

## Outer generalized inverses of centralizers over semiprime rings

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### Abstract

Let  $T$  be a centralizer over a semiprime ring. We find the necessary and sufficient conditions for  $T$  to have an outer generalized inverse with prescribed range and kernel (in the ring of all centralizers). As a corollary we obtain some results from (M. A. Chaudhry and M. S. Samman, *Generalized inverses of centralizers of semiprime rings*, Aequationes Math. 71 (2006), 246–252).

## 1 Introduction

Recently (see [3]), new results appeared concerning generalized inverses of centralizers on semiprime rings. Precisely, if  $T$  is a centralizer, then there exists the commutative generalized inverse of  $T$  if and only if  $T$  is orthogonal. The commutativity condition is a general property of centralizers, and we shall recall the definition of orthogonal centralizers later. Our aim is to consider outer generalized inverses of centralizers on semiprime rings, and then to obtain some results from [3] as a corollary. We recall the interest for outer generalized inverses and related topics in [1, 5, 6, 7, 8, 11]. Generalized inverses are useful in solving overdetermined linear systems and singular operator equations (see [1, 2, 4, 11]).

Recall that the ring  $\mathcal{R}$  is called semiprime, if  $a\mathcal{R}a = \{0\}$  implies  $a = 0$ . Let  $T : \mathcal{R} \rightarrow \mathcal{R}$  be an additive mapping. Then  $T$  is called a centralizer on  $\mathcal{R}$ , if and only if  $T(xy) = xT(y) = T(x)y$  hold for every  $x, y \in \mathcal{R}$ . Equivalently,  $T$  is a centralizer on  $\mathcal{R}$ , if and only if  $T$  is additive and  $T(x)y = xT(y)$  holds for every  $x, y \in \mathcal{R}$ . The set  $\mathcal{M}(\mathcal{R})$  of all centralizers on  $\mathcal{R}$  is a commutative ring. For further properties of centralizers on semiprime rings see [3, 9, 10].

If  $T \in \mathcal{M}(\mathcal{R})$ , then  $\mathcal{R}(T)$  is the range, and  $\mathcal{N}(T)$  is the kernel of  $T$ . Notice that both  $\mathcal{R}(T)$  and  $\mathcal{N}(T)$  are two-sided ideals of  $\mathcal{R}$ .

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2010 *Mathematics Subject Classifications*. 16N10, 15A09.

*Key words and Phrases*. Generalized inverses; centralizers; semiprime rings.

Received: September 4, 2010

Communicated by Dragana Cvetković Ilić

Supported by the grant no. 174007 of the Ministry of Science, Serbia.

Let  $T \in \mathcal{M}(\mathcal{R})$  be given. If there exists some  $S \in \mathcal{M}(\mathcal{R})$  such that  $STS = S$  holds, then  $S$  is an outer generalized inverse of  $T$  in the ring  $\mathcal{M}(\mathcal{R})$ . In this case we denote  $\mathcal{U} = \mathcal{R}(S)$  and  $\mathcal{V} = \mathcal{N}(S)$ . Hence,  $S = T_{\mathcal{U}, \mathcal{V}}^{(2)}$ , i.e.  $S$  is an outer generalized inverse of  $T$  with the fixed (prescribed) range and kernel.

## 2 Results

We are interested in the existence and the uniqueness of outer generalized inverses with fixed range and kernel (in a way that is well-known for complex matrices, or linear bounded operators on Banach spaces). The following theorem represents the existence and the uniqueness of outer generalized inverses with fixed range and kernel, considering the situation in  $\mathcal{M}(\mathcal{R})$ .

**Theorem 2.1.** *Let  $\mathcal{R}$  be a semiprime ring, and let  $\mathcal{M}(\mathcal{R})$  and  $T \in \mathcal{M}(\mathcal{R})$  be as above. Suppose that  $\mathcal{U}$  and  $\mathcal{V}$  are two-sided ideals of  $\mathcal{R}$ . Then there exists  $S = T_{\mathcal{U}, \mathcal{V}}^{(2)} \in \mathcal{M}(\mathcal{R})$ , if and only if the following hold:*

- (a) *there exist two-sided ideals  $\mathcal{U}_1$  and  $\mathcal{V}_1$  of  $\mathcal{R}$ , such that  $\mathcal{R} = \mathcal{U} \oplus \mathcal{U}_1 = \mathcal{V}_1 \oplus \mathcal{V}$ ;*
- (b) *the reduction  $T|_{\mathcal{U}} : \mathcal{U} \rightarrow T(\mathcal{U})$  is invertible;*
- (c)  *$T(\mathcal{U}) = \mathcal{V}_1$ .*

*Moreover, if previous conditions are fulfilled, then such an  $S \in \mathcal{M}(\mathcal{R})$  is unique.*

