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Functional Analysis, Approximation and Computation **2:1** (2010), 53–65

# APPROXIMATE DIAGONALIZATION OF SELF–ADJOINT MATRICES OVER C($M$)

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**Abstract**

Let $M$ be a compact Hausdorff space. We prove that in this paper, every self–adjoint matrix over $C(M)$ is approximately diagonalizable iff $\dim M \leq 2$ and $\mathrm{H}^2(M,\mathbb{Z}) \cong 0$. Using this result, we show that every unitary matrix over $C(M)$ is approximately diagonalizable iff $\dim M \leq 2$, $\mathrm{H}^1(M,\mathbb{Z}) \cong \mathrm{H}^2(M,\mathbb{Z}) \cong 0$ when $M$ is a compact metric space.

## 1 Introduction

For a unital $C^*$–algebra, we write $U(\mathcal{A})$ (resp. $U_0(\mathcal{A})$) to denote the unitary group of $\mathcal{A}$ (resp. the connected component of the unit in $U(\mathcal{A})$). We also denote by $\mathrm{M}_n(\mathcal{A})$ the matrix algebra of $n \times n$ over $\mathcal{A}$. Set $U_1(\mathcal{A}) = U(\mathcal{A})$ (resp. $U_1^0(\mathcal{A}) = U_0(\mathcal{A})$) and $U_n(\mathcal{A}) = U(\mathrm{M}_n(\mathcal{A}))$, $U_n^0(\mathcal{A}) = U_0(\mathrm{M}_n(\mathcal{A}))$.

In [10], Kadison proved that for a Von Neunnam algebra $\mathcal{A}$, every normal matrix $A \in M_n(\mathcal{A})$ can be diagonalized, i.e., there are $U \in U_n(\mathcal{A})$ and $d_1, \cdots, d_n \in \mathcal{A}$ such that $UAU^* = \mathrm{diag}\,(d_1, \cdots, d_n)$. He also showed that if $\mathcal{A}$ is only assumed to be a $C^*$–algebra, the result may fail. At same time, Grove and Pederson in [5] considered the problem of diagonlization of normal matrices over $C(M)$, where $M$ is a compact Hausdorff space. They characterized the $M$ that allows diagonalization of every normal element in $M_n(C(M))$. Of course, these conditions in [5] are very complicated and hard to be verified.

Recently, because of the study of the classification of AI–algebras and AT–algebras, the approximate diagonalization of the self–adjoint matrix over $C([0,1])$ and some kind of unitary matrix over $C(\mathbf{S^1})$ are given respectively (cf. [15, Example 3.1.6]). For more general case, Choi and Elliott showed that if the dimension of the compact Hausdorff space $M$ is no more than two, then every self–adjoint element in $M_n(C(M))$ can been approximated by a self–adjoint element with $n$ distinct eigenvalues in [3]. Using this result, Thomsen proved that if

2000 *Mathematics Subject Classifications.* 46L05

*Key words and Phrases.* approximately diagonal, dimension of compact Huasdorff space, Cěch cohomological group, vector bundle

Received: February 18, 2010

Communicated by Dragan S. Djordjevic

Project supported by Natural Science Foundation of China (no.10771069) and Shanghai Leading Academic Discipline Project(no.B407)

$\dim M \le 2$ and $\mathrm{H}^2(M,\mathbb{Z}) \cong 0$, then two self–adjoint elements $a$, $b \in M_n(C(M))$ are approximately unitarily equivalent iff $a(x)$ and $b(x)$ have same eigenvalues, $\forall\, x \in M$ (cf. [16, Theorem 1.2, Corollary 1.3]).

In this paper, we first show that if every self–adjoint matrix over $C(M)$ is approximately diaonalizable, then $\dim M \le 2$ and $\mathrm{H}^2(M,\mathbb{Z}) \cong 0$ and then we give a constructed proof of Choi and Elliotts' result mentioned above.

## 2 Preliminaries

Throughout the paper, $\mathcal{A}$ is a $C^*$–algebra with unit 1 and $M$ is a compact Haudorff space.

We view $\mathcal{A}^n$ as the set of all $n \times 1$ matrices over $\mathcal{A}$. Set

$$S_n(\mathcal{A}) = \{(a_1,\cdots,a_n)^T \in \mathcal{A}^n |\ \sum_{i=1}^n a_i^* a_i = 1\},$$

$$\mathrm{Lg}_{\,n}(\mathcal{A}) = \{(a_1,\cdots,a_n)^T \in \mathcal{A}^n |\ \sum_{i=1}^n b_i a_i = 1,\ \text{for some } b_1,\cdots,b_n \in \mathcal{A}\}.$$

According to [13] and [14], the topological stable rank, the cnnected stable rank and the general stable rank of $\mathcal{A}$ are defined respectively as follows:

$$\begin{aligned}
\mathrm{tsr}\,(\mathcal{A}) &= \min\{\, n \in \mathbb{N} |\ \mathcal{A}^m \ \text{ is dense in } \mathrm{Lg}_{\,m}(\mathcal{A}), \forall m \ge n\,\}\\
\mathrm{csr}\,(\mathcal{A}) &= \min\{\, n \in \mathbb{N} |\ U_m^0(\mathcal{A}) \ \text{ acts transitively on } S_m(\mathcal{A}), \forall m \ge n\,\}\\
\mathrm{gsr}\,(\mathcal{A}) &= \min\{\, n \in \mathbb{N} |\ U_m(\mathcal{A}) \ \text{ acts transitively on } S_m(\mathcal{A}), \forall m \ge n\,\}.
\end{aligned}$$

If no such integer exists, we set $\mathrm{tsr}\,(\mathcal{A}) = \infty$, $\mathrm{csr}\,(\mathcal{A}) = \infty$ and $\mathrm{gsr}\,(\mathcal{A}) = \infty$ respectively. Those stable ranks of $C^*$–algebras are very usful tools in computing $K$–groups of $C^*$–algebras (cf. [14], [18], [19] and [20] etc.). From [13] and [11], we have

