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Operator equations $ABA = A^2$ and $BAB = B^2$

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Abstract

If $A, B \in B(\mathcal{X})$ are Banach space operators (i.e., bounded linear transformations) such that $ABA = A^2$ and $BAB = B^2$, then any one of A, AB, BA and B has the single–valued extension property, or property $(\beta)_\epsilon$, or Bishop's property (β), or the decomposition property (δ), implies they all have the property. Furthermore, they all have the same left (right) spectrum, the spectrum, upper (lower) Fredholm spectrum, essential spectrum, Browder spectrum, Weyl spectrum, Browder essential approximate point spectrum, Weyl essential approximate point spectrum and Drazin spectrum.

1 Introduction

Given Banach space operators $A, B \in B(\mathcal{X})$, or more generally $A \in B(\mathcal{X}, \mathcal{Y})$ and $B \in B(\mathcal{Y}, \mathcal{X})$, the common spectral properties of the operators AB and BA have been studied by a number of authors [3, 4, 5, 15]. It is known that the non–zero points of the spectrum, and a number of its more distinguished parts, are the same for AB and BA. With additional structure, in particular if $A, B \in B(\mathcal{X})$, $ABA = A^2$ and $BAB = B^2$, it is possible to relate the spectrum of A and B, and some of its distinguished parts, to that of AB and BA. A study to this effect has been carried out by Schmoeger [13, 14, 15]. Thus $\sigma_x(A) \setminus \{0\} = \sigma_x(AB) \setminus \{0\} = \sigma_x(BA) \setminus \{0\} = \sigma_x(B) \setminus \{0\}$, where σ_x stands for either of the point spectrum σ_p or the approximate point spectrum σ_a or the the residual spectrum σ_r or the continuous spectrum σ_c; furthermore, the operators A, AB, BA and B have the same spectrum and the same (Fredholm) essential spectrum [14].

In this paper we apply techniques from *local spectral theory* to prove that if $ABA = A^2$ and $BAB = B^2$ for some operators $A, B \in B(\mathcal{X})$, then any one of A, AB, BA and B has the single–valued extension property (SVEP),

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or Bishop's property (β), or the decomposition property (δ), or the Eschmeier–Putinar–Bishop property $(\beta)_\epsilon$ at a point, implies they all have the property at the point. Using the Sadovskii/Buoni, Harte, Wickstead construction, we prove that $\sigma_x(A) = \sigma_x(AB) = \sigma_x(BA) = \sigma_x(B)$,where σ_x is either of the upper Fredholm spectrum, lower Fredholm spectrum, Fredholm essential spectrum, left spectrum, right spectrum, the spectrum, the Browder spectrum, the Weyl spectrum, the Browder essential approximate point spectrum and the Weyl essential approximate point spectrum. The ascent spectrum, the descent spectrum and the Drazin spectrum are also considered.

In the following, A and B shall denote operators in $B(\mathcal{X})$ such that $ABA = A^2$ and $BAB = B^2$. Evidently $0 \notin \sigma(A) \cap \sigma(B)$ implies $A = B$ is the identity operator I; hence, we shall assume in the following that $0 \in \sigma(A) \cap \sigma(B)$. Most of our terminology is standard, and we refer the reader to [1, 9, 10, 11] for any unexplained terminology. In the following, λ and μ (etc.) shall denote complex numbers; we write $T - \lambda$ for $T - \lambda I$.

2 Results

An operator $T \in B(\mathcal{X})$ has *the single-valued extension property at a point* λ_0, SVEP at λ_0, if for every open disc $\mathcal{D}$ centered at λ_0 the only analytic function $f : \mathcal{D} \longrightarrow \mathcal{X}$ satisfying $(T - \lambda)f(\lambda) = 0$ is the function $f \equiv 0$; T has SVEP if it has SVEP everywhere. Recall, [4], that for every $S, T \in B(\mathcal{X})$, ST has SVEP at a point if and only if TS has SVEP at the point. Let $\sigma_{SVEP}(T) = \{\lambda \in \mathbb{C} : T$ does not have SVEP at $\lambda\}$ denote the SVEP spectrum of T. Observe that $\sigma_{SVEP}(T)$ may be empty.

Theorem 2.1. $\sigma_{SVEP}(A) = \sigma_{SVEP}(AB) = \sigma_{SVEP}(BA) = \sigma_{SVEP}(B)$.

