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ON LOCAL SPECTRAL THEORY II

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Abstract

We offer a local version of "property (β)" for bounded operators, applying the "single valued extension property" to the enlargement.

Recall that a bounded linear operator $T \in B(X)$ has the *single valued extension property at* $0 \in \mathbf{C}$ provided there is implication, writing $z : \mathbf{C} \to \mathbf{C}$ for the complex coordinate,

1.1 $$\forall\ f \in \mathrm{Holo}(0, X)\ :\ (T - zI)f(z) \equiv 0 \Longrightarrow f(z) \equiv 0\ .$$

Here if $K \subseteq \mathbf{C}$ is arbitrary we shall write "$f \in \mathrm{Holo}(K, X)$" to mean that there exists an open set $K \subseteq U \subseteq \mathbf{C}$ for which $f : U \to X$ is holomorphic, and we identify a point $\lambda \in \mathbf{C}$ with the singleton $\{\lambda\}$. To say that $T \in B(X)$ "has the single valued extension property" means that $T - \lambda I$ satisfies (1.1) for every $\lambda \in \mathbf{C}$. Equivalent to (1.1) is ([3] Theorem 9) the condition that $T \in B(X)$ is *locally one one*, in the sense that its only *holomorphic kernel point* is zero:

1.2 $$T^{-1}(0) \cap T^{\omega}(X) = \{0\}\ .$$

Here $x \in T^{\omega}(X)$ means that there is $\xi \in \Xi(X) = \bigcup_{k>0} \Xi_k(X)$ for which

1.3 $$\xi_0 = x \textit{ and } \forall\ n \in \mathbf{N}\ :\ T\xi_n = \xi_{n-1}\ ,$$

where ([3] Definition 1)

1.4 $$\Xi_k(X) = \{\xi \in X^{\mathbf{N}} : \forall\ n \in \mathbf{N} : \|\xi_n\| \le k^n \|\xi_0\|\}\ .$$

More subtle is what is known ([5] Definition 1.2.5) as "Bishop's property (β)": we shall say that $T \in B(X)$ has property (β) at $0 \in \mathbf{C}$ if for arbitrary open $U \in \mathrm{Nbd}(0)$ and arbitrary (f_n) in $\mathrm{Holo}(U, \ell_\infty(X))$ there is implication

1.5 $$(T - zI)f_n(z) \to_U 0\ (n \to \infty) \Longrightarrow f_n(z) \to_U 0\ (n \to \infty)\ ;$$

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here $f_n(z) \to_U 0$ means convergence uniformly on each compact subset $K \subseteq U$. Equivalently ([5] Theorem 1.2.6) the mappings

$$f \mapsto (T - zI)f : \mathrm{Holo}(U, X) \to \mathrm{Holo}(U, X)$$

are one-one and have closed range relative to this topology. Now $T \in B(X)$ "has property (β)" provided $T - \lambda I$ satisfies (1.5) for every $\lambda \in \mathbf{C}$.

If we think of the single valued extension property as a kind of local one-one-ness, then it would be nice to be able to see property (β) as a kind of local boundedness below. In this connection we recall the "enlargement" ([2] Definition 1.9.2)

1.6 $$\mathbf{Q}(X) = \ell_\infty(X)/c_0(X)$$

of a Banach space and hence of a bounded operator. We recall that if $T \in BL(X, Y)$ then ([2] Theorem 3.3.5) the following two conditions are equivalent:

1.7 $$\exists\ h > 0\ :\ \forall\ x \in X\ :\ \|x\| \leq h\ \|Tx\|\ ;$$

1.8 $$\forall\ x \in \ell_\infty(X)\ :\ Tx \in c_0(Y) \Longrightarrow x \in c_0(X)\ .$$

Indeed it is clear that (1.7) implies (1.8); conversely if (1.7) fails then for each $n \in \mathbf{N}$ there is $x_n \in X$ for which

$$\|x_n\| = 1 \geq n\ \|Tx_n\|\ .$$

Thus T is bounded below iff its enlargement $\mathbf{Q}(T) : \ell_\infty(X)/c_0(X) \to \ell_\infty(Y)/c_0(Y)$ is one one, in which case the enlargement is also bounded below. We are therefore encouraged to make the definition that, if $T \in BL(X, Y)$

1.9 $$T \textit{ locally bounded below } \Longleftrightarrow \mathbf{Q}(T) \textit{ locally one one}\ .$$

We have not succeeded (*cf* [3] Definition 11) in translating this equivalence into the language of (1.2); in the language of (1.1) we have

2. Theorem *Necessary and sufficient for $T \in B(X)$ to be locally bounded below is that, for arbitrary open $U \in \mathrm{Nbd}(0)$, there is implication, for pointwise bounded sequences (f_n) in $\mathrm{Holo}(U, X)$,*

2.1 $$(T - zI)f_n(z) \to 0\ (n \to \infty) \Longrightarrow f_n(z) \to 0\ (n \to \infty)$$

Proof. The difference between (2.1) and (1.5) is that we are now dealing in pointwise convergence; we are then simply restating the implication that if the bounded sequence of functions $(T - zI)f_n(z)$ takes each of its values in $c_0(X)$ then so does the bounded sequence of functions $f_n(z)$ •

It follows that applying the enlargement process to the single valued extension property gives rise to a sort of "bounded" analogue of the property (β), in that only bounded sequences of holomorphic functions are involved, with only pointwise convcergence.

