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STRONG CONVERGENCE FOR COMMON FIXED POINTS OF A PAIR OF QUASI-NONEXPANSIVE AND ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we establish some strong convergence theorems of finite step iteration sequences with errors to a common fixed point for a pair of a finite family of quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify a few corresponding results due to Liu et al. [5, 6], Gu and He [3] and others.

1 Introduction and preliminaries

The concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. They proved that every asymptotically nonexpansive mapping of a nonempty closed bounded subset of a uniformly convex Banach always has a fixed point. Since then many authors have studied iterative approximation methods of fixed points for asymptotically nonexpansive mappings. In 1991, Schu [11, 12] introduced the modified Mann iteration method and proved that such iterative sequences converge strongly to a fixed point of an asymptotically nonexpansive mapping in a Hilbert space. In 1994 Rhoades [9] extended the results in [11] to uniformly convex Banach spaces and to the modified Ishikawa iteration method. Recently, Gu and He [3] studied a multi-step iterative sequence involving finite nonexpansive mappings in a uniformly convex Banach space. They obtained weak and strong convergence theorems for approximating common fixed points of nonexpansive mappings. Liu et al. in [5, 6] introduced new iterative methods, the modified two and the modified three-step iteration sequence with errors with respect to a pair of mappings. They also proved some

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convergence theorems which improve and unify many results due to Chang [1], Liu and Kang [4], Osilike and Aniagbosor [8], Rhoades [9], and Schu [11, 12] and others.

Very recently, Saejung and Sithikul [10] studied a finite step iterative sequences with errors involving a finite family of nonexpansive and a finite family of asymptotically nonexpansive mappings. They also obtained weak and strong convergence theorems for approximating common fixed points for said mappings in uniformly convex Banach spaces which improve and unify many results due to Liu et al. [5, 6] and also Gu and He [3].

2 Preliminaries

Let E be a real Banach space, K be a nonempty subset of E and $T\colon K \to K$ be a mapping with the fixed point set $F(T)$, i.e., $F(T) = \{x \in K : x = Tx\}$. We recall the following definitions:

Definition 2.1: A mapping $T\colon K \to K$ is said to be:

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.0.1}$$

for all $x, y \in K$.

(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \tag{2.0.2}$$

for all $x \in K$, $p \in F(T)$.

(3) asymptotically nonexpansive if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n)\,\|x - y\| \tag{2.0.3}$$

for all $x, y \in K$ and $n \geq 1$.

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - p\| \leq (1 + r_n)\,\|x - p\| \tag{2.0.4}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

(5) L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \quad \leq \quad L\,\|x-y\| \tag{2.0.5}$$

for all $x, y \in K$ and $n \geq 1$.

From the above definitions, it follows that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive. But the converse does not hold.

Definition 2.2: Let K be a nonempty closed subset of a Banach space E. A mapping $T\colon K \to K$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\lim_{n\to\infty} x_{n_j} = x \in K$.

Inspired and motivated by the works in [5, 6] and also in [10], we study the following iteration scheme for a pair of a finite family of quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings. Our scheme is as follows:

Definition 2.3: Let K be a nonempty convex subset of a normed linear space E and $S_i, T_i\colon K \to K$ for all $i = 1, 2, \ldots, N$ be two families of mappings. For an arbitrary $x_1 \in K$, the sequence $\{x_n\}_{n\geq 1}$ defined by:

$$\begin{aligned}
x_{n+1} = x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} S_N x_n + \gamma_n^{(N)} u_n^{(N)} \\
x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \beta_n^{(N-1)} S_{N-1} x_n + \gamma_n^{(N-1)} u_n^{(N-1)} \\
\ldots &= \ldots \\
\ldots &= \ldots \\
x_n^{(3)} &= \alpha_n^{(3)} T_3^n x_n^{(2)} + \beta_n^{(3)} S_3 x_n + \gamma_n^{(3)} u_n^{(3)} \\
x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + \beta_n^{(2)} S_2 x_n + \gamma_n^{(2)} u_n^{(2)} \\
x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} S_1 x_n + \gamma_n^{(1)} u_n^{(1)}
\end{aligned} \tag{2.0.6}$$

is called multi-step iterative sequences with errors, where $\{u_n^{(i)}\}$ are bounded sequences in K and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, for all $i = 1, 2, \ldots, N$.

Remark 2.1: In case $S_1 = S_2 = \cdots = S_N = I$, then the sequence $\{x_n\}_{n\geq 1}$ generated in (2.0.6) reduces to the multi-step iteration with errors for N asymptotically quasi-nonexpansive mappings.

The purpose of this paper is to study strong convergence of finite-step iterative sequence with errors defined by (2.0.6) to a common fixed point for a pair of a finite family of quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space. The results presented in this paper generalize, improve and unify the corresponding results of Liu et al. [5, 6], Gu and He [3]and many others.

The following lemmas are our main tool for proving the results.