Proof. The equivalence AB has SVEP at a point $\mu \Longleftrightarrow BA$ has SVEP at μ holds for all $A, B \in B(\mathcal{X})$ [4]. We prove that A has SVEP at $\mu \Longleftrightarrow AB$ has SVEP at μ; the equivalence B has SVEP at $\mu \Longleftrightarrow BA$ has SVEP at μ is similarly proved. Let $\mathcal{U}$ be an open neighbourhood of μ and $f : \mathcal{U} \longrightarrow \mathcal{X}$ an analytic function such that $(A - \lambda)f(\lambda) = 0$ in $\mathcal{U}$. Then

$$(A^2 - \lambda A)f(\lambda) = 0 \Longleftrightarrow A^2 f(\lambda) = \lambda A f(\lambda) = \lambda^2 f(\lambda) \text{ in } \mathcal{U},$$

and so

$$AB(A-\lambda)f(\lambda) = 0 \Longrightarrow (A^2 - \lambda AB)f(\lambda) = 0 \Longrightarrow (AB - \lambda)(-\lambda f(\lambda)) = 0 \text{ in } \mathcal{U}.$$

Thus, if AB has SVEP at μ, then $\lambda f(\lambda) = 0 \Longrightarrow f(\lambda) = 0$ for all λ in $\mathcal{U}$, implies A has SVEP at μ. Conversely, assume that A has SVEP at μ and let $g : \mathcal{U} \longrightarrow \mathcal{X}$ be an analytic function such that $(AB - \lambda)g(\lambda) = 0$ in $\mathcal{U}$. Then

$$\begin{aligned} AB(AB - \lambda) = 0 &\Longleftrightarrow (A^2 B - \lambda AB)g(\lambda) = (A - \lambda)ABg(\lambda) = 0 \text{ in } \mathcal{U} \\ &\Longrightarrow ABg(\lambda) = 0 \Longrightarrow \lambda g(\lambda) = 0 \text{ in } \mathcal{U} \Longrightarrow g(\lambda) = 0 \text{ in } \mathcal{U}. \end{aligned}$$

This implies that A has SVEP at μ. $\square$

A part of the following theorem has been proved by Schmoeger [14, Corollary 2.11] using a different argument: we prove our result using Theorem 2.1 and a construction of Sadovskii/Buoni, Harte and Wickstead [11, Page 159] (which leads to a representation of the Calkin algebra $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ as an algebra of operators on a suitable Banach space). Recall, [1, Corollary 2.24], that a surjective operator is invertible if and only if it has SVEP at 0. In particular:

Lemma 2.2. *If $T \in B(\mathcal{X})$ is left invertible and T^* has SVEP at 0, then T is invertible.*

We shall require the following lemma.

Lemma 2.3. *The implications*

$$\begin{aligned} & (A-\lambda)^{-1}(0) = \{0\} \Longleftrightarrow (AB-\lambda)^{-1}(0) = \{0\} \\ \Longleftrightarrow\ & (BA-\lambda)^{-1}(0) = \{0\} \Longleftrightarrow (B-\lambda)^{-1}(0) = \{0\} \end{aligned}$$

hold for all λ.

Proof. The proof is quite elementary: we start by proving that if $\lambda \neq 0$, then $(A-\lambda)^{-1}(0) = \{0\} \Longrightarrow (BA-\lambda)^{-1}(0) = \{0\} \Longrightarrow (AB-\lambda)^{-1}(0) = \{0\} \Longrightarrow (B-\lambda)^{-1}(0) = \{0\} \Longrightarrow (A-\lambda)^{-1}(0) = \{0\}$.