*Proof.* Suppose that  $S = T_{\mathcal{U}, \mathcal{V}}^{(2)} \in \mathcal{M}(\mathcal{R})$  exists for some two-sided ideals  $\mathcal{U}, \mathcal{V}$  of  $\mathcal{R}$ . Then  $ST$  and  $TS$  belong to  $\mathcal{M}(\mathcal{R})$ . It is easy to see that  $\mathcal{R}(ST) = \mathcal{R}(S) = \mathcal{U}$ . We take  $\mathcal{U}_1 = \mathcal{N}(ST) = \mathcal{R}(I - TS)$ , and get  $\mathcal{R} = \mathcal{U} \oplus \mathcal{U}_1$ . Since  $\mathcal{V} = \mathcal{N}(S)$ , we take  $\mathcal{V}_1 = \mathcal{R}(TS)$ , and obtain  $\mathcal{R} = \mathcal{R}(TS) \oplus \mathcal{N}(TS) = \mathcal{V}_1 \oplus \mathcal{V}$ . By definition, we have  $T(\mathcal{U}) = \mathcal{V}_1$ . We have to prove that  $T$  is one-to-one on  $\mathcal{U}$ . Suppose that  $Tx = 0$  and  $x \in \mathcal{R}(S) = \mathcal{U}$ . Then  $x = Sy$  for some  $y \in \mathcal{R}$ . Hence,  $TSy = 0$  and  $Sy = STSy = 0$ , implying that  $x = 0$ . Hence,  $T|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{V}_1$  is invertible.

We have even more. The restriction  $S|_{\mathcal{V}_1} : \mathcal{V}_1 \rightarrow \mathcal{U}$  is also invertible. Hence,  $S|_{\mathcal{V}_1} = (T|_{\mathcal{U}})^{-1}$ .

On the other hand, suppose that the conditions (a), (b) and (c) hold. Let  $x \in \mathcal{R}$ . Then  $x = u + v$ , where  $u \in \mathcal{V}_1$  and  $v \in \mathcal{V}$ . There exists the unique  $w \in \mathcal{U}$  such that  $T(w) = u$ . Define  $S(x) = w$ . We see that  $STSx = STw = Su = Sx$  for every  $x \in \mathcal{R}$ . Hence,  $STS = S$ .

We have to verify that  $S$  is a centralizer on  $\mathcal{R}$ . First, we show that  $S$  is additive on  $\mathcal{R}$ . Let  $x = u + v$  and  $y = a + b$ , where  $u, a \in \mathcal{V}_1$  and  $v, b \in \mathcal{V}$ . Then there exist the unique  $w, z \in \mathcal{U}$  such that  $T(w) = u$  and  $T(z) = a$ . These facts mean that  $S(u) = w$  and  $S(a) = z$ ,  $T(w + z) = u + a$  (since  $T$  is additive) and  $S(u + a) = w + z$ . Hence,

$$S(x + y) = S((u + a) + (v + b)) = S(u + a) = w + z = S(u) + S(a) = S(x) + S(y).$$

Hence,  $S$  is additive. We have to verify that  $S(x)y = xS(y)$  holds. Notice that  $wT(z) = T(w)z$  implies  $wa = uz$ . On the other hand, we have

$$xS(y) = (u + v)z = uz + vz,$$

and

$$S(x)y = wa + wb.$$

We know that  $wa = uz$ . To prove that  $S$  is a centralizer, it is enough to prove  $vz = wb$ . Notice that  $vz, wb \in \mathcal{V} \cap \mathcal{U}$ . We also have

$$T(vz) = vT(z) = va \in \mathcal{V} \cap \mathcal{V}_1 = \{0\},$$

implying  $vz = 0$ , since  $T$  is one-to-one on  $\mathcal{U}$ . In the same manner we obtain  $wb = 0$ . Hence,  $S$  is a centralizer on  $\mathcal{R}$ .

The last part of the proof is the uniqueness of  $S$ . On the contrary, suppose that there exists one more  $S_1 \in \mathcal{M}(\mathcal{R})$  such that  $S_1TS_1 = S_1$ ,  $\mathcal{R}(S_1) = \mathcal{U}$  and  $\mathcal{N}(S_1) = \mathcal{V}$ . It is enough to prove  $S|_{\mathcal{V}_1} = S_1|_{\mathcal{V}_1}$ . We know that these reductions are one-to-one mappings from  $\mathcal{V}_1$  to  $\mathcal{U}$ . We also know that  $S(\mathcal{V}_1) = S_1(\mathcal{V}_1) = \mathcal{U}$ , so  $S|_{\mathcal{V}_1}, S_1|_{\mathcal{V}_1} : \mathcal{V}_1 \rightarrow \mathcal{U}$  are invertible mappings. Since  $STS = S$ , it is easy to conclude that  $T|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{V}_1$  is ordinary inverse (as a mapping) of both  $S|_{\mathcal{V}_1}$  and  $S_1|_{\mathcal{V}_1}$ . Taking the ordinary inverse of  $T|_{\mathcal{U}}$ , we get that  $S|_{\mathcal{V}_1} = S_1|_{\mathcal{V}_1}$ .  $\square$

