5.8 $$\|q\| = 1\ ;\ \|a^*a\|^2 < 3/4\ .$$

Notice also ([AF] Example 5.9) that therefore $L_p - R_p$ is Hermitian, but

5.9 $$(L_p - R_p)^2 = (L_p + R_p) - 2L_pR_p$$

is not.

It is not clear whether or not either L_pR_p and $(L_p - R_p)^2$ are even in the Palmer subspace: but it follows from a result of Palmer ([P1];[P3] Theorem 2.6.8;[BD] Theorem 6.3) that there must be $T \in B(A)$ for which

5.10 $$T \in \mathrm{Re}\ B(A)\ \textit{and}\ T^2 \notin \mathrm{Reim}\ B(A)\ .$$

When $M = A$ or $M = B$ the inclusion (5.3) partially reverses: for example $\varphi \in A^\dagger$ induces $\Phi \in B(A)^\dagger$ by setting

5.11 $$\Phi(T) = \varphi(T1)\ (T \in B(A))\ .$$

If in particular a Banach (A, B) bimodule M, or the completed tensor product $A \otimes B$, is a Hilbert space then (5.2) of course extends to all of $L_A \circ R_B \subseteq B(M)$, and the subalgebra generated by $\mathrm{Reim}(A) \otimes \mathrm{Reim}(B)$. The analogue of (5.2) holds also on a uniformly cross normed tensor product: if we start with the Vidav involutions in (2.7) then

5.12 $$a^* \otimes 1 - 1 \otimes b^* = (a \otimes 1 - 1 \otimes b)^*\ (a \in A, b \in B)\ .$$

When $0 \neq p^* = p = p^2 \neq 1$ then, since the least uniform cross norm of the tensor $\exp(itp \otimes p)$ is the same as the norm of $\exp(itL_pR_p)$, the Anderson-Foias pathology also applies to all uniformly cross normed tensor products.

6 Fuglede operators

Suppose $H \subseteq A = B(X)$ is a Hermitian subspace: then we shall say that $T \in H + iH$ is *Fuglede* iff

6.1 $$T^{-1}(0) \subseteq T^{\times -1}(0) \subseteq X \ .$$

Segres and Bachir have noticed ([SB] Proposition 3.2) that, with $\times = *$ and $a \in A = B(X)$, $b \in B = B(Y)$ and $Y^\dagger \odot X \subseteq M \subseteq BL(Y, X)$,

6.2 $$a \in A = B(X) \ Fuglede \iff L_a \in B(M) \ Fuglede \ ;$$

6.3 $$b^\dagger \in B(Y^\dagger) \ Fuglede \iff R_b \in B(M) \ Fuglede \ ;$$

6.4 $$L_a, R_b \ Fuglede \implies L_a R_b \ {}^\sim Fuglede \ .$$

Their argument works for more general involutions; for suppose that $a \in H + iH$ is ${}^\times$Fuglede: then for arbitrary $u \in BL(Y, X)$ there is implication

$$L_a(u) = 0 \iff au = 0 \iff au(X) = O \implies a^\times u(X) = O \iff L_{a^\times} u = 0 \ .$$

Conversely if L_a is Fuglede then for arbitrary $x \in X$ and $\varphi \in Y^\dagger$ there is implication

$$a(x) = 0 \implies L_a(\varphi \odot x) = \varphi \odot ax = 0 \implies \varphi \odot a^\times x = L_{a^\times}(\varphi \odot x) = 0 \ .$$

By the Hahn Banach theorem it follows $a^\times(x) = 0$. This proves (6.2) both ways. Towards (6.3), if $b^\dagger \in K^\ddagger + iK^\ddagger$ is Fuglede there is implication

$$R_b(u) = 0 \iff ub = 0 \iff b^\dagger(Y^\dagger u) = Y^\dagger(ub) = O \implies Y^\dagger(ub^\times) = (b^\dagger)^\times(Y^\dagger u) = O \iff R_{b^\times} u = 0 \ .$$

Conversely if R_b is Fuglede then

$$b^\dagger \varphi = 0 \iff (\varphi \odot X)b = O \implies (\varphi \odot X)b^\times = O \iff \varphi b^\times = 0 \ .$$