If $(A-\lambda)^{-1}(0) = \{0\}$ and $(BA-\lambda)x = 0$ for some $(0 \neq)x \in \mathcal{X}$, then the following implications hold:

$$0 = A(BA-\lambda)x = (A-\lambda)Ax \Longrightarrow Ax = 0 \Longrightarrow BAx = \lambda x = 0 \Longrightarrow x = 0.$$

If $(BA-\lambda)^{-1}(0) = \{0\}$ and $(AB-\lambda)y = 0$ for some $(0 \neq)y \in \mathcal{X}$, then

$$\begin{aligned} & 0 = AB(AB-\lambda)y = (A-\lambda)ABy = \lambda(A-\lambda)y \Longrightarrow (A-\lambda)y = 0 \\ \Longrightarrow\ & 0 = B^2(A-\lambda)y = (B^2A - \lambda BAB)y = (B^2A - \lambda^2 B)y = \lambda(BA-\lambda)By \\ \Longrightarrow\ & By = 0 \Longrightarrow ABy = \lambda y = 0 \Longrightarrow y = 0. \end{aligned}$$

Again, if $(AB-\lambda)^{-1}(0) = \{0\}$ and $(B-\lambda)z = 0$ for some $(0 \neq)z \in \mathcal{X}$, then

$$\begin{aligned} 0 &= (AB^2 - \lambda AB)z = (AB-\lambda)ABz \Longrightarrow ABz = 0 \Longrightarrow 0 = BABz \\ &= B^2 z = \lambda^2 z \Longrightarrow z = 0. \end{aligned}$$

Finally, if $(B-\lambda)^{-1}(0) = \{0\}$ and $(A-\lambda)t = 0$ for some $(0 \neq)t \in \mathcal{X}$, then

$$\begin{aligned} 0 &= BA(A-\lambda)t = (B-\lambda)BAt \Longrightarrow BAt = 0 \Longrightarrow 0 = ABAt = A^2 t \\ &= \lambda^2 t \Longrightarrow t = 0. \end{aligned}$$

Now let $\lambda = 0$. Evidently, $(BA)^{-1}(0) = \{0\}$ (resp., $(AB)^{-1}(0) = \{0\}$) implies $A^{-1}(0) = \{0\}$ (resp., $B^{-1}(0) = \{0\}$); if $A^{-1}(0) = \{0\}$ (resp., $B^{-1}(0) = \{0\}$), then $ABA = A^2$ (resp., $BAB = B^2$) implies $(BA)^{-1}(0) = \{0\}$ (resp., $(AB)^{-1}(0) = \{0\}$). Hence $(BA)^{-1}(0) = \{0\} \Longleftrightarrow A^{-1}(0) = \{0\}$ and $(AB)^{-1}(0) =$

$\{0\} \Longleftrightarrow B^{-1}(0) = \{0\}$. Assume now that $(AB)^{-1}(0) = \{0\}$. Then it follows from the implications

$$(AB)(AB-B) = (AB)^2 - AB^2 = 0 \Longrightarrow AB = B \Longrightarrow AB(B-I) = AB^2 - AB = 0$$

that $B = I = AB = A = BA$. A similar argument show that if $(BA)^{-1}(0) = \{0\}$, then (again) $B = I = AB = A = BA$. $\square$

The proof of Lemma 2.2 shows that the only way 0 can fail to be in the spectrum of AB and BA is if 0 is not in the point spectrum of A or B. Evidently, the operators A, B, AB and BA have the same point spectrum; an appeal to the Berberian–Quigley extension theorem [10, Page 255] shows that they also have the same approximate point spectrum. As we shall see later, they (also) have the same spectrum.

The Sadovskii/Buoni, Harte, Wickstead construction [11] may be summarized as follows. Let $\ell^\infty(\mathcal{X})$ denote the Banach space of all bounded sequences $x = (x_n)_{n=1}^\infty$ of elements of $\mathcal{X}$ endowed with the norm $||x||_\infty := \sup_{n\in\mathbb{N}} ||x_n||$, and write T_∞, $T_\infty x := (Tx_n)_{n=1}^\infty$ for all $x = (x_n)_{n=1}^\infty$, for the operator induced by T on $\ell^\infty(\mathcal{X})$. The set $m(\mathcal{X})$ of all precompact sequences of elements of $\mathcal{X}$ is a closed subspace of $\ell^\infty(\mathcal{X})$ which is invariant for T_∞. Let $\mathcal{X}_q := \ell^\infty \mathcal{X})/m(\mathcal{X})$, and denote by T_q the operator T_∞ on $\mathcal{X}_q$. The mapping $T \mapsto T_q$ is then a unital homomorphism from $B(\mathcal{X}) \mapsto B(\mathcal{X}_q)$ with kernel $\mathcal{K}(\mathcal{X})$ (= the ideal of compact operators) which induces a norm decreasing monomorphism from $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ to $B(\mathcal{X}_q)$ with the following properties (see [11, Section 17] for details):

Lemma 2.4. *(i) T is upper semi-Fredholm, $T \in \phi_+$, if and only if T_q is injective, if and only if T_q is bounded below;*

(ii) T is lower semi-Fredholm, $T \in \phi_-$, if and only if T_q is onto, if and only if $T_q \in \phi_-$;

(iii) T is Fredholm, $T \in \phi$, if and only if T_q is invertible.