**Lemma 2.1.** *Let $\mathcal{A}$ be a unital $C^*$–algebra and $M$ be a compact Hausdorff space. Then*

(1) $\mathrm{gsr}\,(\mathcal{A}) \le \mathrm{csr}\,(\mathcal{A}) \le \mathrm{tsr}\,(\mathcal{A}) + 1$;

(2) $\mathrm{tsr}\,(C(M)) = \Big[\dfrac{\dim M}{2}\Big] + 1,\ \mathrm{csr}\,(C(M)) \le \Big[\dfrac{\dim M + 1}{2}\Big] + 1$;

Using topological stable rank and general stable rank, we can deduce a key lemma of this paper as follows:

**Lemma 2.2.** *Suppose that* $\mathrm{tsr}\,(\mathcal{A}) \le 2$ *and* $\mathrm{gsr}\,(\mathcal{A}) \le 2$. *Let $A = (a_{ij})_{n\times n}$ $(n \ge 3)$ be self–adjoint element in $M_n(\mathcal{A})$. Then for any epsilon $> 0$, there are $U \in U_n(\mathcal{A})$ and $b_1,\cdots,b_{n-1} \in \mathcal{A}$ with $b_1,\cdots,b_{n-2} > 0$ such that*

$$\left\| U^*AU - \begin{bmatrix} a_{11} & b_1 & & \\ b_1 & a_{22} & \ddots & \\ & \ddots & \ddots & b_{n-1}^* \\ & & b_{n-1} & a_{nn} \end{bmatrix} \right\| < (n-1)^{3/2}\epsilon.$$

*Proof.* Since $\operatorname{tsr}(\mathcal{A}) \le 2$, there exists $(a_{21}^{(2)}, \cdots, a_{n1}^{(2)})^T \in \operatorname{Lg}_{n-1}(\mathcal{A})$ such that $\|a_{i1} - a_{i1}^{(2)}\| < \epsilon$, $i = 2, \cdots, n$. Since $b_1 = \Big[\sum_{i=2}^{n} \big(a_{i1}^{(2)}\big)^* a_{i1}^{(2)}\Big]^{1/2}$ is invertible, $(a_{21}^{(2)} b_1^{-1}, \cdots, a_{n1}^{(2)} b_1^{-1})^T \in S_{n-1}(\mathcal{A})$. Thus, from $\operatorname{gsr}(\mathcal{A}) \le 2$, we get $U_1 \in U_{n-1}(\mathcal{A})$ such that

$$U_1(a_{21}^{(2)} b_1^{-1}, \cdots, a_{n1}^{(2)} b_1^{-1})^T = (1, 0, \cdots, 0)^T \text{ or } U_1(a_{21}^{(2)}, \cdots, a_{n1}^{(2)})^T = (b_1, 0, \cdots, 0)^T.$$

Set $U^{(1)} = \operatorname{diag}(1, U_1)$. Then using $a_{ij} = a_{ji}^*$, $i, j = 1, \cdots$, we have

$$U^{(1)} \begin{bmatrix} a_{11} & \big(a_{i1}^{(2)}\big)^* & \cdots & \big(a_{n1}^{(2)}\big)^* \\ a_{21}^{(2)} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}^{(2)} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \Big(U^{(1)}\Big)^* = \begin{bmatrix} a_{11} & b_{(1)}^T \\ b_{(1)} & A_1 \end{bmatrix},$$

where $b_{(1)} = (b_1, 0, \cdots, 0)^T \in \mathcal{A}^{n-1}$, $A_1 = A_1^* \in \mathrm{M}_{n-1}\mathcal{A}$. Simple computation shows that

$$\left\| U^{(1)} A (U^{(1)})^* - \begin{bmatrix} a_{11} & b_{(1)}^T \\ b_{(1)} & A_1 \end{bmatrix} \right\| < \sqrt{n-1}\,\epsilon.$$

Using the same way as above to $A_1$, after $n-2$ times, we can find a $U \in U_n(\mathcal{A})$ and invertible positive elements $b_1, \cdots, b_{n-2} \in \mathcal{A}$ and an element $b_{n-1} \in \mathcal{A}$ such that

$$\left\| U^* A U - \begin{bmatrix} a_{11} & b_1 & & & \\ b_1 & a_{22} & \ddots & & \\ & \ddots & \ddots & b_{n-1}^* \\ & & b_{n-1} & a_{nn} \end{bmatrix} \right\| < (\sqrt{2} + \cdots + \sqrt{n-1}) < (n-1)^{3/2}\epsilon.$$

□

The following lemma concers the extension of continuous map.

**Lemma 2.3.** *Let $M_0$ be a closed subset of $M$ and $f_0 \colon M_0 \to U(\mathcal{A})$ be a continuous map. Then there are an open subset $O$ in $M$ containing $M_0$ and a continuous map $f \colon O \to U(\mathcal{A})$ such that $f\big|_{M_0} = f_0$.*

*Proof.* By [4, P360], there is a continuous map $g \colon M \to \mathcal{A}$ such that $g\big|_{M_0} = f_0$. Thus for any $\epsilon \in (0, \frac{1}{2})$ and any $x_0 \in M_0$, there is an open subset $V(x_0)$ in $M$ containing $x_0$ such that $\|g(x) - g(x_0)\| < \epsilon$ whenever $x \in V(x_0)$. Since $M_0 = \bigcup_{x_0 \in M_0} (V(x_0) \cap M_0)$ and $M_0$ is compact, it follows that there are $x_1, \cdots, x_n \in M_0$ such that $M_0 = \bigcup_{i=1}^{n} (V(x_i) \cap M_0)$. Set $O = \bigcup_{i=1}^{n} V(x_i)$. Then $O$ is open and contains $M_0$. Furthermore, for any $x \in M$, there is $x_i$ such that

$$\|g(x) - g(x_i)\| = \|g(x) - f_0(x_i)\| < \epsilon < \frac{1}{2}. \tag{2.1}$$

(2.1) implies that $(f_0(x_i))^* g(x)$ is invertible in $\mathcal{A}$ and so is the $g(x)$ and moreover,

$$\|(g(x))^{-1}\| = \|[f_0(x_i))^* g(x)]^{-1}\| < 2, \ \forall\, x \in O.$$

Now set $f(x) = g(x)[(g(x))^* g(x)]^{-1/2}$, $x \in O$. Then $f\colon O \to U(\mathcal{A})$ is continuous and $f\big|_{M_0} = f_0$ (for $g\big|_{M_0} = f_0$). □

The following results are well–known in Matrix Thoery, which come from [17].