Lemma 2.1([14]; Lemma 1): Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists. In particular, $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2([12]): Let E be a uniformly convex Banach space, $\{t_n\} \subseteq [b, c] \subset (0, 1)$, $\{x_n\}$ and $\{y_n\}$ be sequences in E. If $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Recall that a mapping $T\colon K \to K$ where K is a subset of E, is said to satisfy Condition (A) [13] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Senter and Dotson [13] approximated fixed points of a nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [7] and Tan and Xu [14] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same Condition (A) which is weaker than the requirement that T is demicompact. We modify this condition for N mappings $T_1, T_2, \ldots, T_N\colon K \to K$ as follows.

A finite family $\{T_1, T_2, \ldots, T_N\}$ of N self mappings of K where K is a subset of E, is said to satisfy Condition (B) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $a_1\|x - T_1x\| + a_2\|x - T_2x\| + \cdots + a_N\|x - T_Nx\| \geq f(d(x, F))$ for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F = \cap_{i=1}^{N} F(T_i)\}$ and $a_1, a_2, \ldots, a_N$ are N nonnegative real numbers such that $a_1 + a_2 + \cdots + a_N = 1$.

Again we modify the above condition for two finite families of mappings $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\colon K \to K$ as follows:

Two finite families $\{S_1, S_2, \ldots, S_N\}$ and $\{T_1, T_2, \ldots, T_N\}$ of self mappings of K where K is a subset of E, is said to satisfy Condition (C) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $a_1 \|x - T_1x\| + a_2 \|x - T_2x\| + \cdots + a_N \|x - T_Nx\| + b_1 \|x - S_1x\| + b_2 \|x - S_2x\| + \cdots + b_N \|x - S_Nx\| \geq f(d(x, F(S,T)))$ for all $x \in K$, where $d(x, F(S,T)) = \inf\{\|x - p\| : p \in F(S,T) = \bigcap_{i=1}^{N} F(T_i) \cap F(S_i)\}$ and $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N$ are $2N$ nonnegative real numbers such that $a_1 + a_2 + \cdots + a_N = 1$ and $b_1 + b_2 + \cdots + b_N = 1$.

Remark 1.2: (i) Condition (B) reduces to Condition (A) when $T_1 = T_2 = \cdots = T_N = T$.

(ii) Condition (C) reduces to Condition (B) when $S_1 = S_2 = \cdots = S_N = I$(identity map).

(iii) Condition (C) reduces to Condition (A) when $S_1 = S_2 = \cdots = S_N = I$(identity map) and $T_1 = T_2 = \cdots = T_N = T$.

3 Main results

Lemma 3.1: Let K be a nonempty convex subset of a normed linear space E. Let $S_1, S_2, \ldots, S_N\colon K \to K$ be a finite family of quasi-nonexpansive mappings and $T_1, T_2, \ldots, T_N\colon K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\} \subset [0,\infty)$ satisfying $\lim_{n\to\infty} \lambda_{in} = 0$ such that $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ for all $i = 1, 2, \ldots, N$ and $F(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}_{n\geq 1}$ defined by (2.0.6) with the restrictions $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \ldots, N$. Then $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F(S,T)$.

Proof: (a) Let $q \in F(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i)$. Since $\{u_n^{(i)}\}$ for all $i = 1, 2, \ldots, N$ are bounded sequences in K. So we can set

$$D = \max\{\sup_{n\geq 1} \left\| u_n^{(i)} - q \right\| : i = 1, 2, \ldots, N\}.$$

Since $S_1, S_2, \ldots, S_N$ are quasi-nonexpansive and $T_1, T_2, \ldots, T_N$ are asymptotically quasi-nonexpansive mappings, it follows from (2.0.6), that

$$\begin{aligned}
\left\|x_n^{(1)}-q\right\| &= \left\|\alpha_n^{(1)}T_1^n x_n+\beta_n^{(1)}S_1x_n+\gamma_n^{(1)}u_n^{(1)}-q\right\| \\
&\le \alpha_n^{(1)}\left\|T_1^n x_n-q\right\|+\beta_n^{(1)}\left\|S_1x_n-q\right\|+\gamma_n^{(1)}\left\|u_n^{(1)}-q\right\| \\
&\le \alpha_n^{(1)}(1+\lambda_{1n})\left\|x_n-q\right\|+\beta_n^{(1)}\left\|x_n-q\right\|+\gamma_n^{(1)}\left\|u_n^{(1)}-q\right\| \\
&\le [\alpha_n^{(1)}+\beta_n^{(1)}](1+\lambda_{1n})\left\|x_n-q\right\|+\gamma_n^{(1)}\left\|u_n^{(1)}-q\right\| \\
&= [1-\gamma_n^{(1)}](1+\lambda_{1n})\left\|x_n-q\right\|+\gamma_n^{(1)}\left\|u_n^{(1)}-q\right\| \\
&\le (1+\lambda_{1n})\left\|x_n-q\right\|+\gamma_n^{(1)}D \\
&\le (1+\lambda_{1n})\left\|x_n-q\right\|+d_n^{(1)}, \qquad (3.0.7)
\end{aligned}$$

where $d_n^{(1)}=\gamma_n^{(1)}D$. Since $\sum_{n=1}^{\infty}\gamma_n^{(1)}<\infty$, we can see that $\sum_{n=1}^{\infty}d_n^{(1)}<\infty$. It follows from (3.0.7) that