As a corollary, we obtain the main result from [3]. Recall that  $T \in \mathcal{M}(\mathcal{R})$  is called orthogonal, if  $\mathcal{R} = \mathcal{R}(T) \oplus \mathcal{N}(T)$ . In [3] a centralizer  $S$  is a generalized inverse of  $T \in \mathcal{M}(\mathcal{R})$ , if  $TST = T$  and  $STS = S$ , i.e.  $S$  is a commuting reflexive generalized inverse of  $T$  in  $\mathcal{M}(\mathcal{R})$ . Now, we get the following result.

**Corollary 2.1.** *Let  $\mathcal{R}$  be semiprime and  $T \in \mathcal{M}(\mathcal{R})$ . Then there exists a generalized inverse  $S \in \mathcal{M}(\mathcal{R})$  of  $T$  if and only if  $T$  is orthogonal.*

*Proof.*  $\Leftarrow$ : Let  $T$  be orthogonal, i.e.  $\mathcal{R} = \mathcal{R}(T) \oplus \mathcal{N}(T)$ . We apply Theorem 2.1 with  $\mathcal{U} = \mathcal{V}_1 = \mathcal{R}(T)$  and  $\mathcal{V} = \mathcal{U}_1 = \mathcal{N}(T)$ . According to the construction of  $S$ , we easily obtain  $TST = T$ .

$\Rightarrow$ : Let  $TST = T$  and  $STS = S$ . Take  $\mathcal{U} = \mathcal{R}(S) = \mathcal{R}(ST)$  and  $\mathcal{V} = \mathcal{R}(T) = \mathcal{R}(TS)$ . Then we can also take  $\mathcal{U}_1 = \mathcal{N}(ST) = \mathcal{N}(T)$  and  $\mathcal{V}_1 = \mathcal{R}(TS) = \mathcal{R}(T)$ . Since  $T_{\mathcal{U}, \mathcal{V}}^{(2)}$  is unique, we obtain that  $T_{\mathcal{U}, \mathcal{V}}^{(2)} = S$ . Finally,  $\mathcal{R} = \mathcal{R}(T) \oplus \mathcal{N}(T)$  and  $T$  is orthogonal.  $\square$

Now, we consider the Drazin invertibility of  $T \in \mathcal{M}(\mathcal{R})$ . A centralizer  $S \in \mathcal{M}(\mathcal{R})$  is a Drazin inverse of  $T$ , if the following hold:

$$STS = S, \quad T^{n+1}S = T^n,$$

for some non-negative integer  $n$ . It is known that the Drazin inverse is unique whenever it exists. The Drazin inverse of  $T$  is denoted by  $T^D$ , and  $T$  is called Drazin invertible. In this case  $P = TS - I$  is a centralizer and a projection. Hence,  $\mathcal{R} = \mathcal{N}(P) \oplus \mathcal{R}(P)$ . Now,  $P$  commutes with  $T$ , the reduction  $T_1 = T|_{\mathcal{N}(P)} : \mathcal{N}(P) \rightarrow \mathcal{N}(P)$  is invertible, and the reduction  $T_2 = T|_{\mathcal{R}(P)} : \mathcal{R}(P) \rightarrow \mathcal{R}(P)$  is nilpotent. Hence,  $T_1$  is a centralizer,  $T_1^{-1}$  is a centralizer, and  $T^D|_{\mathcal{N}(P)} = T_1^{-1}$ ,  $T^D|_{\mathcal{R}(P)} = 0$ . Moreover, the following result holds.

**Corollary 2.2.** *Let  $\mathcal{R}$  be a semiprime ring and  $T \in \mathcal{M}(\mathcal{R})$ . Then  $T$  is Drazin invertible in  $\mathcal{M}(\mathcal{R})$ , if and only if there exist ideals  $\mathcal{M}$  and  $\mathcal{N}$  of  $\mathcal{R}$ , such that  $\mathcal{R} = \mathcal{M} \oplus \mathcal{N}$ , this decomposition reduces  $T$ , the restriction  $T|_{\mathcal{M}}$  is invertible and the restriction  $T|_{\mathcal{N}}$  is nilpotent. In this case  $T^D = T_{\mathcal{M},\mathcal{N}}^{(2)}$ .*

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