Finally, for (6.4) there is implication

$$aub = 0 \implies a^\times ub = 0 \implies a^\times ub^\times = 0 \ .$$

Of course, thanks to Anderson/Foias, when $\times = *$ "Fuglede" in the right hand side of (4.4) may not be also in the sense of Vidav. It holds instead with $\sim$: there is implication for arbitrary $m \in M$

6.5 $$amb = 0 \in M \implies a^* m b^* = 0 \in M \ .$$

This of course is a statement involving the Vidav Hermitian property for the participating $a \in A$, $b \in B$.

7 Normal operators

We declare $a \in H + iH \subseteq A$ to be ${}^\times$*normal* if

7.1 $$a^\times a = aa^\times \in A \ :$$

there is no assumption that this product is again in the Palmer subspace. Evidently if M is an (A, B) bimodule

7.2 $$a^\times a = aa^\times \in A\ ,\ b^\times b = bb^\times \in B \implies L_a - R_b \in B(M)\ {}^\sim normal\ .$$

Specialising to $A = B = M = B(X)$ and the Vidav definition $\times = *$, Fong ([F] Lemma 3) has shown, for $T \in B(X)$,

7.3 $$T\ normal \implies T\ Fuglede\ .$$

From (7.2) and (7.3) it follows

7.4 $$a^*a = aa^* \in A\ ,\ b^*b = bb^* \in B \implies (L_a - R_b)^{-1}(0) \subseteq (L_a - R_b)^{*-1}(0) \subseteq M\ :$$

if $a \in A$ and $b \in B$ are both normal then there is implication, for arbitrary $m \in M$,

7.5 $$am = mb \Longrightarrow a^*m = mb^*\ .$$

We also have implication

7.6 $$a^*a = aa^* \in A\ ,\ b^*b = bb^* \in B \implies (L_aR_b)^\sim(L_aR_b) = (L_aR_b)(L_aR_b)^\sim\ .$$

While we cannot apply Fong (7.3) to deduce that L_aR_b is ${}^\sim$Fuglede, we recall that this follows from (6.4) above. More generally if $a \in A^n$ and $b \in B^n$ are commuting normal tuples then

7.7 $$(L_a \circ R_b)(L_a \circ R_b)^\sim = (L_a \circ R_b)^\sim(L_a \circ R_b)\ .$$

Also, analagous to (1.5) and (1.6), if

7.8 $$(ab)(b^*a^*) = (b^*a^*)(ab)$$

then, using (7.5), there is implication

7.9 $$ab, b^*a^*\ normal \iff ab + b^*a^*, i(ab - b^*a^*)\ normal\ .$$

If $H \subseteq A$ is a Lie subspace with the property that

7.10 $$\forall\ a \in H\ :\ a^2 \in H + iH\ normal$$

then it follows

7.11 $$\forall\ a \in H\ :\ a^2 \in H\ .$$

The argument, and more ([BD] Theorem 6.3), is due to Palmer ([P1]; [P3] Theorem 2.6.8): if $a^2 = h + ik \in H$ with $a, h, k \in H$ then

$$a(h + ik) = a^2 = (h + ik)a$$

giving $ah - ha = i(ka - ak)$, $ka - ak = i(ha - ah)$ and hence $\{ah - ha, ak - ka\} \subseteq H \cap iH = \{0\}$. Thus $ah - ha = ak - ka = 0$ and hence $hk - kh = a^2k - ka^2 = 0$. Now $i\sigma(a) \subseteq \sigma(a^2) - \sigma(h) \subseteq \mathbf{R}$ giving $\sigma(k) \subseteq i\mathbf{R}_\cap\mathbf{R} = \{0\}$ and hence $k = 0$.