$$\begin{aligned}
&\left\|x_n^{(2)}-q\right\| \le \alpha_n^{(2)}\left\|T_2^n x_n^{(1)}-q\right\|+\beta_n^{(2)}\left\|S_2x_n-q\right\|+\gamma_n^{(2)}\left\|u_n^{(2)}-q\right\| \\
&\le \alpha_n^{(2)}(1+\lambda_{2n})\left\|x_n^{(1)}-q\right\|+\beta_n^{(2)}\left\|x_n-q\right\|+\gamma_n^{(2)}\left\|u_n^{(2)}-q\right\| \\
&\le \alpha_n^{(2)}(1+\lambda_{2n})[(1+\lambda_{1n})\left\|x_n-q\right\|+d_n^{(1)}]+\beta_n^{(2)}\left\|x_n-q\right\|+\gamma_n^{(2)}\left\|u_n^{(2)}-q\right\| \\
&\le [\alpha_n^{(2)}+\beta_n^{(2)}](1+\lambda_{1n})(1+\lambda_{2n})\left\|x_n-q\right\|+\alpha_n^{(2)}(1+\lambda_{2n})d_n^{(1)} \qquad (3.0.8) \\
&\quad +\gamma_n^{(2)}\left\|u_n^{(2)}-q\right\| \\
&= [1-\gamma_n^{(2)}](1+\lambda_{1n})(1+\lambda_{2n})\left\|x_n-q\right\|+\alpha_n^{(2)}(1+\lambda_{2n})d_n^{(1)}+\gamma_n^{(2)}D \\
&\le (1+\lambda_{1n})(1+\lambda_{2n})\left\|x_n-q\right\|+(1+\lambda_{2n})d_n^{(1)}+\gamma_n^{(2)}D \\
&\le [1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n})]\left\|x_n-q\right\|+d_n^{(2)} \qquad (3.0.9)
\end{aligned}$$

where $d_n^{(2)}=(1+\lambda_{2n})d_n^{(1)}+\gamma_n^{(2)}D$. Since $\sum_{n=1}^{\infty}\gamma_n^{(2)}<\infty$ and $\sum_{n=1}^{\infty}d_n^{(1)}<\infty$, we can see that $\sum_{n=1}^{\infty}d_n^{(2)}<\infty$. It follows from (3.0.9), that

$$\begin{aligned}
\left\|x_n^{(3)}-q\right\| &\leq \alpha_n^{(3)}\left\|T_3^n x_n^{(2)}-q\right\|+\beta_n^{(3)}\left\|S_3 x_n-q\right\|+\gamma_n^{(3)}\left\|u_n^{(3)}-q\right\| \\
&\leq \alpha_n^{(3)}(1+\lambda_{3n})\left\|x_n^{(2)}-q\right\|+\beta_n^{(3)}\left\|x_n-q\right\|+\gamma_n^{(3)}\left\|u_n^{(3)}-q\right\| \\
&\leq \alpha_n^{(3)}(1+\lambda_{3n})[(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n}))\left\|x_n-q\right\|+d_n^{(2)}] \\
&\quad +\beta_n^{(3)}\left\|x_n-q\right\|+\gamma_n^{(3)}\left\|u_n^{(3)}-q\right\| \\
&\leq (\alpha_n^{(3)}+\beta_n^{(3)})[(1+\lambda_{3n})(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n}))]\left\|x_n-q\right\| \\
&\quad +\alpha_n^{(3)}(1+\lambda_{3n})d_n^{(2)}+\gamma_n^{(3)}\left\|u_n^{(3)}-q\right\| \\
&= [1-\gamma_n^{(3)}][(1+\lambda_{3n})(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n}))]\left\|x_n-q\right\| \\
&\quad +\alpha_n^{(3)}(1+\lambda_{3n})d_n^{(2)}+\gamma_n^{(3)}D \\
&\leq [(1+\lambda_{3n})(1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n}))]\left\|x_n-q\right\| \\
&\quad +(1+\lambda_{3n})d_n^{(2)}+\gamma_n^{(3)}D \\
&\leq [1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n})+\lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n})]\left\|x_n-q\right\| \\
&\quad +d_n^{(3)}, \qquad (3.0.10)
\end{aligned}$$

where $d_n^{(3)}=(1+\lambda_{3n})d_n^{(2)}+\gamma_n^{(3)}D$. Since $\sum_{n=1}^{\infty}\gamma_n^{(3)}<\infty$ and $\sum_{n=1}^{\infty}d_n^{(2)}<\infty$, we can see that $\sum_{n=1}^{\infty}d_n^{(3)}<\infty$. Continuing the above process, we get

$$\begin{aligned}
&\left\|x_{n+1}-q\right\|=\left\|x_n^{(N)}-q\right\| \\
&\leq [1+\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n})+\lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n}) \\
&+\cdots+\lambda_{Nn}(1+\lambda_{1n})(1+\lambda_{2n})\ldots(1+\lambda_{(N-1)n})]\left\|x_n-q\right\|+d_n^{(N)} \\
&\leq (1+t_n)\left\|x_n-q\right\|+d_n^{(N)} \qquad (3.0.11)
\end{aligned}$$

where $t_n=\lambda_{1n}+\lambda_{2n}(1+\lambda_{1n})+\lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n})+\cdots+\lambda_{Nn}(1+\lambda_{1n})(1+\lambda_{2n})\ldots(1+\lambda_{(N-1)n})$. Since $\sum_{n=1}^{\infty}\lambda_{in}<\infty$ for all $i=1,2,\ldots,N$, it follows that $\sum_{n=1}^{\infty}t_n<\infty$ and by assumptions of the theorem $\sum_{n=1}^{\infty}d_n^{(N)}<\infty$. From Lemma 2.1, we know that $\lim_{n\to\infty}\left\|x_n-q\right\|$ exists. This completes the proof.