Let $\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \phi_+\}$ denote the upper semi–Fredholm spectrum of T, $\sigma_{SF_-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \phi_-\}$ the lower semi–Fredholm spectrum of T and let $\sigma_e(T) = \sigma_{SF_+}(T) \cup \sigma_{SF_-}(T)$ denote the (Fredholm) essential spectrum of T.

Theorem 2.5. *$\sigma_x(A) = \sigma_x(AB) = \sigma_x(BA) = \sigma_x(B)$, where $\sigma_x = \sigma_{SF_+}$ or σ_{SF_-} or σ_e.*

Proof. We prove the equality of the spectra for the case in which $\sigma_x = \sigma_{SF_+}$; the proof for σ_{SF_-} follows from a duality argument and the proof for σ_e is then a straightforward consequence. Apparently,

$$A_q B_q A_q = A_q^2 \text{ and } B_q A_q B_q = B_q^2.$$

Let $T - \lambda \in \phi_+$, where T stands for one of the operators A, AB, BA and B. Then, see Lemma 2.4, $T_q - \lambda I_q$ is left invertible , in particular injective. This implies by Lemma 2.3 that the operators $A_q - \lambda I_q$, $A_q B_q - \lambda I_q$, $B_q A_q - \lambda I_q$

and $B_q - \lambda I_q$ are all injective, hence all left invertible (by Lemma 2.4). Another application of Lemma 2.4 now proves that $A - \lambda$, $AB - \lambda$, $BA - \lambda$ and $B - \lambda$ are all upper semi–Fredholm. $\square$

Let $\sigma_{left}(T) = \{\lambda : T - \lambda$ is not left invertible$\}$ and $\sigma_{right}(T) = \{\lambda : T - \lambda$ is not right invertible$\}$. Part of the following corollary appears in [14, Proposition 2.6 and Corollary 2.7]

Corollary 2.6. *$\sigma_x(A) = \sigma_x(AB) = \sigma_x(BA) = \sigma_x(B)$, where $\sigma_x = \sigma_{left}$ or σ_{right} or σ.*

Proof. We prove the case $\sigma_x = \sigma_{left}$; the proof for σ_{right} follows from a duality argument, and the proof for σ is a consequence of $\sigma = \sigma_{left} \cup \sigma_{right}$. If either of $A-\lambda$, $AB-\lambda$, $BA-\lambda$ and $B-\lambda$ is left invertible (hence upper semi–Fredholm and injective), then they are all upper semi–Fredholm and injective, hence left invertible. $\square$

An operator $T \in B(\mathcal{X})$ is said to have a *generalized Drazin inverse* at λ if λ is an accumulation point of $\sigma(T)$ [6, Theorem 2.2.1]. If we denote the generalized Drazin spectrum of T by $\sigma_{GD}(T)$, then it is immediate from Corollary 2.6 that:

Corollary 2.7. *$\sigma_{GD}(A) = \sigma_{GD}(AB) = \sigma_{GD}(BA) = \sigma_{GD}(B)$.*

The ascent (resp., descent) of an operator $T \in B(\mathcal{X})$, asc(T) (resp., dsc(T)), is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ (resp., $T^n\mathcal{X} = T^{n+1}\mathcal{X}$); if no such integer exists, then asc$(T) = \infty$ (resp., dsc$(T) = \infty$). Let $\sigma_a(T)$ denote the approximate point spectrum of T. The Browder, the Weyl , the Browder essential approximate point and the Weyl essential approximate point spectrum of an operator $T \in B(\mathcal{X})$ are the sets $\sigma_b(T) = \{\lambda \in \sigma(T) : T - \lambda \notin \phi$ or one of asc$(T-\lambda)$ and dsc$(T-\lambda)$ is not finite$\}$, $\sigma_w(T) = \{\lambda \in \sigma(T) : T-\lambda \notin \phi$ or ind$(T - \lambda) \neq 0\}$, $\sigma_{ab}(T) = \{\lambda \in \sigma_a(T) : T - \lambda \notin \phi_+$ or asc$(T - \lambda \not< \infty\}$ and $\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) : T - \lambda \notin \phi_+$ or ind$(T - \lambda \neq 0\}$, respectively. Recall that $T - \lambda$ is said to be Weyl (resp., a-Weyl) at λ if $\lambda \notin \sigma_w(T)$ (resp., $\lambda \notin \sigma_{aw}(T)$).

