Idempotents related to
the weighted Moore–Penrose inverse

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Abstract
We investigate necessary and sufficient conditions for $aa^\dagger_{e,f} = bb^\dagger_{e,f}$ to hold in rings with involution. Here, $a^\dagger_{e,f}$ denotes the weighted Moore–Penrose inverse of $a$, related to invertible and Hermitian elements $e, f \in R$. Thus, some recent results from [7] are extended to the weighted Moore–Penrose inverse.

1 Introduction
Let $R$ be an associative ring with the unit 1. An involution $a \mapsto a^*$ in a ring $R$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.\$$

An element $a \in R$ is selfadjoint (or Hermitian) if $a^* = a$. An element $a \in R$ is regular if there exists some inner inverse (or 1-inverse) $a^{-} \in R$ satisfying $aa^{-}a = a$. The set of all inner inverses (or 1-inverses) is denoted by $a{\{1\}}$. Hence, $a$ is regular if $a{\{1\}} \neq \emptyset$. A reflexive inverse $a^{\dagger}$ of $a$ is a 1-inverse of $a$ such that $a^{\dagger}a^{\dagger} = a^{\dagger}$.

**Definition 1.1.** Let $R$ be a ring with involution, and let $e, f$ be invertible Hermitian elements in $R$. The element $a \in R$ has the weighted Moore-Penrose inverse (weighted MP-inverse) with weights $e, f$ if there exists $b \in R$ such that

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$

The unique weighted MP-inverse with weights $e, f$, will be denoted by $a^\dagger_{e,f}$ if it exists [4]. The set of all weighted MP-invertible elements of $R$ with weights $e, f$, will be denoted by $R^\dagger_{e,f}$. If $e = f = 1$, then the weighted MP-inverse reduces to the ordinary MP-inverse of $a$, denoted by $a^\dagger$.

If $a \in R^\dagger_{e,f}$, then $aa^\dagger_{e,f}$ and $a^\dagger_{e,f}a$ are idempotents related to $a$ and $a^\dagger_{e,f}$.

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Notice that if \( R \) is a \( C^* \)-algebra, if \( e, f \) are selfadjoint, invertible and positive elements in a \( C^* \)-algebra \( R \), and if \( a \in R \) is regular, then the following formula holds:

\[
a_{e,f}^\dagger = f^{-1/2}(a e^{1/2} a f^{-1/2})^e 1/2. 
\]

Hence, the existence of an inner inverse of \( a \) implies the existence of the MP-inverse and the weighted MP-inverse of \( a \).

However, if \( R \) is a general ring with involution, then we do not have the existence of a square root of a positive element. Hence, in this case we always have to assume that the weighted MP-inverse of \( a \) exists.

Theorem 1.1. Let \( R \) be a ring with involution and let \( e, f \) be invertible Hermitian elements in \( R \). For any \( a \in R_{e,f} \), the following is satisfied:

(a) \((a_{e,f})^\dagger_{f,e} = a;\)
(b) \((a_{e,f}^* e)^\dagger = (a_{e,f}^\dagger)^* e f;\)
(c) \(a_{e,f}^* e = a_{e,f}^\dagger a a_{e,f}^* e = a_{e,f}^* e a a_{e,f}^\dagger;\)
(d) \(a_{e,f}^* e (a_{e,f}^\dagger)^* e f = a_{e,f}^\dagger e;\)
(e) \((a_{e,f}^\dagger)^* e f a_{e,f}^* e = aa_{e,f}^\dagger;\)
(f) \((a_{e,f}^* e a)_{f,f}^\dagger = a_{e,f}^\dagger (a_{e,f}^\dagger)^* e f;\)
(g) \((aa_{e,f}^* e)_{e,e}^\dagger = (a_{e,f}^\dagger)^* e f a_{e,f}^\dagger;\)
(h) \(a_{e,f}^\dagger = (a_{e,f}^* e a)_{f,f}^\dagger a_{e,f}^* e = a_{e,f}^* e (aa_{e,f}^* e)_{e,e}^\dagger;\)
(i) \((a_{e,f}^* e a)_{f,f}^\dagger = a(a_{e,f}^* e a)_{f,f}^\dagger = (aa_{e,f}^* e)_{e,e}^\dagger a.\)

For \( a \in R \) consider two annihilators

\( a^\circ = \{x \in R : ax = 0\}, \quad {^\circ}a = \{x \in R : xa = 0\} \).