Lemma 3.2: Let K be a nonempty convex subset of a uniformly convex Banach space E. Let $S_1,S_2,\ldots,S_N\colon K\to K$ be a finite family of quasi-nonexpansive mappings and $T_1,T_2,\ldots,T_N\colon K\to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\}\subset[0,\infty)$ satisfying $\lim_{n\to\infty}\lambda_{in}=0$ such that $\sum_{n=1}^{\infty}\lambda_{in}<\infty$ for all $i=1,2,\ldots,N$ and $F(S,T)=\bigcap_{i=1}^{N}F(S_i)\cap F(T_i)\neq\emptyset$. Suppose that

$$\left\|x-T_iy\right\| \leq \left\|S_ix-T_iy\right\| \qquad (3.0.12)$$

for all $x, y \in K$ and $i = 1, 2, \dots, N$. Let $\{x_n\}$ be the sequence as defined in (2.0.6) and for some $\delta_1, \delta_2 \in (0,1)$ with the following restrictions:

(C_1) $0 < \delta_1 \leq \alpha_n^{(i)} \leq \delta_2 < 1$, $\forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.

(C_2) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, for all $i = 1, 2, \dots, N$.

Then

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$$

for all $i = 1, 2, \dots, N$.

Proof: For any $q \in F(S,T)$. By Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q\|$ exists. Let $\lim_{n\to\infty} \|x_n - q\| = a$ for some $a \geq 0$. We note that

$$\begin{aligned}\|x_n^{(N-1)} - q\| \leq & [1 + \lambda_{1n} + \lambda_{2n}(1+\lambda_{1n}) + \lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n}) \\ & + \cdots + \lambda_{(N-1)n}(1+\lambda_{1n})(1+\lambda_{2n})\dots(1+\lambda_{(N-2)n})]\|x_n - q\| \\ & + d_n^{(N-1)} \end{aligned} \tag{3.0.13}$$

where $\sum_{n=1}^{\infty} d_n^{(N-1)} < \infty$ and $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ for all $i = 1, 2, \dots, N$. It follows that

$$\limsup_{n\to\infty} \|x_n^{(N-1)} - q\| \tag{3.0.14}$$

$$\begin{aligned} & \leq \limsup_{n\to\infty} [\{1 + \lambda_{1n} + \lambda_{2n}(1+\lambda_{1n}) + \lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n}) \\ & + \cdots + \lambda_{(N-1)n}(1+\lambda_{1n})(1+\lambda_{2n})\dots(1+\lambda_{(N-2)n})\}\|x_n - q\| \\ & + d_n^{(N-1)}] \\ \leq & a \end{aligned} \tag{3.0.15}$$

and so

$$\begin{aligned}\limsup_{n\to\infty} \|T_N^n x_n^{(N-1)} - q\| \leq & \limsup_{n\to\infty} (1 + \lambda_{Nn})\|x_n^{(N-1)} - q\| \\ = & \limsup_{n\to\infty} \|x_n^{(N-1)} - q\| \\ \leq & a. \end{aligned} \tag{3.0.16}$$

Now, since S_N is quasi nonexpansive, so we have

$$\|S_N x_n - q\| \leq \|x_n - q\|$$

and so

$$\begin{aligned}\limsup_{n\to\infty} \|S_N x_n - q\| \leq & \limsup_{n\to\infty} \|x_n - q\| \\ \leq & a. \qquad (3.0.17)\end{aligned}$$

Next, consider

$$\|T_N^n x_n^{(N-1)} - q + \gamma_n^{(N)}(u_n^{(N)} - x_n)\| \leq \|T_N^n x_n^{(N-1)} - q\| + \gamma_n^{(N)} \left\| u_n^{(N)} - x_n \right\|.$$

Thus,

$$\limsup_{n\to\infty} \|T_N^n x_n^{(N-1)} - q + \gamma_n^{(N)}(u_n^{(N)} - x_n)\| \leq a. \qquad (3.0.18)$$

Also,

$$\begin{aligned}\|S_N x_n - q + \gamma_n^{(N)}(u_n^{(N)} - x_n)\| &\leq \|S_N x_n - q\| + \gamma_n^{(N)}\|u_n^{(N)} - x_n\| \\ &\leq \|x_n - q\| + \gamma_n^{(N)}\|u_n^{(N)} - x_n\|\end{aligned}$$

gives that

$$\limsup_{n\to\infty} \|S_N x_n - q + \gamma_n^{(N)}(u_n^{(N)} - x_n)\| \leq a, \qquad (3.0.19)$$

and we observe that

$$\begin{aligned}x_n^{(N)} - q \quad = \quad & \alpha_n^{(N)}(T_N^n x_n^{(N-1)} - q \\ & + \gamma_n^{(N)}(u_n^{(N)} - x_n)) + (1 - \alpha_n^{(N)})(S_N x_n - q + \gamma_n^{(N)}(u_n^{(N)} - x_n)).\end{aligned}$$