Corollary 2.8. *$\sigma_x(A) = \sigma_x(AB) = \sigma_x(BA) = \sigma_x(B)$, where $\sigma_x = \sigma_b$ or σ_w or σ_{ab} or σ_{aw}.*

Proof. Let T represent either of A, AB, BA and B. Recall that asc$(T - \lambda) < \infty \Longrightarrow T$ has SVEP at λ, dsc$(T - \lambda) < \infty \Longrightarrow T^*$ has SVEP at λ, and if asc$(T-\lambda)$ and dsc$(T-\lambda)$ are both finite, then (they are equal and) both T and T^* have SVEP at λ ([1, Theorems 3.3 and 3.8] (see also [9])); recall also that a Fredholm operator $T - \lambda$ such that both T and T^* have SVEP at λ satisfies asc$(T - \lambda) =$ dsc$(T - \lambda) < \infty$ [1, Corollary 3.21]. Now apply Theorems 2.5 and 2.1 to prove the equality for the case in which $\sigma_x = \sigma_b$ or σ_{ab}. To prove the equality of the spectra for the case $\sigma_x = \sigma_w$ (resp., σ_{aw}), we observe that if $\lambda \notin \sigma_w(T)$ (resp., $\lambda \notin \sigma_{aw}(T)$) for a choice of T, then $T - \lambda \in \phi$ (resp., ϕ_+) for every choice of T. Recall now from [14, Corollary 2.12] that if $T - \lambda \in \phi$ for a choice of T, then ind$(A - \lambda) =$ ind$(AB - \lambda) =$ ind$(BA - \lambda) =$ ind$(B - \lambda)$. Hence if $T - \lambda$ is Weyl for a choice of T, then $A - \lambda$, $AB - \lambda$, $BA - \lambda$ and $B - \lambda$

are all Weyl. If instead $T-\lambda \in \phi_+$ and $\mathrm{ind}(T-\lambda) > 0$ for a choice of T, say T', other than the one we started with, then $T'-\lambda \in \phi \Longrightarrow T-\lambda \in \phi$ and $\mathrm{ind}(T-\lambda) > 0$ for every choice of T; hence $A-\lambda$, $AB-\lambda$, $BA-\lambda$ and $B-\lambda$ are all a-Weyl. □

For a Banach space $\mathcal{X}$ and open subset $\mathcal{U}$ of $\mathtt{C}$, let $\mathcal{E}(\mathcal{U},\mathcal{X})$ (resp., $\mathcal{O}(\mathcal{U},\mathcal{X})$) denote the Fréchet space of all infinitely differentiable $\mathcal{X}$–valued functions on $\mathcal{U}$ endowed with the topology of uniform convergence of all derivatives on compact subsets of $\mathcal{U}$ (resp., of all analytic $\mathcal{X}$–valued functions on $\mathcal{U}$ endowed with the topology of uniform convergence on compact subsets of $\mathcal{U}$). We say that $T \in B(\mathcal{X})$ satisfies (Eschmeier–Putinar–Bishop) property $(\beta)_\epsilon$ at λ if there exists a neighbourhood $\mathcal{N}$ of λ such that, for each open subset $\mathcal{U}$ of $\mathcal{N}$ and sequence $\{f_n\}$ of $\mathcal{X}$–valued functions in $\mathcal{E}(\mathcal{U},\mathcal{X})$,

$$(T-z)f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}) \Longrightarrow f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X})$$