8 Orthogonality

If we call $T \in H + iH \subseteq B(X)$ *reduced* iff

8.1 $$T^{-1}(0) \subseteq (TT^{\times})^{-1}(0) \subseteq X \ ,$$

and *natural* iff

8.2 $$(TT^{\times})^{-1}(0) \subseteq T^{\times -1}(0) \subseteq X \ ,$$

then evidently if $T \in H + iH$ there is implication

8.3 $$T \text{ reduced natural } \Longrightarrow T \text{ Fuglede } \Longrightarrow T \text{ reduced} \ .$$

For example if X is a Hilbert space and $H \subseteq A = B(X)$ is the usual Hermitians then every operator $T \in A$ is "natural", since

8.4 $$T^{-1}(0) \perp T^*(X) \ .$$

Shulman and Turowska [ST] call this the "non commutative Fuglede theorem for Hilbert space operators". Now for $T \in B(X)$ with a Banach space X there is implication

8.5 $$T \text{ normal } \Longrightarrow T \text{ reduced} \ ,$$

and hence also

8.6 $$T \text{ normal natural } \Longrightarrow T \text{ Fuglede} \ .$$

Thus if $a \in A^n$ and $b \in B^n$ are commuting normal n-tuples there is implication, with $T = L_a \circ R_b$,

8.7 $$T^{-1}(0)_\cap T^\sim(M) = O \Longrightarrow T^{-1}(0) \subseteq T^{\sim -1}(0) \subseteq M \ .$$

This compares with Shulman's observation that, if $a \in A^2$ and $b \in A^2$ are commuting normal pairs in A,

8.8 $$(L_a \circ R_b)^{-1}(0)_\cap (L_a \circ R_b)B(A) = O \Longrightarrow (L_a \circ R_b)^{-1}(0) \subseteq (L_{a^*} \circ R_{b^*})^{-1}(0) \ .$$

We may also recall another Fong result ([F] Theorem A): if $T \in \text{Reim } B(X)$ then

8.9 $$T \text{ normal } \Longrightarrow T^{-1}(0)_\cap T^{*-1}(0) \perp T(X) \ :$$

for arbitrary $x \in X$ there is implication

8.10 $$Tx = T^*x = 0 \Longrightarrow \|x\| = \text{dist}(x, T(X)) \ .$$

Together with (7.3) this says that normal operators "have ascent ≤ 1". This in turn ([M1] Theorem 4.7) shows that (3.9) extends to normal operators on Banach spaces:

8.11 $$\|\cdot\| \leq k\|T(\cdot)\| \ , \ T^{\dagger -1}(0)_\cap T^\dagger(X^\dagger) = \{0\} \Longrightarrow T \in B(X)^{-1} \ .$$

Indeed if $T \in B(X)$ is bounded below then $T^\dagger \in B(X^\dagger)$ is onto, and if $T^\dagger$ has ascent one is also one-one and therefore invertible. The extension of (8.11) to Banach algebra elements follows by considering left multiplications.

9 Strong Fuglede

We shall call $T \in B(X)$ *strongly Fuglede* iff $T^\times$ is *majorised by* T: there is ([H1] Chapter 10) $\ell > 0$ for which

9.1 $$\forall x \in X \ : \|T^\times(x)\| \le \ell \ \|T(x)\| \ ,$$

and *algebraically Fuglede* iff $T^\times$ factors through T: there is $U \in B(X)$ for which

9.2 $$T^\times = UT \ .$$

Essentially the same argument shows that (6.2), (6.3) and (6.4) extend to strongly Fuglede operators:

9.3 $$a \in A = B(X) \ strongly\ Fuglede \iff L_a \in B(M) \ strongly\ Fuglede \ ;$$

9.4 $$b^\dagger \in B(Y^\dagger) \ strongly\ Fuglede \iff R_b \in B(M) \ strongly\ Fuglede \ ;$$

9.5 $$L_a, R_b \ strongly\ Fuglede \implies L_a R_b \ strongly\ Fuglede \ .$$

For example if $a \in A = B(X)$ is strongly Fuglede then for arbitrary $u \in A$ and arbitrary $x \in X$

$$\|L_{a^\times}(u)x\| = \|a^\times ux\| \le \ell \|aux\| = \ell \|L_a(u)x\|$$

giving $\|L_{a^\times}u\| \le \ell \|L_a u\|$. Conversely if L_a is Fuglede then for arbitrary $x \in X$ and $\varphi \in Y^\dagger$

$$\|\varphi\| \ \|a^\times x\| = \|L_{a^\times}(\varphi \odot x)\| \le \ell \|L_a(\varphi \odot x)\| = \ell \|\varphi\| \ \|ax\| \ .$$