**Lemma 2.4.** *Let $A$ be a self–adjoint matrix in $M_n(\mathbb{C})$.*

(1) *If $A$ is tri–diagonal such that the elements in subdiagonal line are nonzero, then $A$ has $n$ distinct eigenvalues*;

(2) *Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A$, ordered non–increasingly and each eigenvalue repeated according to its multiplicity. For every nonzero subspace $V \subset \mathbb{C}^n$, set $\lambda_A(V) = \min\{(Ax, x) \,|\, x \in V,\ \|x\| = 1\}$. Then*

$$\lambda_j = \max\{\lambda_A(V) \,|\, V \subset \mathbb{C}^n, \dim V = j\},\ j = 1, \cdots, n.$$

Now applying Lemma 2.4 to self–adjoint matrices in $\mathrm{M}_n(C(M))$, we have

**Corollary 2.5.** *Let $A$, $B$ be two self–adjoint matrices in $M_n(C(M))$. For each $x \in M$, let $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$, $\mu_1(x) \geq \cdots \geq \mu_n(x)$ be the eigenvalues of $A(x)$ and $B(x)$, ordered non–increasingly and counted with its multiplicity, respectively. Then, for every $x$, $y \in M$ and $j = 1, \cdots, n$,*

(1) $|\lambda_j(x) - \lambda_j(y)| \leq \|A(x) - A(y)\|$;

(2) $|\lambda_j(x) - \mu_j(x)| \leq \|A(x) - B(x)\|$.

*Proof.* We only prove (2). The proof of (1) is similar.

Let $V$ be any nonzero subspace of $\mathbb{C}^n$. Then for each $x \in M$ and $\xi \in V$,

$$(A(x)\xi, \xi) \leq \|A(x) - B(x)\|\|\xi\|^2 + (B(x)\xi, \xi),$$
$$(A(x)\xi, \xi) \geq -\|A(x) - B(x)\|\|\xi\|^2 + (B(x)\xi, \xi)$$

Thus,

$$\lambda_{A(x)}(V) \leq \|A(x) - B(x)\| + \lambda_{B(x)}(V), \quad \lambda_{A(x)}(V) \geq -\|A(x) - B(x)\| + \lambda_{B(x)}(V)$$

$\forall\, x \in M$ and consequently, by Lemma 2.4,

$$\lambda_j(x) \leq \|A(x) - B(x)\| + \lambda_j(x), \quad \mu_j(x) \leq \|A(x) - B(x)\| + \lambda_j(x),$$

i.e., $|\lambda_j(x) - \mu_j(x)| \leq \|A(x) - B(x)\|$, $\forall\, x \in M$ and $j = 1, \cdots, n$. □

Let Det (resp. Tr) denote the determinant (resp. trace) on $\mathrm{M}_n(\mathbb{C})$. Define functions $\det,\ \mathrm{tr} : \mathrm{M}_n(C(M)) \to C(M)$ by

$$\det(A)(x) = \mathrm{Det}(A(x)), \quad \mathrm{tr}(A)(x) = \mathrm{Tr}(A(x)),\ \forall\, A \in \mathrm{M}_n(C(M)),\ x \in M$$

respectively. By means of some theory in Linear Algebra, we have

**Lemma 2.6.** *Let* $A$, $B \in M_n(C(M))$. *Then*

(1) $\det(AB) = \det(A)\det(B)$, $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$;

(2) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\|\mathrm{tr}(A) - \mathrm{tr}(B)\| \leq n\|A-B\|$;

(3) $\|\det(A) - \det(B)\| \leq n!\Big(\sum\limits_{k=0}^{n-1}\|A\|^k\|B\|^{n-k-1}\Big)\|A-B\|$;

(4) *$A$ is invertible in $M_n(C(M))$ iff* $\mathrm{Det}(A(x)) \neq 0$, $\forall\, x \in M$.

# 3 A necessary condition

**Definition 3.1.** *An element $A \in M_n(\mathcal{A})$ $(n \geq 2)$ is said to be approximately diagonalizable, if for any $\epsilon > 0$, there are $U \in U_n(\mathcal{A})$ and $a_1, \cdots, a_n \in \mathcal{A}$ such that*

$$\|UAU^* - \mathrm{diag}(a_1, \cdots, a_n)\| < \epsilon. \tag{3.1}$$

*$\mathcal{A}$ is called to be approximately diagonal* (**AD**)*, if for any $n \geq 2$, every self–adjoint element in $M_n(\mathcal{A})$ $(n \geq 2)$ can be approximate diagonalization.*

Clearly, if $A$ is self–adjoint (or unitary), then $a_1, \cdots, a_n$ in (3.1) can be chosen as self–adjoint (or unitary).

**Lemma 3.2.** *Let $P$ be a approximately diagonalizable projection in $M_n(\mathcal{A})$ $(n \geq 2)$. Then there are $U \in U_n(\mathcal{A})$ and projections $p_1, \cdots, p_n \in \mathcal{A}$ such that* $UPU^* = \mathrm{diag}(p_1, \cdots, p_n)$.