[7]; T satisfies (Bishop's) property (β) at $\lambda \in \mathtt{C}$ if there exists an $r > 0$ such that, for every open subset $\mathcal{U}$ of the open disc $\mathbf{D}(\lambda; r)$ of radius r centered at λ and sequence $\{f_n\}$ of $\mathcal{X}$–valued functions in $\mathcal{O}(\mathcal{U},\mathcal{X})$,

$$(T-z)f_n(z) \longrightarrow 0 \text{ in } \mathcal{O}(\mathcal{U},\mathcal{X}) \Longrightarrow f_n(z) \longrightarrow 0 \text{ in } \mathcal{O}(\mathcal{U},\mathcal{X})$$

[10, Page 11]. Property $(\beta)_\epsilon$ implies property (β) [7]. It is well known that property (β) implies SVEP, and that T satisfies property (β) if and only if T^* satisfies property (δ) [10]. (We shall have no more than a passing interest in the (decomposition) property (δ): the interested reader is invited to consult [10], in particular Definitions 1.2.28.) Let $\sigma_{(\beta)_\epsilon}(T) = \{\lambda \in \mathtt{C} : T$ does not satisfy property $(\beta)_\epsilon$ at $\lambda\}$, $\sigma_{(\beta)}(T) = \{\lambda \in \mathtt{C} : T$ does not satisfy property (β) at $\lambda\}$ and $\sigma_\delta(T) = \{\lambda \in \mathtt{C} : T$ does not satisfy property (δ) at $\lambda\}$.

Theorem 2.9. *$\sigma_x(A) = \sigma_x(AB) = \sigma_x(BA) = \sigma_x(B)$, where $\sigma_x = \sigma_{(\beta)_\epsilon}$ or $\sigma_{(\beta)}$ or $\sigma_{(\delta)}$.*

Proof. It is known [4] that $\sigma_{(\beta)_\epsilon}(AB) = \sigma_{(\beta)_\epsilon}(BA)$ for all $A, B \in B(\mathcal{X})$. We prove: $\lambda \notin \sigma_{(\beta)_\epsilon}(A) \Longrightarrow \lambda \notin \sigma_{(\beta)_\epsilon}(AB) \Longrightarrow \lambda \notin \sigma_{(\beta)_\epsilon}(BA) \Longrightarrow \lambda \notin \sigma_{(\beta)_\epsilon}(B) \Longrightarrow \lambda \notin \sigma_{(\beta)_\epsilon}(A)$.

Thus suppose that $A-\lambda \in (\beta)_\epsilon$ (i.e., $\lambda \notin \sigma_{(\beta)_\epsilon}(A)$) and that $\{f_n\}$ is a sequence in $\mathcal{E}(\mathcal{U},\mathcal{X})$ such that

$$(AB-z)f_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}).$$

Then

$$\begin{aligned} AB(AB-z)f_n(z) &= (A-z)ABf_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}) \\ &\Longrightarrow ABf_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}). \end{aligned}$$

Hence $zf_n(z)$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$, and so $f_n(z), \longrightarrow 0$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$. Suppose now that $BA-z \in (\beta)_\epsilon$ and that $\{g_n\}$ is a sequence in $\mathcal{E}(\mathcal{U},\mathcal{X})$ such that

$$(B-z)g_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}).$$

Then

$$\begin{aligned} BA(B-z)g_n(z) &= (BA-z)Bg_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}) \\ &\Longrightarrow Bg_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}). \end{aligned}$$

Hence $zg_n(z)$, so also $g_n(z)$, $\longrightarrow 0$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$. Finally suppose that $B - z \in (\beta)_\epsilon$ and that $\{F_n\}$ is a sequence in $\mathcal{E}(\mathcal{U},\mathcal{X})$ such that

$$(A-z)F_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}).$$

Then

$$\begin{aligned} BA(A-z)F_n(z) &= B(AB-z)AF_n(z) = (B-z)BAF_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}) \\ &\Longrightarrow BAF_n(z) \longrightarrow 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}). \end{aligned}$$

But then $ABAF_n(z) = A^2F_n(z) \longrightarrow 0$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$. Since $(A-z)F_n(z) \longrightarrow 0$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$ also implies $(A^2 - zA)F_n(z) \longrightarrow 0$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$, it follows that $zAF_n(z) \longrightarrow 0 \Longrightarrow AF_n(z) \longrightarrow 0$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$. Hence $zF_n(z)$, so also $F_n(z)$, $\longrightarrow 0$ in $\mathcal{E}(\mathcal{U},\mathcal{X})$.