The local single valued extension property, and the bounded version of (β), extend to left and right multiplication operators $L_T : U \mapsto TU$ and $R_T : U \mapsto UT$:

3. Theorem *If $T \in B(X)$ and $S^* \in B(X^*)$ are locally one one at then the same is true of $L_T \in B(B(X))$ and $R_S \in B(B(X))$, and conversely. Also there is implication*

3.1

$$T \in B(X) \textit{ locally bounded below } \Longrightarrow L_T \in B(B(X)) \textit{ locally bounded below} .$$

Proof: If $F \in \mathrm{Holo}(0, B(X))$ and $x \in X$ and $\varphi \in X^*$ are arbitrary then $f(z) \equiv F(z)x \in \mathrm{Holo}(0, X)$ and hence

$$(T - zI)F(z) \equiv 0 \in B(X) \Longrightarrow (T - zI)F(z)x \equiv 0 \in X \Longrightarrow F(z)x \equiv 0 \in X \ ,$$

and since $x \in X$ is arbitrary it follows that $F(z) \equiv 0 \in B(X)$. Conversely if $f \in \mathrm{Holo}(0, X)$ then $F = \varphi \odot f : x \to \varphi(x)f$ is in $\mathrm{Holo}(0, B(X))$ and hence

$$\begin{aligned}(T - zI)f(z) \equiv 0 \in X \Longrightarrow (T - zI)F(z) \equiv 0 \in B(X)\\ \Longrightarrow \varphi \odot f(z) \equiv F(z) \equiv 0 \in B(X) \ ,\end{aligned}$$

and since $\varphi \in X^*$ is arbitrary $f(z) \equiv 0 \in X$. Also $g(z) \equiv \varphi F(z) \in \mathrm{Holo}(0, X^*)$ and hence

$$F(z)(S - zI) \equiv 0 \in B(X) \Longrightarrow \varphi F(z)(S - zI) \equiv 0 \in X^* \Longrightarrow \varphi F(z) \equiv 0 \in X^* \ ,$$

and since $\varphi \in X^*$ is arbitrary it follows that $F(z) \equiv 0$. Conversely if $g \in \mathrm{Holo}(0, X^*)$ then $F = g \odot x : \varphi \to \varphi(x)f$ is in $\mathrm{Holo}(0, B(X^*))$ and hence

$$\begin{aligned}g(z)(T - zI) \equiv 0 \in X^* \Longrightarrow F(z)(T - zI) \equiv 0 \in B(X)\\ \Longrightarrow g(z) \odot x \equiv F(z) \equiv 0 \in B(X) \ ,\end{aligned}$$

and since $x \in X$ is arbitrary $g(z) \equiv 0 \in X^*$.

This gives the argument for local one-one-ness, and local boundedness below (3.1) follows by considering enlargements. Associated with $T \in B(X)$ is $\ell_\infty(T) \in B(B(\ell_\infty(X)))$ and hence $\mathbf{Q}(T) \in B(B(\mathbf{Q}(X)))$. Now firstly, by definition,

$$T \textit{ locally bounded below } \Longleftrightarrow \mathbf{Q}(T) \textit{ locally one one} ;$$

secondly, since $\mathbf{Q}(L_T) \in B(B(\mathbf{Q}(X)))$ is in effect a restriction of $L_{\mathbf{Q}(T)} \in B(\mathbf{Q}(B(X)))$,

$$\mathbf{Q}(T) \textit{ locally one one} \Longleftrightarrow L_{\mathbf{Q}(T)} \textit{ locally one one } \Longrightarrow \mathbf{Q}(L_T) \textit{ locally one one} ;$$

and thirdly, again by definition,

$$\mathbf{Q}(L_T) \textit{ locally one one } \Longleftrightarrow L_T \textit{ locally bounded below} \bullet$$

We have not settled whether **4. Problem** *Is there implication, for $S \in$*

$B(X)$,
4.1
$S^* \in B(X^*)$ *locally bounded below* $\Longrightarrow$ $R_S \in B(B(X))$ *locally bounded below*.

We recall ([4];[2] Theorem 5.7.3) the implication

4.2 $$\mathbf{Q}(S)^* \text{ one one } \Longrightarrow \mathbf{Q}(S^*) \text{ one one },$$

which suggests looking for local analogue. We can certainly argue

$$S^* \text{ locally bounded below } \Longleftrightarrow \mathbf{Q}(S^*) \text{ locally one one },$$

that

$$\mathbf{Q}(S)^* \text{ locally one one} \Longleftrightarrow R_{\mathbf{Q}(S)} \text{ locally one one } \Longrightarrow \mathbf{Q}(R_S) \text{ locally one one },$$

and that

$$\mathbf{Q}(R_S) \text{ locally one one } \Longleftrightarrow R_S \text{ locally bounded below };$$

what os needed to complete the circle is implication

4.3 $$\mathbf{Q}(S^*) \text{ locally one one } \Longrightarrow \mathbf{Q}(S)^* \text{ locally one one }.$$

This would indeed be a local analogue of (4.2) if the implication were the other way round.

If operators S and T in $B(X)$ commute then ([5] Theorem 3.6.3, Note 3.6.19)
4.4
S, T *have property* (β) $\Longrightarrow$ $S - T, ST$ *have the single valued extension property*;

thus, assuming (4.1), if $T \in B(X)$ and $S^* \in B(X^*)$ both have property (β) then $L_T - R_S$ and $L_T R_S$ in $B(B(X))$ both have the single valued extension property. We would however be unable to extend Theorem 3 to elementary operators such as $L_T - R_S$ or $L_T R_S$: Bourhim and Miller [1] have examples in which commuting operators A and B fail to pass on the single valued extension property to their sum or product.

We take the opportunity to correct a misprint in the proof of Theorem 9 of [3], which relates the two versions of local one-one-ness: in each of the formulae (9.4) and (9.5) we have written "$T(\xi) = 0$" where we should have written, in our tensor product notation, "$(I \otimes u)\xi = (T \otimes 1)\xi$".

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