Therefore,

$$\begin{aligned}a &= \lim_{n\to\infty} \|x_n^{(N)} - q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N)}(T_N^n x_n^{(N-1)} - q \qquad (3.0.20) \\ &\quad + \gamma_n^{(N)}(u_n^{(N)} - x_n)) + (1 - \alpha_n^{(N)})(S_N x_n - q + \gamma_n^{(N)}(u_n^{(N)} - x_n))\|. \qquad (3.0.21)\end{aligned}$$

By (3.0.18)-(3.0.21) and Lemma 2.2, we have

$$\lim_{n\to\infty}\left\|T_N^n x_n^{(N-1)} - S_N x_n\right\| = 0. \tag{3.0.22}$$

It follows from (3.0.12) that

$$\lim_{n\to\infty}\left\|T_N^n x_n^{(N-1)} - x_n\right\| = 0. \tag{3.0.23}$$

Now, we shall show that $\lim_{n\to\infty}\left\|T_{N-1}^n x_n^{(N-2)} - S_{N-1} x_n\right\| = 0$. For each $n \geq 1$,

$$\begin{aligned}\|x_n - q\| &\leq \left\|T_N^n x_n^{(N-1)} - x_n\right\| + \left\|T_N^n x_n^{(N-1)} - q\right\| \\ &\leq \left\|T_N^n x_n^{(N-1)} - x_n\right\| + (1 + \lambda_{Nn}\left\|x_n^{(N-1)} - q\right\|.\end{aligned}$$

Using (3.0.23), we have

$$a = \lim_{n\to\infty}\|x_n - q\| \leq \liminf_{n\to\infty}\left\|x_n^{(N-1)} - q\right\|.$$

It follows that

$$a \leq \liminf_{n\to\infty}\left\|x_n^{(N-1)} - q\right\| \leq \limsup_{n\to\infty}\left\|x_n^{(N-1)} - q\right\| \leq a.$$

This implies that

$$\lim_{n\to\infty}\left\|x_n^{(N-1)} - q\right\| = a. \tag{3.0.24}$$

On the other hand, we have

$$\begin{aligned}&\|x_n^{(N-2)} - q\| \leq [1 + \lambda_{1n} + \lambda_{2n}(1 + \lambda_{1n}) + \lambda_{3n}(1 + \lambda_{1n})(1 + \lambda_{2n}) \\ &+ \cdots + \lambda_{(N-2)n}(1 + \lambda_{1n})(1 + \lambda_{2n}) \ldots (1 + \lambda_{(N-3)n})]\|x_n - q\| \\ &+ d_n^{(N-2)}\end{aligned} \tag{3.0.25}$$

where $\sum_{n=1}^{\infty} d_n^{(N-2)} < \infty$ and $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ for all $i = 1, 2, \ldots, N$. Therefore

$$\limsup_{n\to\infty} \|x_n^{(N-2)} - q\| \tag{3.0.26}$$
$$\leq \limsup_{n\to\infty}[\{1 + \lambda_{1n} + \lambda_{2n}(1 + \lambda_{1n}) + \lambda_{3n}(1 + \lambda_{1n})(1 + \lambda_{2n})$$
$$+ \cdots + \lambda_{(N-2)n}(1 + \lambda_{1n})(1 + \lambda_{2n}) \dots (1 + \lambda_{(N-3)n})\}\|x_n - q\|$$
$$+ d_n^{(N-2)}]$$
$$\leq a \tag{3.0.27}$$

and hence

$$\begin{aligned}\limsup_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - q\| &\leq \limsup_{n\to\infty}(1 + \lambda_{(N-1)n})\|x_n^{(N-2)} - q\| \\ &= \limsup_{n\to\infty} \|x_n^{(N-2)} - q\| \\ &\leq a. \end{aligned} \tag{3.0.28}$$

Now, since S_{N-1} is quasi nonexpansive, so we have

$$\|S_{N-1}x_n - q\| \leq \|x_n - q\|$$

and so

$$\begin{aligned}\limsup_{n\to\infty} \|S_{N-1}x_n - q\| &\leq \limsup_{n\to\infty} \|x_n - q\| \\ &\leq a. \end{aligned} \tag{3.0.29}$$

Next, consider

$$\|T_{N-1}^n x_n^{(N-2)} - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\|$$
$$\leq \|T_{N-1}^n x_n^{(N-2)} - q\| + \gamma_n^{(N-1)} \left\|u_n^{(N-1)} - x_n\right\|.$$

Thus,

$$\limsup_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\| \leq a. \tag{3.0.30}$$

Also,

$$\begin{aligned}\|S_{N-1}x_n - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\| &\leq \|S_{N-1}x_n - q\| + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\| \\ &\leq \|x_n - q\| + \gamma_n^{(N-1)}\|u_n^{(N-1)} - x_n\|\end{aligned}$$

gives that

$$\limsup_{n\to\infty} \|S_{N-1}x_n - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\| \leq a, \tag{3.0.31}$$

and we observe that

$$\begin{aligned} x_n^{(N-1)} - q &= \alpha_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)) \\ &\quad +(1-\alpha_n^{(N-1)})(S_{N-1}x_n - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)). \end{aligned}$$