The proof for $\sigma_{(\beta)}$ follows from a similar argument, and then the proof for σ_δ follows from a duality argument. □

Let $\sigma_{\text{asc}}(T) = \{\lambda : \text{asc}(T-\lambda) = \infty\}$ and $\sigma_{\text{dsc}}(T) = \{\lambda : \text{dsc}(T-\lambda) = \infty\}$ denote the ascent spectrum and the descent spectrum of T, respectively. (We remark here that this definition of the ascent spectrum and the descent spectrum differs from the usual topological definition, wherein one assumes also that $(T-\lambda)^{\text{asc}(T-\lambda)+1}\mathcal{X}$, respectively $(T-\lambda)^{\text{dsc}(T-\lambda)}\mathcal{X}$, is closed; see, for example [8].) The following theorem shows that A, B, AB and BA have the same non-zero ascent and descent spectrum.

Theorem 2.10. $\sigma_x(A) \setminus \{0\} = \sigma_x(B) \setminus \{0\} = \sigma_x(AB) \setminus \{0\} = \sigma_x(BA) \setminus \{0\}$, *where* $\sigma_x = \sigma_{asc}$ *or* σ_{dsc}.

Proof. In view of the fact that $\text{asc}(AB-\lambda) = \text{asc}(BA-\lambda)$ and $\text{dsc}(AB-\lambda) = \text{dsc}(BA-\lambda)$ for all $A, B \in B(\mathcal{X})$ [5, Theorem 1], it will suffice to prove: (i) $\text{asc}(A-\lambda) = \text{asc}(AB-\lambda) = \text{asc}(B-\lambda$, and (ii) $\text{dsc}(A-\lambda) = \text{dsc}(AB-\lambda) = \text{dsc}(B-\lambda)$ for all $\lambda \neq 0$. Let $\lambda \neq 0$.

(i) Suppose that $(B-\lambda)^n x = 0$ and $(B-\lambda)^{n-1}x \neq 0$ for some $(0 \neq)$ $x \in \mathcal{X}$ and some integer $n \geq 1$. Then

$$0 = B(B-\lambda)^n x = (BA-\lambda)^n Bx \Longrightarrow Bx \in (BA-\lambda)^{-n}(0).$$

If $(BA-\lambda)^{n-1}Bx = 0$, then $B(B-\lambda)^{n-1}x = 0$, and hence

$$0 = (B-\lambda)^n x = B(B-\lambda)^{n-1}x - \lambda(B-\lambda)^{n-1}x \Longrightarrow (B-\lambda)^{n-1}x = 0.$$

This is a contradiction; hence $\text{asc}(AB-\lambda) = \text{asc}(BA-\lambda) \leq \text{asc}(B-\lambda)$. Conversely, assume that $(AB-\lambda)^n x = 0$ and $(AB-\lambda)^{n-1}x \neq 0$ for some $(0 \neq)$ $x \in \mathcal{X}$ and some integer $n \geq 1$. Then

$$0 = B(AB-\lambda)^n = (B-\lambda)^n Bx \Longrightarrow Bx \in (B-\lambda)^{-n}(0).$$

If $(B-\lambda)^{n-1}Bx=0$, then $B(AB-\lambda)^{n-1}x=0$, and hence

$$0=(AB-\lambda)^n x = AB(AB-\lambda)^{n-1}-\lambda(AB-\lambda)^{n-1}x \Longrightarrow (AB-\lambda)^{n-1}x=0.$$

This contradiction implies that $\mathrm{asc}(BA-\lambda)\leq \mathrm{asc}(B-\lambda)$. The proof of $\mathrm{asc}(A-\lambda)=\mathrm{asc}(AB-\lambda)$ follows from a similar argument.

(*ii*) Suppose that $y=(B-\lambda)^n x$ for some $(0\neq)\ x\in\mathcal{X}$ and integer $n\geq 1$, but y is not in the range of $(B-\lambda)^{n+1}$. Then $By=(BA-\lambda)^n Bx$, and so $By\in(BA-\lambda)^n\mathcal{X}$. If $By\in(BA-\lambda)^{n+1}\mathcal{X}$, then there exists a $w\in\mathcal{X}$ such that