*Proof.* By assumption, there are $W \in U_n(\mathcal{A})$ and self–adjoint elements $a_1, \cdots,$ $a_n \in A$ such that

$$\|WPW^* - \mathrm{diag}(a_1, \cdots, a_n)\| < \frac{1}{2}.$$

Put $A = \mathrm{diag}(a_1, \cdots, a_n)$. Then by [9, Lemma 2.5.4], there exists a projection $Q$ in the $C^*$–algebra generated by $A$ such that

$$\|WPW^* - Q\| < 2\,\|WPW^* - A\| < 1. \tag{3.2}$$

Since $A$ is diagonal, $Q$ has the form $Q = \mathrm{diag}(p_1, \cdots, p_n)$, where $p_1, \cdots, p_n \in \mathcal{A}$ are projections. Finally, Using [9, Lemma 2.5.1] to (3.2), we can find $W_0 \in U_n^0(\mathcal{A})$ such that $W_0WPW^*W_0^* = \mathrm{diag}(p_1, \cdots, p_n)$. Put $U = W_0W$. Then we get the assertion. □

Recall from [7] that a compact Hausdorff space $M$ is of $\dim M \leq n$ iff for each closed subset $A$ of $M$, any continuous map $f\colon A \to \mathbf{S^n}$ can be extended to $M$ and if $\dim M < \infty$, then $\dim M \leq n$ iff $i^*\colon \mathrm{H}^n(M, \mathbb{Z}) \to \mathrm{H}^n(A, \mathbb{Z})$ is epimorphic for any closed subset $A$ of $M$, where $\mathrm{H}^n(M, \mathbb{Z})$ is the n'th Cěch Cohomology of $M$ and $i^*$ is the induced homomorphism of the inclusion $i\colon A \to M$ on $\mathrm{H}^n(M, \mathbb{Z})$.

**Lemma 3.3.** *Suppose that $C(M)$ is* (AD). *Then* $\dim M \leq 3$.

*Proof.* If $\dim M > 3$, then exist a closed subset $M_0$ of $M$ and a continuous map $f_0\colon M_0 \to \mathbf{S^3}$ which can not be extended to $M$. Write $f_0(x) = (f_1(x), f_2(x))^T$, $x \in M_0$, where $f_1, f_2\colon M_0 \to \mathbb{C}$ are continuous and $|f_1(x)|^2 + |f_2(x)|^2 = 1$, $\forall\, x \in M_0$. Put

$$U_0(x) = \begin{bmatrix} f_1(x) & -\overline{f_2(x)} \\ f_2(x) & \overline{f_1(x)} \end{bmatrix}, \quad x \in M_0.$$

Then $U_0 \in U_2(C(M_0))$ and it follows from Lemma 2.3 that there are an open subset $O$ in $M$ containing $M_0$ and a continuous map $V\colon O \to U_2(\mathbb{C})$ such that $V\big|_{M_0} = U_0$. Pick a continuous function $h\colon M \to [0,1]$ such that $h(x) = 1$, if $x \in M_0$ and $h(x) = 0$ when $x \in O\backslash M_0$. Now define a normal element $N \in \mathrm{M}_2(C(M))$ by

$$N(x) = \begin{cases} h(x)V(x) & x \in O \\ 0 & x \in M\backslash O \end{cases}$$

and put $A_1 = \dfrac{1}{2}(N + N^*)$, $A_2 = \dfrac{1}{2i}(N - N^*)$. By hypothesis, there are $W \in U_2(C(M))$ and continuous real valued functions $\lambda_1$, $\lambda_2$ on $M$ such that

$$\|WA_2W^* - \mathrm{diag}\,(\lambda_1, \lambda_2)\| < \frac{1}{6}. \tag{3.3}$$

Note that $A_1 = \mathrm{diag}\,(h\mathrm{Re}(f_1), h\mathrm{Re}(f_1))$. So if we set $\mu_j = h\mathrm{Re}(f_1) + i\lambda_j$, $j = 1, 2$, then

$$\|WAW^* - \mathrm{diag}\,(\mu_1, \mu_2)\| < \frac{1}{6}, \tag{3.4}$$

by (3.3). Applying Lemma 2.6 to (3.4), we get that

$$\begin{aligned} \|h^2 - \mu_1\mu_2\| \leq & 2\,(\|WNW^*\| + \|\mathrm{diag}\,(\mu_1, \mu_2)\|) \\ \leq & 2(1 + 1 + \frac{1}{6})\,\frac{1}{6} < 1 \end{aligned}$$

Set

$$X = \begin{bmatrix} \mu_1 & -(1 - h^2)^{1/2} \\ (1 - h^2)^{1/2} & \mu_2 \end{bmatrix} \in \mathrm{M}_2(C(M)).$$

Since $\|1 - \det(X)\| = \|h^2 - \mu_1\mu_2\| < 1$, $X$ is invertible by Lemma 2.6. Set $W_1 = X(X^*X)^{-1/2} \in U_2(C(M))$. Noting that $h(x) = 1$, when $x \in M_0$, we have from (3.4),

$$\|W(x)V(x)(W(x))^* - X(x)\| < \frac{1}{6}, \quad x \in M_0.$$

Simple computation shows that

$$\|1_2 - (X(x))^*X(x)\| < \frac{13}{36}, \quad \|1_2 - ((X(x))^*X(x))^{-1/2}\| < \frac{1}{2},\ \forall\, x \in M_0.$$

Thus,

$$\|W(x)V(x)(W(x))^* - W_1(x)\| < \frac{1}{6} + \frac{1}{2}(1 + \frac{1}{6}) < 1, \quad x \in M_0.$$

This means that there is a self–adjoint element $B_0$ in $\mathrm{M}_2(C(M_0))$ such that

$$W(x)V(x)(W(x))^* = \exp(i\,B_0(x))W_1(x), \quad \forall\, x \in M_0.$$

Choose self–adjoint element $B$ in $\mathrm{M}_2(C(M))$ such that $B\big|_{M_0} = B_0$ and put

$$U = W^*\exp(i\,B)W_1W = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in U_2(C(M)),$$

$f = (u_{11}, u_{21})^T$. Then $f\colon M \to \mathbf{S^3}$ is continuous with $f\big|_{M_0} = f_0$, a contradiction. □