Therefore,

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_n^{(N-1)} - q\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)) \\ &\quad + (1-\alpha_n^{(N-1)})(S_{N-1}x_n - q + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n))\|. \end{aligned} \tag{3.0.32}$$

By (3.0.30) - (3.0.32) and Lemma 2.2, we have

$$\lim_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - S_{N-1}x_n \right\| = 0. \tag{3.0.33}$$

It follows from (3.0.12) that

$$\lim_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - x_n \right\| = 0. \tag{3.0.34}$$

Similarly, using the same argument as in the proof above, we have

$$\lim_{n\to\infty} \left\| T_{N-2}^n x_n^{(N-3)} - x_n \right\| = 0. \tag{3.0.35}$$

Continuing the similar process, we have

$$\lim_{n\to\infty} \left\| T_{N-i}^n x_n^{(N-i-1)} - x_n \right\| = 0, \quad 0 \leq i \leq (N-2). \tag{3.0.36}$$

Now,

$$\left\| T_1^n x_n - q + \gamma_n^{(1)}(u_n^{(1)} - x_n) \right\| \leq \|T_1^n x_n - q\| + \gamma_n^{(1)} \left\| u_n^{(1)} - x_n \right\|.$$

Thus,

$$\limsup_{n\to\infty}\left\|T_1^n x_n - q + \gamma_n^{(1)}(u_n^{(1)} - x_n)\right\| \le a. \tag{3.0.37}$$

Now, since S_1 is quasi nonexpansive, so we have

$$\|S_1 x_n - q\| \le \|x_n - q\|$$

and so

$$\begin{aligned}\limsup_{n\to\infty}\|S_1 x_n - q\| &\le \limsup_{n\to\infty}\|x_n - q\| \\ &\le a. \end{aligned} \tag{3.0.38}$$

Also,

$$\begin{aligned}\left\|S_1 x_n - q + \gamma_n^{(1)}(u_n^{(1)} - x_n)\right\| &\le \|S_1 x_n - q\| + \gamma_n^{(1)}\left\|u_n^{(1)} - x_n\right\| \\ &\le \|x_n - q\| + \gamma_n^{(1)}\left\|u_n^{(1)} - x_n\right\|\end{aligned}$$

gives that

$$\limsup_{n\to\infty}\left\|S_1 x_n - q + \gamma_n^{(1)}(u_n^{(1)} - x_n)\right\| \le a, \tag{3.0.39}$$

and hence

$$\begin{aligned} a &= \lim_{n\to\infty}\|x_n^{(1)} - q\| \\ &= \lim_{n\to\infty}\|\alpha_n^{(1)}(T_1^n x_n - q + \gamma_n^{(1)}(u_n^{(1)} - x_n)) \\ &\quad + (1-\alpha_n^{(1)})(S_1 x_n - q + \gamma_n^{(1)}(u_n^{(1)} - x_n))\|. \end{aligned} \tag{3.0.40}$$

By (3.0.38) - (3.0.40) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|T_1^n x_n - S_1 x_n\| = 0. \tag{3.0.41}$$

It follows from (3.0.12) that

$$\lim_{n\to\infty}\|T_1^n x_n - x_n\| = 0. \tag{3.0.42}$$

From (3.0.22), (3.0.23), (3.0.33) - (3.0.36), (3.0.41) and (3.0.42), we have

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0. \tag{3.0.43}$$

for all $i = 1, 2, \ldots, N$.

On the other hand, we also have

$$\begin{aligned}
\|x_n^{(N-1)} - x_n\| =& \|\alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \beta_n^{(N-1)} S_{N-1} x_n + \gamma_n^{(N-1)} u_n^{(N-1)} - x_n\| \\
=& \|\alpha_n^{(N-1)} (T_{N-1}^n x_n^{(N-2)} - S_{N-1} x_n) + \beta_n^{(N-1)} (S_{N-1} x_n - x_n) \\
& + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n)\| \\
\leq& \alpha_n^{(N-1)} \|T_{N-1}^n x_n^{(N-2)} - S_{N-1} x_n\| + (1 - \gamma_n^{(N-1)}) \|S_{N-1} x_n - x_n\| \\
& + \gamma_n^{(N-1)} \|u_n^{(N-1)} - x_n\|.
\end{aligned}$$

Using (3.0.33), (3.0.43) and condition $\sum_{n=1}^{\infty} \gamma_n^{(N-1)} < \infty$, we have

$$\lim_{n\to\infty} \left\| x_n^{(N-1)} - x_n \right\| = 0. \tag{3.0.44}$$

Now observe that

$$\begin{aligned}
\|x_n - T_N^n x_n\| \leq& \|T_N^n x_n^{(N-1)} - T_N^n x_n\| + \|T_N^n x_n^{(N-1)} - x_n\| \\
\leq& (1 + \lambda_{Nn}) \|x_n^{(N-1)} - x_n\| + \|T_N^n x_n^{(N-1)} - x_n\|.
\end{aligned}$$