Notice that,

\[(a^*)^\circ = a^\circ \Leftrightarrow {^\circ}(a^*) = {^\circ}a, \quad aR = a^* R \Rightarrow Ra = Ra^*.\]

Lemma 1.1. Let \( a \in A^* \), and let \( e, f \) be invertible positive elements in \( A \). Then

\[
a_{e,f}^\dagger = (a_{e,f}^* e a + 1 - a_{e,f}^\dagger a)^{-1} a_{e,f}^* e = a_{e,f}^* e (aa_{e,f}^* e + 1 - a_{e,f}^\dagger a)^{-1}, \tag{1}
\]

\[
a_{e,f}^* e A^{-1} = a_{e,f}^\dagger A^{-1} \text{ and } A^{-1} a_{e,f}^* e = A^{-1} a_{e,f}^\dagger, \tag{2}
\]

\[
(a_{e,f}^* e)^\circ = (a_{e,f}^\dagger)^\circ \text{ and } {^\circ}(a_{e,f}^* e) = {^\circ}(a_{e,f}^\dagger). \tag{3}
\]
Proof. By Theorem 1.1, we can verify
\[
(a^{*,f,e}a + 1 - a^\perp_{e,f}a)a^\perp_{e,f} = a^\perp_{e,f}(aa^{*,f,e} + 1 - aa^\perp_{e,f}),
\]
and
\[
(a^{*,f,e}a + 1 - a^\perp_{e,f}a)^{-1} = a^\perp_{e,f}(a^\perp_{e,f})^{*,f,e} + 1 - a^\perp_{e,f}a
\]
Thus, the part (1) holds and it implies the equalities (2) and (3).

Now, we state an useful result from [7].

Lemma 1.2. [7, Lemma 2.1] Let \( a, b \in \mathcal{R} \) be regular elements.

1. There exist \( a^- \in a[1], b^- \in b[1] \) for which \( (1 - bb^-)aa^- = 0 \) if and only if \( (1 - bb^-)aa^- = 0 \) for all \( a^- \in a[1], b^- \in b[1] \).

2. There exist \( a^- \in a[1], b^- \in b[1] \) for which \( (1 - bb^-)(1 - a^-a) = 0 \) if and only if \( (1 - bb^-)(1 - a^-a) = 0 \) for all \( a^- \in a[1], b^- \in b[1] \).

In [7], necessary and sufficient conditions for \( aa^\perp = bb^\perp \) in ring with involution are investigated. In this paper we generalized this results to the weighted Moore-Penrose in rings with involution.

2 Results

A semigroup is a regular, if every elements of that semigroup has an inner generalized inverse. The notion extends to rings also.

In a regular semigroup, the natural partial order is defined by ([2], [5], [6])

\[
a \leq_- b \text{ if } aa^- = ba^- \text{ and } a^-a = a^-b \text{ for some inner inverse } a^- \text{ of } a.
\]

See also [3] for intuitionistic fuzzy matrices. Notice that \( \leq_- \) is a partial order in regular rings.

A semigroup with involution \( x \mapsto x^* \) is proper, if the following implication holds:

\[
a^*a = a^*b = b^*a = b^*b \implies a = b.
\]

Notice that if the semigroup has the zero element 0, then a semigroup is a proper with respect to the involution \( x \mapsto x^* \), if and only if \( a^*a = 0 \implies a = 0 \). The last implication is called \( * \)-cancellability. For example, every element of a \( C^* \)-algebra is \( * \)-cancellable, so every \( C^* \)-algebra is proper (with respect to multiplication).

Drazin [1] presented a partial order on a proper \( * \)-semigroup in the following way

\[
a \leq_\ast b \text{ if } aa^\ast = ba^\ast \text{ and } a^\ast a = a^\ast b.
\]

If \( a \in \mathcal{R} \) is MP invertible, then \( \leq_\ast \) implies \( \leq_- \). Indeed, \( aa^\ast = ba^\ast \Rightarrow aa^\dagger = aa^\ast(a^\dagger)^*a^\dagger = ba^\ast(a^\dagger)^*a^\dagger = ba^\dagger \) and similarly \( a^\ast a = a^\ast b \Rightarrow a^\dagger a = a^\dagger b \).
In this paper we introduce the \( \preceq_{*,e,f} \) as follows:

\[
a \preceq_{*,e,f} b \text{ if } aa^{*,e,f} = ba^{*,e,f} \text{ and } a^{*,e,f} a = a^{*,e,f} b.
\]

Here \( e, f \) are Hermitian invertible elements in a ring \( R \) with involution \( x \mapsto x^* \). We like to see that \( \preceq_{*,e,f} \) is a partial ordering in \( R \).