Let $\mathrm{M}_\infty(C(M)) = \bigcup_{n=1}^{\infty} \mathrm{M}_n(C(M))$ under the inclusion

$$i_n\colon \mathrm{M}_n(C(M)) \to \mathrm{M}_{n+1}(C(M)) \text{ given by } i_n(a) = \mathrm{diag}\,(a, 0).$$

It is well–known that there is an one–to–one and onto correspodence between $n$–dimensional complex vector bundles over $M$ and projections in $\mathrm{M}_\infty(C(M))$ with rank $n$ (i.e., $p$ is a projection in $\mathrm{M}_m(C(M))$ with m large enough such that $\mathrm{tr}\,(p) = n$). Let $\mathrm{V}^1_{\mathbb{C}}(M)$ denote the isomorphic class of all 1–dimensional complex vector bundles over $M$. Then from [10], $\mathrm{V}^1_{\mathbb{C}}(M) \cong \mathrm{H}^2(M, \mathbb{Z})$. Thus $\mathrm{H}^2(M, \mathbb{Z}) \cong 0$ iff all 1–dimensional complex vector bundles over $M$ is trivial iff ever projection with rank one in $\mathrm{M}_\infty(C(M))$ is equivalent to the form $\mathrm{diag}\,(1, 0_s) \in \mathrm{M}_{s+1}(C(M))$ for sufficiently large $s$.

**Theorem 3.4.** *Let $M$ be a compact Hausdorff space such that $C(M)$ is* (AD)*. Then* $\dim M \le 2$ *and* $\mathrm{H}^2(M, \mathbb{Z}) \cong 0$.

*Proof.* Let $A$ be any closed subspace of $M$ and let $a$ be any self–adjoint element in $\mathrm{M}_n(C(A))$ for $n \ge 2$. Then we can find a self–adjoint element $\hat{a} \in \mathrm{M}_n(C(M))$ such that $\hat{a}\big|_A = a$. This means that $C(A)$ is (AD) when $C(M)$ is (AD).

Now let $p$ be a projection in $\mathrm{M}_n(C(A))$ with $\mathrm{rank}\,(p(x)) = 1$, $\forall\, x \in A$. Then by Lemma 3.2, there are projections $p_1, \cdots, p_n \in C(A)$ and $W \in \mathrm{M}_n(C(A))$ such that

$$WpW^* = \mathrm{diag}\,(p_1, \cdots, p_n) \tag{3.5}$$

Thus,

$$p_1(x) + \cdots + p_n(x) = \mathrm{Tr}(\mathrm{diag}\,(p_1(x), \cdots, p_n(x)) = \mathrm{Tr}(p(x)) = 1,$$

$\forall\, x \in A$ and hence $p_ip_j = 0,\ i \ne j,\ i, j = 1, \cdots, n$. Set

$$S = \begin{bmatrix} p_1 & \cdots & p_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} W.$$

Then $S^*S = p,\ SS^* = \mathrm{diag}\,(1, 0_{n-1})$ by (3.5). This shows that $\mathrm{H}^2(A, \mathbb{Z}) \cong 0$.

Since $\dim M \le 3$ by Lemma 3.3 and $i^*\colon \mathrm{H}^2(M, \mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Z})$ is surjective, it follows that $\dim M \le 2$. □

# 4 A sufficient condition

In this section, we will prove following theorm:

**Theorem 4.1.** *Let $M$ be compact Hausdorff space with* $\dim M \le 2$ *and* $\mathrm{H}^2(M,\mathbb{Z}) \cong 0$. *Then* $C(M)$ *is* (AD). *Precisely, let $A$ be a self–adjoint element in $M_n(C(M))$ ($n \ge 2$) and let $\lambda_1(x) \ge \cdots \ge \lambda_n(x)$ be the eigenvalues of $A(x)$, ordered non–increasingly and counted with their multiplicity for each $x \in M$. Then for any $\epsilon > 0$, there is $U \in U_n(C(M))$ such that*

$$\|U^*(x)A(x)U(x) - \mathrm{diag}\,(\lambda_1(x), \cdots, \lambda_n(x))\| < \epsilon,\ \forall\, x \in M.$$

To prove this theorm, we need two lemmas.

**Lemma 4.2.** [1, Proposition 1.1] *Let $M$ be a compact Hausdorff space with* $\dim M \le n$. *Then for any real–valued continuous fuctions $f_1, \cdots, f_{n+1}$ on $M$, there are real–valued continuous functions $g_1, \cdots, g_{n+1}$ on $M$ such that $\sum\limits_{i=1}^{n+1} g_i^2$ is invertible in $C(M)$ and $\|f_i - g_i\| < \epsilon,\ i = 1, \cdots, n+1$.*

Let $H_1, H_2$ be two complex Hilbert spaces and let $B(H_1, H_2)$ denote the set of all bounded linear operators from $H_1$ to $H_2$. For $T \in B(H_1, H_2)$, we denote by $\mathrm{Ker}\, T$ (resp. $\mathrm{Ran}\,(T)$) the null space (resp. range) of $T$.

**Lemma 4.3.** *Let $T(x)$ be a continuous map from $M$ to $B(H_1, H_2)$ such that* $\dim \mathrm{Ker}\, T(x) = n$ *and* $\mathrm{Ran}\,(T(x))$ *is closed in $H_2$, $\forall\, x \in M$. Set*

$$E(T) = \{(x, \xi) \in M \times H_1 |\, T(x)\xi = 0\},\ \pi(x, \xi) = x,\ \forall\, (x, \xi) \in E_T.$$

*Then $(E(T), \pi, M)$ is an $n$–dimensional complex vector bundle over $M$.*

*Proof.* Let $x_0 \in M$ be an arbitrary point. Let $P$ (resp. $Q$) be the projection of $H_1$ (resp. $H_2$) onto $\mathrm{Ker}\, T(x_0)$ (resp. $\mathrm{Ran}\,(T(x_0))$. Then there is $G \in B(H_2, H_1)$ such that $GT(x_0) = I_{H_1} - P$, $T(x_0)G = Q$ ($G$ is called the generalized inverse of $T(x_0)$, denoted by $T(x_0)^+$).