Using (3.0.23) and (3.0.44), we have

$$\lim_{n\to\infty} \|x_n - T_N^n x_n\| = 0. \tag{3.0.45}$$

Next, we consider

$$\begin{aligned}
\|x_{n+1} - x_n\| =& \|\alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1 - \alpha_n^{(N)} - \gamma_n^{(N)}) S_N x_n + \gamma_n^{(N)} u_n^{(N)} - x_n\| \\
\leq& \alpha_n^{(N)} \|T_N^n x_n^{(N-1)} - S_N x_n\| + (1 - \gamma_n^{(N)}) \|S_N x_n - x_n\| \\
& + \gamma_n^{(N)} \|u_n^{(N)} - x_n\|
\end{aligned}$$

From (3.0.22) and (3.0.43), we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.0.46}$$

Therefore, we have

$$\begin{aligned}
\|x_n - T_N x_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_N^{n+1} x_{n+1}\right\| \\
&\quad + \left\|T_N^{n+1} x_{n+1} - T_N x_n\right\| \\
&\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_N^{n+1} x_{n+1}\right\| \\
&\quad +(1+\lambda_{N1}) \left\|T_N^n x_{n+1} - x_n\right\| \\
&\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_N^{n+1} x_{n+1}\right\| \\
&\quad +(1+\lambda_{1n}) \left\|T_N^n x_{n+1} - T_N^n x_n + T_N^n x_n - x_n\right\| \\
&\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_N^{n+1} x_{n+1}\right\| \\
&\quad +(1+\lambda_{N1})(1+\lambda_{Nn}) \|x_{n+1} - x_n\| + (1+\lambda_{N1}) \left\|T_N^n x_n - x_n\right\|
\end{aligned}$$

From (3.0.45) and (3.0.46), we have

$$\lim_{n\to\infty} \|x_n - T_N x_n\| = 0. \tag{3.0.47}$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty} \|x_n - T_{N-1} x_n\| = 0. \tag{3.0.48}$$

Continuing similar process, we have

$$\lim_{n\to\infty} \|x_n - T_{N-i} x_n\| = 0, \tag{3.0.49}$$

for all $i = 0, 1, 2, \ldots, (N-2)$. Now,

$$\begin{aligned}\|x_n - T_1x_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_1^{n+1}x_{n+1}\right\| \\ &\quad + \left\|T_1^{n+1}x_{n+1} - T_1x_n\right\| \\ &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_1^{n+1}x_{n+1}\right\| \\ &\quad +(1+\lambda_{11})\left\|T_1^n x_{n+1} - x_n\right\| \\ &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_1^{n+1}x_{n+1}\right\| \\ &\quad +(1+\lambda_{11})(\|T_1^n x_{n+1} - T_1^n x_n\| + \|T_1^n x_n - x_n\|) \\ &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_1^{n+1}x_{n+1}\right\| \\ &\quad +(1+\lambda_{11})(1+\lambda_{1n})\|x_{n+1} - x_n\| + (1+\lambda_{11})\|T_1^n x_n - x_n\|.\end{aligned}$$

From (3.0.42) and (3.0.46), we have

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = 0, \tag{3.0.50}$$

and hence

$$\lim_{n\to\infty} \|x_n - T_ix_n\| = 0. \tag{3.0.51}$$

for all $i = 1, 2, \dots, N$. This completes the proof.

Theorem 3.1: Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S_1, S_2, \dots, S_N \colon K \to K$ be a finite family of quasi-nonexpansive mappings and $T_1, T_2, \dots, T_N \colon K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\} \subset [0, \infty)$ satisfying $\lim_{n\to\infty} \lambda_{in} = 0$ such that $\sum_{n=1}^{\infty} \lambda_{in} < \infty$ for all $i = 1, 2, \dots, N$ and $F(S,T) = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$. If the conditions (3.0.12), (C_1) and (C_2) of Lemma 3.2 are satisfied. Suppose $\{S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N\}$ satisfies Condition (C). Then the sequence $\{x_n\}$ as defined in (2.0.6) converges strongly to a common fixed of the mappings $\{S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N\}$.

Proof: Since form Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F(S,T)$. Let $\lim_{n\to\infty} \|x_n - q\| = r$ for some $r \geq 0$. If $r = 0$, there is nothing to prove. Assume that $r > 0$. As proved in Lemma 3.1, we have

$$\|x_{n+1} - q\| \leq (1+t_n)\|x_n - q\| + d_n^{(N)}$$

where $t_n = \lambda_{1n} + \lambda_{2n}(1+\lambda_{1n}) + \lambda_{3n}(1+\lambda_{1n})(1+\lambda_{2n}) + \dots + \lambda_{Nn}(1+\lambda_{1n})(1+$