$$(BAB-\lambda^2)y = BA((BA-\lambda)^{n+1}w)-\lambda^2 y.$$

But then

$$\begin{aligned}
\lambda^2 y &= BA(BA-\lambda)^{n+1}w-(BAB-\lambda^2)y\\
&= BA(BA-\lambda)^{n+1}w-(BAB-\lambda B)y-\lambda(B-\lambda)y\\
&= (B-\lambda)^{n+1}BAw-B(B-\lambda)y-\lambda(B-\lambda)y\\
&= (B-\lambda)^{n+1}\{BAw-Bx-\lambda x\},
\end{aligned}$$

i.e., $y\in(B-\lambda)^{n+1}\mathcal{X}$ — a contradiction. Hence $\mathrm{dsc}(BA-\lambda)=\mathrm{dsc}(AB-\lambda)\leq \mathrm{dsc}(B-\lambda)$. Conversely, suppose that $y=(BA-\lambda)^n x$ for some $(0\neq)\ x\in\mathcal{X}$ and integer $n\geq 1$, but y is not in the range of $(BA-\lambda)^{n+1}$. Then

$$BAy=(B-\lambda)^n BAx \Longrightarrow BAx\in(B-\lambda)^{\mathcal{X}}.$$

If $BAx\in(B-\lambda)^{n+1}\mathcal{X}$, then (upon arguing as above) there exists a $w\in\mathcal{X}$ such that

$$\begin{aligned}
\lambda^2 y &= B(B-\lambda)^{n+1}w-(B^2A-\lambda BA)y-\lambda(BA-\lambda)y\\
&= (BA-\lambda)^{n+1}\{Bw-BAx-\lambda x\},
\end{aligned}$$

i.e., $y\in(BA-\lambda)^{n+1}\mathcal{X}$ — a contradiction. Hence $\mathrm{dsc}(B-\lambda)\leq\mathrm{dsc}(BA-\lambda)=\mathrm{dsc}(AB-\lambda)$. A similar argument proves $\mathrm{dsc}(A-\lambda)=\mathrm{dsc}(AB-\lambda)$. $\square$

The Drazin spectrum of $T\in B(\mathcal{X})$ is the set $\sigma_D(T)=\{\lambda\in\mathbb{C}:\lambda\in\sigma_{\mathrm{asc}}(T)\cup\sigma_{\mathrm{dsc}}(T)\}$. Theorem 2.10 shows that the operators A, B, AB and BA have the same non-zero Drazin spectrum. Recall, [12, Theorem 1], however that $\sigma_D(ST)=\sigma_D(TS)$ for every $S,T\in B(\mathcal{X})$; the following theorem shows that this equality extends to our operators A, B, AB and BA.

Theorem 2.11. $\sigma_D(A)=\sigma_D(AB)=\sigma_D(BA)=\sigma_D(B)$.

Proof. Evidently, 0 is in the spectrum of the operators A, B, AB and BA. The equations $ABA=A^2$ and $BAB=B^2$ imply that $(AB)^2=A^2B=AB^2$. Recall that the Drazin spectrum is a regularity [11, Theorem 10, Page 195], and so satisfies the spectral mapping theorem for every function which is analytic on an open neighbourhood of, and non-constant on connected components of, the spectrum of the operator [11, Theorem 7, Page 52]. Thus, if $0 \in \sigma_D(AB)$ (equivalently, $0 \in \sigma_D(BA)$), then $0 \in \sigma_D(A^2B)$ and $\sigma_D(AB^2)$. Since $\sigma_D(ST) = \sigma_D(TS)$ for every $S, T \in B(\mathcal{X})$, $\{\sigma_D(AB)\}^2 = \sigma_D((AB)^2) = \sigma_D(A^2B) = \sigma_D(ABA) = \sigma_D(A^2) = \{\sigma_D(A)\}^2 = \sigma_D(AB^2) = \sigma_D(BAB) = \sigma_D(B^2) = \{\sigma_D(B)\}^2$. But then $0 \in \sigma_D(A) \iff 0 \in \sigma_D(B) \iff 0 \in \sigma_D(AB)$. $\square$

We note here that unlike the case of $\lambda \neq 0$ (when $A - \lambda$, $B - \lambda$, $AB - \lambda$ and $BA - \lambda$ have the same Drazin index whenever $\lambda \notin \sigma_D(A)$), $0 = \lambda \notin \sigma_D(A)$ does not imply that the operators A, B, AB and BA have the same Drazin index [15].

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