Since $T(x)$ is continuous at $x_0$, we can find a closed neighbourhood $U(x_0)$ of $x_0$ in $M$ such that $\|T(x) - T(x_0)\| < \dfrac{1}{\|A\|}$ whenever $x \in U(x_0)$. Thus, $\phi(x) = (I_{H_1} + G(T(x) - T(x_0)))^{-1}$ is a continuous map from $U(x_0)$ to the the group of invertible operators in $B(H_1)$ and furthermore, $\mathrm{Ker}\, T(x) = \phi(x)\mathrm{Ker}\, T(x_0)$ by [2, Proposition 3.1]. Now let $\{e_1, \cdots, e_n\}$ be a basis for $\mathrm{Ker}\, T(x_0)$ and put $e_j(x) = \phi(x)e_j$, $j = 1, \cdots, n$, $x \in U(x_0)$. Then $\{e_1(x), \cdots, e_n(x)\}$ forms a continuous basis for $\mathrm{Ker}\, T(x)$, $x \in U(x_0)$. This shows that $(E(T), \pi, M)$ is a vector bundle of dimension $n$ (cf.[6]). □

Proof of Theorem 4.1. We firt assume that $A = \begin{bmatrix} a_{11} & a_{12}^* \\ a_{21} & a_{22} \end{bmatrix}$ is self–adjoint in $\mathrm{M}_2(C(M))$. Write $a_{12} = a_{12}^{(1)} + i\, a_{12}^{(2)}$. Since $\dim M \le 2$, it follows from Lemma

4.2 that for any $\epsilon > 0$, there are continuous functions $b_1, b_2, b_3\colon M \to \mathbb{R}$ such that

$$\|a_{11} - a_{22} - b_1\| < \epsilon,\ \|a_{12}^{(1)} - b_2\| < \epsilon,\ \|a_{12}^{(2)} - b_3\| < \epsilon$$

and $b_1^2 + b_2^2 + b_3^2$ is invertible. Set $b = b_2 + ib_3$ and $B = \begin{bmatrix} a_{11} & b^* \\ b & a_{11} - b_1 \end{bmatrix}$. Then

$$\|A - B\| = \left\| \begin{bmatrix} 0 & (b - a_{12})^* \\ b - a_{12} & a_{22} - a_{11} - b_1 \end{bmatrix} \right\| < 2\,\epsilon \tag{4.1}$$

and for each $x \in M$, two eigenvalues of $B(x)$

$$\mu_1(x) = \frac{1}{2}\left[2a_{11}(x) - b_1(x) + \sqrt{b_1^2(x) + 4|b(x)|^2}\right],$$
$$\mu_2(x) = \frac{1}{2}\left[2a_{11}(x) - b_1(x) - \sqrt{b_1^2(x) + 4|b(x)|^2}\right]$$

are not equal for each $x \in M$. Thus, $\mathrm{Ker}\,(B(x) - \mu_j(x)) = 1$, $\forall\, x \in M$, $j = 1, 2$ and hence by Lemma 4.3 $(E(B - \mu_j), \pi, M)$ is an one–dimensional complex vector bundle which is also trivial for $\mathrm{H}^2(M, \mathbb{Z}) \cong 0$, $j = 1, 2$. This implies that there are continuous maps $\xi_1, \xi_2\colon M \to \mathbb{C}^2$ with $\|\xi_j\| = 1$ such that $B(x)\xi_j(x) = \mu_j(x)\xi_j(x)$, $\forall\, x \in M$ and $j = 1, 2$. Moreover, from $\mu_1(x) > \mu_2(x)$, we have $(\xi_1(x), \xi_2(x)) = 0$, $\forall\, x \in M$. Put $U(x) = (\xi_1(x), \xi_2(x))$, $x \in M$. Then $U \in U_2(C(M))$ and

$$U^*(x)B(x)U(x) = \begin{bmatrix} \mu_1(x) & \\ & \mu_2(x) \end{bmatrix}. \tag{4.2}$$

Let $\lambda_1(x), \lambda_2(x)$ be the eigenvalus of $A(x)$, for each $x \in M$. Then by Corollary 2.5 (2) and (4.1), $\|\lambda_j - \mu_j\| < 2\epsilon$, $j = 1, 2$. Finally, combining this with (4.1) and (4.2), we obtain $\|U^*AU - \mathrm{diag}\,(\lambda_1, \lambda_2)\| < 4\epsilon$.

Now Let $A = (a_{ij})_{n\times n}$ with $a_{ij} \in C(M)$ and $a_{ij}^* = a_{ji}$, $i, j = 1, \cdots, n$, $n \geq 3$. By Lemma 2.1, $\mathrm{tsr}\,(C(M)) \leq 2$ and $\mathrm{gsr}\,(C(M)) \leq 2$ when $\dim M \leq 2$. Thus by Lemma 2.2, for any $\epsilon \in (0, \frac{1}{2})$, there are $b_1, \cdots, b_{n-1} \in C(M)$ with $b_1, \cdots, b_{n-2} > 0$ and $U_1 \in U_n(C(M))$ such that

$$\left\| U_1^* A U_1 - \begin{bmatrix} a_{11} & b_1 & & \\ b_1 & a_{22} & \ddots & \\ & \ddots & \ddots & b_{n-1}^* \\ & & b_{n-1} & a_{nn} \end{bmatrix} \right\| < (n-1)^{3/2} \frac{\epsilon}{(n-1)^{3/2}} = \epsilon. \tag{4.3}$$

