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Exactness and the Jordan form

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Abstract

The Jordan classification of nilpotent matrices is described in terms of exactness.

1 Introduction

There is an unexpected intervention of *exactness* in the process of diagonalizing a matrix: if a linear operator $T : X \to X$ satisfies a nontrivial polynomial identity

$$p(T) = 0 \ ,$$

then the spectrum

$$\sigma(T) = \pi^{left}(T) \subseteq p^{-1}(0) \tag{1}$$

will consist entirely of eigenvalues, zeroes of the polynomial p. In finite dimensional linear algebra, after solving the equation $p(z) = 0$, there is a second stage problem to identify the eigenspaces $(T - \lambda I)^{-1}(0)$ for each point $\lambda \in p^{-1}(0)$. At this point [5] exactness says that it is not necessary to solve any more equations: the Euclidean algorithm for polynomials says

$$p = q\ r\ ,\ hcf(q, r) = 1 \implies 1 = q'\ q\ +\ r\ r'\ , \tag{2}$$

and hence

$$q(T)^{-1}(0) \subseteq r(T)(X)\ . \tag{3}$$

This applies in particular if one factor q takes up all the occurrences of a particular eigenvalue λ of T:

$$q = (z - \lambda)^k\ ,\ hcf(z - \lambda, r) = 1\ . \tag{4}$$

In words, all the eigenvectors of a matrix T corresponding to λ, and more, will be found among the columns of the matrix $r(T)$.

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In practice it will be numerically more sensible to solve the usual equations for eigenvectors rather than go to the trouble of computing the matrices $r(T)$ corresponding to $q(T) = (T - \lambda I)^k$, but theoretically this is a beguiling observation. Of course not every matrix will have enough eigenvectors to enable the process of "diagonalization"; this opens the door to the discussion of "Jordan form".

Suppose $T : X \to X$ is a linear operator on a real or complex vector space: then an "invariant subspace" for T satisfies $T(Y) \subseteq Y \subseteq X$. The trivial invariant subspaces are $O = \{0\}$ and X. Among the invariant subspaces of T we shall identify [8] its "platforms":

1. Definition *A subplatform for the linear operator $T : X \to X$ is a linear subspace $X' \subseteq X$ for which*

$$T^{-1}(0)_\cap X' \subseteq T(X') \subseteq X' \ , \tag{5}$$

and a platform is a maximal subplatform. A coplatform for the subplatform X' is a linear subspace $X'' \subseteq X$ for which

$$T(X'') \subseteq X'' \ ; \ X'_\cap X'' = O \ ; \ X' + X'' = X \ , \tag{6}$$

compatible provided also

$$X'' \subseteq T^{-1}(0) \ . \tag{7}$$

Zorn's condition is easily checked for subplatforms, and O is a subplatform, so that platforms always exist. Necessary and sufficient for the null space $T^{-1}(0)$ to be a subplatform for T is that T be one-one; necessary and sufficient for the range $T(X)$ to be a subplatform is the interesting condition

$$T^{-1}(0)_\cap T(X) \subseteq T^2(X) \ . \tag{8}$$

We shall describe T as *strictly nilpotent* provided

$$T^2 = 0 \ . \tag{9}$$

2. Theorem *If T is strictly nilpotent and X' is a subplatform for T then, for arbitrary $z \in X \setminus X'$, there is implication*

$$Tz \notin X' \Longrightarrow X' + \mathbf{K}z + \mathbf{K}Tz \text{ is a subplatform for } T \Longrightarrow z \notin X' + T^{-1}(0) \ . \tag{10}$$

Proof. If $Tz \notin X'$ then if $x \in X'$ and $\lambda, \mu \in \mathbf{K}$ there is implication

$$Tx - \lambda Tz = T(x - \lambda z - \mu Tz) = 0 \Longrightarrow Tx = 0 = \lambda Tz \ ,$$

so that $\lambda = 0$ and $x \in X'_\cap T^{-1}(0) \subseteq T(X')$, giving $x' \in X'$ for which

$$x - \lambda z - \mu Tz = T(x' - \mu z) \in T(X' + \mathbf{K}z + \mathbf{K}Tz) \ .$$