By the Hahn Banach theorem it follows $\|a^\times x\| \le \ell \|ax\|$. This proves (9.3) both ways. Towards (9.4), if $b^\dagger \in K^\ddagger + iK^\ddagger$ is Fuglede there is implication

$$\|\varphi R_{b^\times}(u)\| = \|\varphi u b^\times\| \le \ell \|\varphi u b\| = \ell \|\varphi R_b(u)\| \ .$$

Conversely if R_b is Fuglede then for arbitrary $x \in X$ and $\varphi \in Y^\dagger$

$$\|\varphi b^\times\| \ \|x\| = \|R_{b^\times}(\varphi \odot x)\| \le \ell \|R_b(\varphi \odot x)\| = \ell \|\varphi a\| \ \|x\| \ .$$

Finally, for (9.5)

$$\|a^\times u b^\times\| \le \ell \|a^\times u b\| \le \ell' \ell \|a^\times u b^\times\| \ .$$

For the algebraic version there is only one way implication:

9.6
$$a \in A = B(X) \ algebraically\ Fuglede \implies L_a \in B(M) \ algebraically\ Fuglede \ ;$$

9.7
$$b^\dagger \in B(Y^\dagger) \ algebraically\ Fuglede \implies R_b \in B(M) \ algebraically\ Fuglede \ ;$$

9.8 $\quad L_a, R_b$ *algebraically Fuglede* $\implies L_aR_b$ *algebraically Fuglede* .

On the strength of (6.2) and (9.3) we offer a definition; for arbitrary A we define $a \in A$ to be *(strongly) left Fuglede* iff $L_a \in B(A)$ is (strongly) Fuglede, and *(strongly) right Fuglede* iff $R_a \in B(A)$ is (strongly) Fuglede. For $a \in A$ to be *algebraically left Fuglede*, and $b \in B$ to be *algebraically right Fuglede*, we would ask, analagous to (9.2), that

9.9 $$a^\times \in Aa \ , \ b^\times \in bB.$$

Evidently

$$a \in A \ algebraically \ left \ Fuglede \ , \ b \in B \ algebraically \ right \ Fuglede$$

9.10 $$\implies L_aR_b \ algebraically \ Fuglede \ .$$

10 Hyponormality

The "hyponormal" concept associated with the Vidav concept of Hermitian is as follows: $a \in A$ is declared to be *positive* iff

10.1 $$V_A(a) \subseteq [0.\infty) \subseteq \mathbf{R} \ ,$$

and *hyponormal* iff

10.2 $$a^*a - aa^* \ positive \ .$$

Kirsti Mattila has [M2] a strengthened version of hyponormality: $a \in A$ is ** hyponormal* iff

10.3 $$\forall \ \lambda \in \mathbf{C} \ : \ \|e^{\lambda a^*} e^{\overline{\lambda} a}\| \leq 1 \ .$$

If $a \in H + iH \subseteq A$ and $b \in K + iK \subseteq B$ then, recalling (5.2),

10.4 $$a, b^* \ hyponormal \ \implies \ L_a - R_b \ hyponormal \ .$$

Kirsti Mattila has shown ([M2] Theorem 4)

10.5 $$a, b^* \ {}^*hyponormal \ \implies \ L_a - R_b \ {}^*hyponormal \ .$$

There is implication ([M2] Theorem 3), for $a \in A = B(X)$,

10.6 $$a \ {}^*hyponormal \ \implies \ a \ Fuglede \ .$$

When $T \in A = B(X)$ for a Hilbert space then

10.7 $$T \ hyponormal \ \implies \ T \ strongly \ Fuglede \ .$$

In general however we are unable to decide upon the relationship between the * hyponormal, strong Fuglede and hyponormal conditions. Crabb and Spain [CS] have an example of a normal operator which is not strongly Fuglede.

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