$\lambda_{2n})\ldots(1+\lambda_{(N-1)n})$. Since $\sum_{n=1}^{\infty}\lambda_{in}<\infty$ for all $i=1,2,\ldots,N$, it follows that $\sum_{n=1}^{\infty}t_n<\infty$ and by assumptions of the theorem $\sum_{n=1}^{\infty}d_n^{(N)}<\infty$. This gives that

$$d(x_{n+1},F(S,T))\leq(1+t_n)d(x_n,F(S,T))+d_n^{(N)}.$$

Since $\sum_{n=1}^{\infty}t_n<\infty$ and $\sum_{n=1}^{\infty}d_n^{(N)}<\infty$. So by Lemma 2.1, we obtain that $\lim_{n\to\infty}d(x_n,F)$ exists. Also by Lemma 3.2, $\lim_{n\to\infty}\|x_n-S_ix_n\|=\lim_{n\to\infty}\|x_n-T_ix_n\|=0$ for all $i=1,2,\ldots,N$. Since the mappings $\{S_1,S_2,\ldots,S_N,T_1,T_2,\ldots,T_N\}$ satisfy Condition (C), we conclude that $\lim\limits_{n\to\infty}d(x_n,F(S,T))=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Indeed, since $\lim_{n\to\infty}d(x_n,F(S,T))$ exists, we can suppose $\lim_{n\to\infty}d(x_n,F(S,T))=r_0>0$. Thus, there exists a natural number N, $\forall n>N$, we have $d(x_n,F(S,T))>\frac{r_0}{2}$. Since f is a nondecreasing function, $f(0)=0$ and $f(r)>0$ for $\forall r\in(0,\infty)$, we can obtain $f(d(x_n,F(S,T)))\geq f(\frac{r_0}{2})>0$ for $\forall n>N$. On the other hand, we have $\lim_{n\to\infty}f(d(x_n,F(S,T)))=0$. This derives a contradiction. Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\}\subset F(S,T)$ such that $\left\|x_{n_j}-y_j\right\|<2^{-j}$. Then following the method of Tan and Xu [14], we get that $\{y_j\}$ is a Cauchy sequence in $F(S,T)$ and so it converges. Let $y_j\to y$. Since $F(S,T)$ is closed, therefore $y\in F(S,T)$ and then $x_{n_j}\to y$. As $\lim_{n\to\infty}\|x_n-q\|$ exists, $x_n\to y\in F(S,T)$ thereby completing the proof.

Theorem 3.2: Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S_1,S_2,\ldots,S_N\colon K\to K$ be a finite family of quasi-nonexpansive mappings and $T_1,T_2,\ldots,T_N\colon K\to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a sequence $\{\lambda_{in}\}\subset[0,\infty)$ satisfying $\lim_{n\to\infty}\lambda_{in}=0$ such that $\sum_{n=1}^{\infty}\lambda_{in}<\infty$ for all $i=1,2,\ldots,N$ and $F(S,T)=\bigcap_{i=1}^{N}F(S_i)\cap F(T_i)\neq\emptyset$. If the conditions (3.0.12), (C_1) and (C_2) of Lemma 3.2 are satisfied. Suppose one of the mappings in $\{T_1,T_2,\ldots,T_N\}$ is semi-compact. Then the sequence $\{x_n\}$ as defined in (2.0.6) converges strongly to a common fixed of the mappings $\{S_1,S_2,\ldots,S_N,T_1,T_2,\ldots,T_N\}$.

Proof: Suppose T_{i_0} is semi-compact for some $i_0\in\{1,2,\ldots,N\}$. By Lemma 3.2, we have

$$\lim_{n\to\infty}\|x_n-T_{i_0}x_n\|=0.$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty}x_{n_j}=x^*\in K$. Now Lemma 3.2 guarantees that $\|x^*-T_ix^*\|=\lim_{n_j\to\infty}\left\|x_{n_j}-T_ix_{n_j}\right\|=0$ and $\|x^*-S_ix^*\|=\lim_{n_j\to\infty}\left\|x_{n_j}-S_ix_{n_j}\right\|=0$ for all $i=1,2,\ldots,N$. This implies that $x^*\in F(S,T)=\bigcap_{i=1}^{N}F(T_i)\cap F(S_i)$. Since $\lim_{n\to\infty}d(x_n,F(S,T))=0$, it follows, as in the proof of Theorem 3.1, that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N\}$. This completes the proof.

Remark 3.1: Theorem 3.2 extend, improve and unify Theorem 3.2 of Liu et al. [5] in the following ways:

(i) the nonexpansive and asymptotically nonexpansive mappings in [5] is replaced by the more general quasi-nonexpansive mapping and asymptotically quasi-nonexpansive mappings.

(ii) the three-step iteration scheme with errors in [5], for two mappings are extended to the finite-step iteration scheme with errors with respect to a pair of a family of mappings.

Remark 3.2: Our results also extend, improve and unify corresponding results of Liu et al. [6] in the following aspect:

(i) the nonexpansive and asymptotically nonexpansive mappings in [6] is replaced by the more general quasi-nonexpansive mapping and asymptotically quasi-nonexpansive mappings.

(ii) the two-step iteration scheme with errors in [6], for two mappings are extended to the finite-step iteration scheme with errors with respect to a pair of a family of mappings.

Remark 3.3: Our results also extend, improve and unify corresponding results of Gu and He [3] in the following aspect:

(i) the identity mapping in [3] is replaced by the family of quasi-nonexpansive mappings.

(ii) the family of nonexpansive mappings in [3] is replaced by the family of more general class of asymptotically quasi-nonexpansive mappings.

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