Conversely if $x \in X'$ and $z \in x + T^{-1}(0)$ then, if $X' + \mathbf{K}z + \mathbf{K}Tz$ is a subplatform,

$$z - x \in T^{-1}(0) \cap (X' + \mathbf{K}z + \mathbf{K}Tz) \subseteq X' + \mathbf{K}Tz \subseteq X' + X' \ ,$$

contradicting $z \in X \setminus X'$ •

In particular if O is a platform for strictly nilpotent T then $T = 0$.

The whole space X is a platform for T iff T is *self-exact*, in the sense that (T, T) is *exact*, where (S, T) is exact (whether or not $ST = 0$) iff

$$S^{-1}(0) \subseteq T(X) \ ; \tag{11}$$

generally a subplatform is just an invariant subspace supporting a self-exact restriction.

More generally we shall say that $X' \subseteq X$ is an *n-subplatform* for T provided

$$T^{-n}(0) \cap X' \subseteq T(X') \subseteq X' \subseteq X \ , \tag{12}$$

and that a coplatform X'' for X' is *n-compatible* provided

$$X'' \subseteq T^{-n}(0) \ . \tag{13}$$

Thus the whole space X is an n-subplatform for T if and only if T is *n-exact*, in the sense that

$$(T^n, T) \text{ is exact, equivalently } (T, T^n) \text{ is exact} \ . \tag{14}$$

The equivalence is ([2] Theorem 10.9.2;[6]) very simple: if $U : W \to X$, $T : X \to Y$ and $V : Y \to Z$ then

$$V^{-1}(0) \subseteq TU(W) \ , \ T^{-1}(0) \subseteq U(W) \Longrightarrow (VT)^{-1}(0) \subseteq U(W) \tag{15}$$

and

$$(VT)^{-1}(0) \subseteq U(W) \ , \ V^{-1}(0) \subseteq T(X) \Longrightarrow V^{-1}(0) \subseteq TU(W) \ . \tag{16}$$

An n-subplatform in turn is an invariant subspace on which the restricted operator is n-exact. If $T^{n+1} = 0$ and X' is an n-subplatform for T then we have an extension of (2.1): if $z \in X \setminus X'$ there is implication

$$T^n z \notin X' \Longrightarrow X' + \mathbf{K}z + \sum_{j=1}^{n} \mathbf{K}T^j z \text{ is an } n\text{-subplatform for } T \Longrightarrow z \notin X' + T^{-n}(0) \ . \tag{17}$$

Platforms of nilpotent operators have compatible coplatforms:

3. Theorem *If $T : X \to X$ is strictly nilpotent then all its subplatforms $X' \subseteq X$ have coplatforms. Necessary and sufficient for a subplatform to be a platform is that it has a compatible coplatform.*

Proof. The first part of this is given by Herstein ([9] Lemma 6.5.4): look at $X'' \subseteq X$ which is maximal with respect to the first two conditions of (1.2) and claim that there is inclusion

$$T^{-1}(X' + X'') \subseteq X' + X'' \ . \tag{18}$$

Indeed it is clear that, if $x \in X$ is arbitrary,

$$Tx = y + z \in X' + X'' \Longrightarrow 0 = T^2x = Ty + Tz \Longrightarrow Tz = -Ty \in X'_\cap X'' = O$$

giving $y = Ty'$ with $y' \in X'$ by the subplatform property (1.1): now consider the subspace

$$W = X'' + \mathbf{K}(x - y') \ .$$

We note that $TW \subseteq W$, since $T(x - y') = z \in X''$, and claim that

$$x \notin X' + X'' \Longrightarrow X'_\cap W = O \ .$$

To see this argue

$$z + \lambda(x - y') \in W_\cap X' \Longrightarrow \lambda x \in X' + \lambda X' - X'' \subseteq X' + X'' \ ,$$

and if $x \notin X' + X''$ this forces $\lambda = 0$ and hence $z \in X'_\cap X'' = O$. The maximality of X'' thus forces $x \in X' + X''$, as required by (3.1). Now if for a contradiction $X' + X'' \neq X$ then there is $z \in X$ for which $z \notin X' + X''$, in which case by (3.1) also $Tz \notin X' + X''$; but then

$$X'' + \mathbf{K}z + \mathbf{K}Tz \neq X''$$

satisfies the first two conditions of (1.2), contradicting the maximality of X''.