Write $c_j = a_{jj} - a_{nn}$, $j = 1, \cdots, n$ and

$$B_0 = \begin{bmatrix} c_1 & b_1 & & \\ b_1 & c_2 & \ddots & \\ & \ddots & \ddots & b_{n-1}^* \\ & & b_{n-1} & 0 \end{bmatrix}, \quad B_k = \begin{bmatrix} c_1 & b_1 & & \\ b_1 & c_2 & \ddots & \\ & \ddots & \ddots & b_{k-1}^* \\ & & b_{k-1} & c_k, \end{bmatrix}$$

$k = 2, \cdots, n-1$. Put $q_k = \det(B_k)$, $q_1 = c_1$, $q_0 = 1$, $k = 2, \cdots, n-1$. Then $q_1, \cdots, q_{n-1}$ are real–valued functions in $C(M)$ and

$$q_k = c_k q_{k-1} - b_{k-1}^2 q_{k-2},\ k = 2, \cdots, n-1. \tag{4.4}$$

From (4.3), we can deduce that $q_{k-1}^2 + q_{k-2}^2$ is invertible in $C(M)$, $k = 2, \cdots, n-1$. Write $b_{n-1} = b_{n-1}^{(1)} + i b_{n-1}^{(2)}$. Since $\dim M \le 2$, it follows from Lemma 4.2 that there are real continuous functions $d_{n-1}^{(1)}, d_{n-1}^{(2)}, b$ on $M$ such that

$$\|b_{n-1}^{(j)} - d_{n-1}^{(j)}\| < \epsilon, \quad \|q_{n-1} - b\| < \frac{\min\{\epsilon, m\epsilon, 0.5m^2\}}{(\|q_{n-2}\| + \|q_{n-3}\|)\|(q_{n-2}^2 + q_{n-3}^2)^{-1}\|},$$

$j = 1, 2$ and $b^2 + (d_{n-1}^{(1)})^2 + (d_{n-1}^{(2)})^2$ is invertible in $C(M)$, where $m = \min\limits_{x \in M} b_{n-2}(x) > 0$. Note that

$$\begin{aligned} b_{n-2}^2(x) - \frac{(b(x) - q_{n-1}(x))q_{n-3}(x)}{q_{n-2}^2(x) + q_{n-3}^2(x)} &\ge m^2 - \|(q_{n-2}^2 + q_{n-3}^2)^{-1}(b - q_{n-1})q_{n-3}\| \\ &\ge m^2 - \|(q_{n-2}^2 + q_{n-3}^2)^{-1}\|\|(b - q_{n-1})\|\|q_{n-3}\| \\ &\ge m^2 - 0.5m^2 = 0.5m^2 > 0 \end{aligned}$$

Put $d_{n-1} = d_{n-1}^{(1)} + i\, d_{n-1}^{(2)}$ and

$$\begin{aligned} c'_{n-1} &= c_{n-1} + \frac{(b - q_{n-1})q_{n-2}}{q_{n-2}^2 + q_{n-3}^2}, \quad & b'_{n-2} &= \Big(b_{n-2}^2 - \frac{(b - q_{n-1})q_{n-3}}{q_{n-2}^2 + q_{n-3}^2}\Big)^{1/2} > 0 \\ B'_{n-1} &= \begin{bmatrix} B_{n-2} & X \\ X^T & c'_{n-1} \end{bmatrix}, & B &= \begin{bmatrix} B'_{n-1} & Y \\ Y^T & 0 \end{bmatrix}, \end{aligned}$$

where $X = (0, \cdots, 0, b'_{n-2})^T \in (C(M))^{n-2}$, $Y = (0, \cdots, 0, d_{n-1})^T \in (C(M))^{n-1}$. Then

$$\begin{aligned} \|c'_{n-1} - c_{n-1}\| &< \epsilon,\ \|b'_{n-2} - b_{n-2}\| < \epsilon,\ \|B'_{n-1} - B_{n-1}\| < 3\epsilon \\ \|B - B_0\| &\le \|B'_{n-1} - B_{n-1}\| + 2\|d_{n-1} - b_{n-1}\| < 6\epsilon \text{ and} \\ \det(B'_{n-1}) &= c'_{n-1} q_{n-2} - (b'_{n-2})^2 q_{n-3} = b. \end{aligned}$$

Now let $\mu(x)$ be an eigenvalue of $B(x)$ for each $x \in M$. Put $M_1 = \{x \in M |\, d_{n-1}(x) = 0\}$. Let $x_0 \in M \backslash M_1$. Then $d_{n-1}(x_0) = |d_{n-1}(x_0)| \exp(i\theta)$ for some $\theta \in \mathbb{R}$ and

$$\begin{bmatrix} 1_{n-1} & \\ & \exp(-i\theta) \end{bmatrix} B(x_0) \begin{bmatrix} 1_{n-1} & \\ & \exp(i\theta) \end{bmatrix} = \begin{bmatrix} B'_{n-1}(x_0) & Z \\ Z^T & 0 \end{bmatrix},$$

where $Z = (0, \cdots, 0, |d_{n-1}(x_0)|)^T \in (C(M))^{n-1}$. In this case, $\dim \mathrm{Ker}\,(B(x_0) - \mu(x_0) I_n) = 1$ by Lemma 2.4 (1). Suppose that $x_0 \in M_1$. If $\mu(x_0) \ne 0$, then $\mu(x_0)$ must be an eigenvalue of $B'_{n-1}(x_0)$. Thus, by Lemma 2.4 (1), $\dim \mathrm{Ker}\,(B'_{n-1}(x_0) - \mu(x_0) I_{n-1}) = 1$ and hence $\dim \mathrm{Ker}\,(B(x_0) - \mu(x_0) I_n) = 1$;

If $\mu(x_0) = 0$, then from $b^2(x) + |d_{n-1}(x)|^2 \neq 0,\ \forall\, x \in M$ and $\mathrm{Det}(B'_{n-1}(x_0)) = b(x_0),\ d_{n-1}(x_0) = 0$, we have $\mathrm{Det}(B'_{n-1}(x_0)) \neq 0$, i.e., 0 is not the eigenvalue of $B'_{n-1}(x_0)$, so $\dim \mathrm{Ker}\,(B(x_0) - \mu(x_0)I_n) = 1$.