For the second part suppose X' is a subplatform, with a coplatform X'', and that there is $z \in X''$ for which $Tz \neq 0$: then by (2.1) $X' + \mathbf{K}z + \mathbf{K}Tz$ would also be a subplatform, with necessarily $Tz \notin X'$: thus X' cannot be maximal. Conversely if $X' \subseteq Y \subseteq X$ is contained in a subplatform Y, and also has a compatible coplatform X'', then $T(X) \subseteq X'$, and hence

$$Y \subseteq X' + X'' \subseteq Y_\cap(X' + T^{-1}(0)) \subseteq X' + TY \subseteq X' \ \bullet$$

Theorem 3 extends to n-subplatforms when $T^{n+1} = 0$, as indeed is proved by Herstein [9]:

4. Corollary *If $T^{n+1} = O$ then*

$$T \sim \begin{pmatrix} T' & O \\ rO & T'' \end{pmatrix} \ \textit{with} \ (T')^{-n}(0) \subseteq T'(X') \ ; \ (T'')^n = O'' \ . \tag{19}$$

For example if

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{20}$$

then

$$T(X) = T^{-2}(0) = \begin{pmatrix} \mathbf{K} \\ \mathbf{K} \\ O \end{pmatrix} \; ; \; T^2(X) = T^{-1}(0) = \begin{pmatrix} \mathbf{K} \\ O \\ O \end{pmatrix} \; , \tag{21}$$

and hence, in this case, $X = X'$ and $X'' = \{0\}$. If instead

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; or \; T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

then $T^2 = 0$ and hence $X = X''$ and $X' = \{0\}$. More generally for an $n{+}1 \times n{+}1$ nilpotent matrix an n-platform is either all or nothing. However in (4.4) $T^2 = 0$ and in the first option the platform is nontrivial:

$$X' = \begin{pmatrix} \mathbf{K} \\ \mathbf{K} \\ O \end{pmatrix} \; , \; X'' = \begin{pmatrix} O \\ O \\ \mathbf{K} \end{pmatrix} \; . \tag{23}$$

We leave it to the reader to determine the platform in the second option.

In a sense however we are able to reduce the general nilpotent to the strictly nilpotent:

5. Example *If $T : X \to Y$ and $S : Y \to Z$ satisfy*

$$ST = 0 : X \to Z \; , \tag{24}$$

and

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 \\ T & 0 & 0 \\ 0 & S & 0 \end{pmatrix} : \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{X} \to \mathbf{X}, \tag{25}$$

then

$$\mathbf{T}(\mathbf{X}) = \begin{pmatrix} O \\ TX \\ SY \end{pmatrix} \subseteq \begin{pmatrix} T^{-1}(0) \\ S^{-1}(0) \\ Z \end{pmatrix} = \mathbf{T}^{-1}(\mathbf{O}) \; . \tag{26}$$

For $\mathbf{X}' \subseteq \mathbf{X}$ to be a subplatform for $\mathbf{T}$ we require

$$\mathbf{X}' = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} \Longrightarrow \mathbf{T}^{-1}(\mathbf{O})_\cap \mathbf{X}' = \begin{pmatrix} T^{-1}(0)_\cap X' \\ S^{-1}(0)_\cap Y' \\ Z' \end{pmatrix} \subseteq \begin{pmatrix} O \\ TX' \\ SY' \end{pmatrix} = \mathbf{T}(\mathbf{X}') \; . \tag{27}$$