The above shows that $\dim \mathrm{Ker}\,(B(x) - \mu(x)I_n) = 1$ for each $x \in M$. Let $\mu_1(x) \geq \cdots \geq \mu_n(x)$ be the eigenvalues of $B(x)$, ordered non–increasingly and counted with its multiplicity, for each $x \in M$. Then $\mu_1(x) > \cdots > \mu_n(x),\ \forall\, x \in M$ and there are continuous maps $\xi_1, \cdots, \xi_n \colon M \to \mathbb{C}^n$ with $\|\xi_i\| = 1$ such that $B(x)\xi_i(x) = \mu_i(x)\xi_i(x)$, $\forall\, x \in M$ and $i = 1, \cdots, n$ by Lemma 4.3 and the assumption $\mathrm{H}^2(M, \mathbb{Z}) \cong 0$. Moreover, $(\xi_i(x), \xi_j(x)) = 0,\ i \neq j,\ i, j = 1, \cdots, n$, $\forall\, x \in M$. Put $U_2(x) = (\xi_1(x), \cdots, \xi_n(x)),\ t \in M$. Then $U_2 \in U_n(C(M))$ and

$$U_2^*(x)B(x)U_2(x) = \mathrm{diag}\,(\mu_1(x), \cdots, \mu_n(x)),\ \forall\, x \in M.$$

Put $U = U_1U_2$. Then by (4.3),

$$\begin{aligned}\|U^*AU - &\mathrm{diag}\,(\mu_1(x) + a_{nn}, \cdots, \mu_n(x) + a_{nn})\| \\ &\leq \|U_2^*(U_1^*AU_1 - B_0 - a_{nn}1_n)U_2 + U_2^*(B_0 - B)U_2 \\ &\quad + U_2^*BU_2 - \mathrm{diag}\,(\mu_1(x), \cdots, \mu_n(x))\| \\ &< \epsilon + 6\epsilon = 7\epsilon.\end{aligned}$$

Let $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ be the eigenvalues of $A(x)$, ordered non–increasingly and counted with its multiplicity, for each $x \in M$. Then we have

$$\|\lambda_j - \mu_j - a_{nn}\| < 7\epsilon, \quad j = 1, \cdots, n$$

by Corollary 2.5 and so that $\|U^*AU - \mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)\| < 14\epsilon$.

Now we consider the approximate diagonalization of unitary matrices over $C(M)$. We have

**Proposition 4.4.** *Let $M$ be compact metric space.*

(1) *If $\dim M \leq 2$ and $\mathrm{H}^2(M, \mathbb{Z}) \cong 0$, then every unitary element in $U_n^0(C(M))$ is approximately diagonalizable*;

(2) *Every unitary matrix over $C(M)$ is approximately diagonalizable iff $\dim M \leq 2$ and $\mathrm{H}^j(M, \mathbb{Z}) \cong 0,\ j = 1, 2$.*

*Proof.* (1) Let $U \in U_n^0(C(M))$. By [12, Theorem 2.1], for any $\epsilon \in (0, 1)$, there is self–adjoint element $A$ in $\mathrm{M}_n(C(M))$ such that $\|U - \exp(iA)\| < \epsilon/2$. Since $A$ can be approximate diagonalization by Theorem 4.1, we get that by simple computation, $U$ is approximately diagonalizable.

(2) Assume that $\dim M \leq 2$ and $\mathrm{H}^j(M, \mathbb{Z}) \cong 0,\ j = 1, 2$. Then we have $\mathrm{csr}\,(C(M)) \leq 2$ and for any $U \in U_n(C(M))$ there are $U_0 \in U_n^0(CM))$ and $a \in U(C(M))$ such that $U = U_0\mathrm{diag}\,(1_{n-1}, a)$ by [21, Lemma 2.1]. Note that $U(C(M))/U_0(C(M)) \cong \mathrm{H}^1(M, \mathbb{Z})$ (Arens and Royden's Theorem). So $U_n(C(M))$ is connected $n \geq 2$ when $\mathrm{H}^1(M, \mathbb{Z}) \cong 0$. Then the claim follows from (1).

On the other hand, if ever unitary matrix over $C(M)$ is approximately diagonalizable, then for any self–adjoint matrix $A \in \mathrm{M}_n(C(M))$, $n \geq 2$, we can deduce from following equation $V = \dfrac{A}{\|A\|} + i\sqrt{1_n - \Big(\dfrac{A}{\|A\|}\Big)^2} \in U_n(C(M))$ that $A = \dfrac{1}{2}\|A\|(V + V^*)$ is approximately diagonalizable and hence $\dim M \leq 2$ and $\mathrm{H}^2(M, \mathbb{Z}) \cong 0$ by Theorem 3.4.

Now we prove $\mathrm{H}^1(M, \mathbb{Z}) \cong 0$. Let $f \in U(C(M))$ such that the equivalece class $[f]$ of $f$ in $U(C(M))/U_0(C(M))$ is a generator. Since the unitary matrix $U = \begin{bmatrix} 0 & 1 \\ f & 0 \end{bmatrix}$ is approximately diagonalizable, there are $\lambda_1, \lambda_2 \in U(C(M))$ and $W \in U_2(C(M))$ such that

$$\|WUW^* - \mathrm{diag}\,(\lambda_1, \lambda_2)\| < \frac{1}{6}. \tag{4.5}$$

By using Lemma 2.6 to (4.5), we get that

$$\|\lambda_1 + \lambda_2\| < \frac{1}{3}, \quad \| - f - \lambda_1\lambda_2\| < \frac{2}{3}$$

and hence $\|f - \lambda_1^2\| < 1$. Put $\lambda = |\lambda_1|^{-1}\lambda_1$. Then $[f] = 2[\lambda]$ in $U(C(M))/U_0(C(M))$. Noting that $\mathrm{H}^1(M, \mathbb{Z})$ is torsion–free, we have $[f] = 0$. □

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