While not every subplatform $\mathbf{X}'$ for $\mathbf{T}$ need respect the direct sum decomposition here, Theorem 3 enables us to recognise when such a subplatform is maximal. If for example

$$X = Y = Z \; , \; S = T^n \tag{28}$$

then the analysis of the strictly nilpotent case $T^2 = 0$ extends to the generally nilpotent. If instead $Y = X^2$, $Z = X$ and the chain $(0, S, T, 0)$ is derived from the *Koszul complex* [2],[3],[4] of a commuting pair of operators (T_1, T_2) on X then the analysis offers a reduction of the Taylor singular case into the direct sum of a zero and a nonsingular component. If in particular

$$T'T = I = SS' \ , \ \textit{with} \ T'S' = O \tag{29}$$

then the range of a projection $\mathbf{P}$ will give a platform for $\mathbf{T}$, where

$$\mathbf{P} = \begin{pmatrix} I & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{pmatrix} \ \textit{with} \ Q = TT' + S'S \ . \tag{30}$$

For a specific example take $T = v = (v_1, v_2)$ to be the shifts of [1].

More complicated but still strictly nilpotent matrices represent commuting n-tuples of operators. Inductively if a commuting n-tuple of operators $T = (T_1, T_2, \ldots, T_n)$ is represented by the strictly nilpotent matrix $\mathbf{T}$, and if an operator U commutes with each of the operators T_j, then the $n+1$ tuple (T, U) will be represented by a matrix of the form

$$\begin{pmatrix} \mathbf{T} & \mathbf{O} \\ \mathbf{U} & \mathbf{T} \end{pmatrix} \ , \tag{31}$$

where $\mathbf{U}$ is diagonal with entries $\pm U$. We claim ([3] Theorem 10.9.5, Theorem 10.9.6) that platforms for the matrix (5.8) can be derived from those for $\mathbf{T}$:

6. Theorem *Suppose* $T : X \to Y$, $S : Y \to Z$, $R : Z \to W$, $U : Y \to Y$ *and* $V : Z \to Z$ *satisfy*

$$RS = ST = O = VS + SU \ ; \tag{32}$$

then if

$$R^{-1}(0)_\cap Z' \subseteq S(Y') \subseteq Z' \ \textit{and} \ S^{-1}(0)_\cap Y' \subseteq T(X') \subseteq Y' \ , \tag{33}$$

it follows

$$\begin{pmatrix} R & O \\ V & S \end{pmatrix}^{-1} \begin{pmatrix} O \\ O \end{pmatrix} \cap \begin{pmatrix} Z' \\ Y' \end{pmatrix} \subseteq \begin{pmatrix} S & O \\ U & T \end{pmatrix} \begin{pmatrix} Y' \\ X' \end{pmatrix} \ , \tag{34}$$

which in turn implies

$$\begin{pmatrix} R \\ V \end{pmatrix}^{-1} \begin{pmatrix} O \\ O \end{pmatrix} \cap Z' \subseteq S(Y') \ \textit{and} \ S^{-1}(0)_\cap Y' \subseteq \begin{pmatrix} U & T \end{pmatrix} \begin{pmatrix} Y' \\ X' \end{pmatrix} \ . \tag{35}$$

Proof. If

$$\begin{pmatrix} z' \\ y' \end{pmatrix} \in \begin{pmatrix} R & O \\ V & S \end{pmatrix}^{-1} \begin{pmatrix} O \\ O \end{pmatrix}$$

then since $Rz' = 0$ there is $y' \in Y'$ for which $z' = Sy'$, giving $Sy = -Vz = -VSy' = SUy'$, so that $S(y - Uy') = 0$ and hence there is $x' \in X'$ for which $y - Uy' = Tx'$. as required by (6.3). Conversely if (6.3) holds then

$$R^{-1}(0)_\cap V^{-1}(0)_\cap Z' \subseteq S(U^{-1}(TX')) \subseteq S(Y')$$

and

$$S^{-1}(0)_{\cap}Y' \subseteq T(X') + U(S^{-1}(0)_{\cap}Y') \subseteq T(X') + U(Y') \bullet$$

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