If \( a \in R_{e,f}^+ \), then \( \preceq_{*,e,f} \) implies \( \preceq_{-} \). Indeed, from \( aa^{*,e,f} = ba^{*,e,f} \) we get

\[
aa^* = aa^{*,e,f}(a_e^{*})^{*,e,f}a_e^* = ba^{*,e,f}(a_e^{*})^{*,e,f}a_e^* = ba^{*,e,f}.
\]

Similarly, \( a^{*,e,f} a = a^{*,e,f} b \) gives \( a_{e,f}^* a = a_{e,f}^* b \).

In the rest of the paper we assume that \( e, f \in R \) are Hermitian end invertible.

The ring \( R \) is \( (*, e, f) \)-proper if the following implication holds:

\[
a^{*,e,f} a = a^{*,e,f} b = b^{*,e,f} a = b^{*,e,f} b \implies a = b.
\]

If \( R \) is a \( C^* \)-algebra and \( e, f \) are positive Hermitian elements in \( R \), then \( R \) is \( (*, e, f) \)-proper. Indeed, \( a^{*,e,f} a = a^{*,e,f} b = b^{*,e,f} a \) gives \( (a-b)^{*,e,f} (a-b) = 0 \) which gives that \( \langle f^1/2 (a-b) \rangle \parallel f^1/2 (a-b) = 0 \). Since every element in \( C^* \)-algebra is \( \cap \)-cancellable, then \( f^1/2 (a-b) = 0 \), that is \( a = b \).

**Theorem 2.1.** Let \( R \) be: \( (*, e, f) \)-proper, \( (*, e, f) \)-proper and \( (*, f, e) \)-proper. Then \( \preceq_{*,e,f} \) is a partial ordering in \( R \).

**Proof.** Since \( a \preceq_{*,e,f} a \), then \( \preceq_{*,e,f} \) is reflexive.

From \( a \preceq_{*,e,f} b \) and \( b \preceq_{*,e,f} a \) we get \( a^{*,e,f} a = a^{*,e,f} b \) and \( b^{*,e,f} a = b^{*,e,f} b \). Observe that

\[
a^{*,e,f} a = (a^{*,e,f} b)^{*,e,f} = (a^{*,e,f} b)^{*,e,f} = b^{*,e,f} a
\]

So, we deduce \( a^{*,e,f} a = a^{*,e,f} b = b^{*,e,f} a = b^{*,e,f} b \) which gives \( a = b \).

If \( a \preceq_{*,e,f} b \) and \( b \preceq_{*,e,f} c \), we obtain (4) and, applying (4) for \( b \) and \( c \) instead of \( a \) and \( b \), we have \( b^{*,e,f} b = c^{*,e,f} b \). Further,

\[
c^{*,e,f} (a^{*,e,f} c) = (c^{*,e,f} b) a^{*,e,f} c = b^{*,e,f} (b^{*,e,f} c) = (b^{*,e,f} a) a^{*,e,f} c = a^{*,e,f} a a^{*,e,f} c,
\]

\[
(a^{*,e,f} a) a^{*,e,f} a = b^{*,e,f} (a^{*,e,f} b) a^{*,e,f} a = c^{*,e,f} (b^{*,e,f} a) a^{*,e,f} a = a^{*,e,f} a c^{*,e,f} a a^{*,e,f} a
\]

and

\[
a^{*,e,f} a a^{*,e,f} a = (a^{*,e,f} a a^{*,e,f} a)^{*,e,f} = (a^{*,e,f} a a^{*,e,f} a)^{*,e,f} = a^{*,e,f} a a^{*,e,f} c.
\]

Since \( (a^{*,e,f} a)^{*,e,f} = a^{*,e,f} a \) and \( (a^{*,e,f} c)^{*,e,f} = c^{*,e,f} a \), by the previous tree equalities, we conclude

\[
(a^{*,e,f} a)^{*,e,f} = (a^{*,e,f} a)^{*,e,f} = (a^{*,e,f} a)^{*,e,f} = (a^{*,e,f} a)^{*,e,f} = a^{*,e,f} a a^{*,e,f} a\]

which implies \( a^{*,e,f} a = a^{*,e,f} c \), because ring \( R \) is \( *, e, f \)-proper. Similarly, by \( *, f, e \)-proper of \( R \), we can verify that \( a a^{*,e,f} = (a a^{*,e,f} f)^{*,f} \) which yields \( a a^{*,e,f} = (a a^{*,e,f} f)^{*,f} = ((a a^{*,e,f})^{*,f})^{*,f} = a c^{*,e,f} \). Thus, \( a^{*,e,f} a = a^{*,e,f} c \) and \( a a^{*,e,f} = c a^{*,e,f} a \) give that \( a \preceq_{*,e,f} c \).

\( \square \)
In the following theorem, we present some equivalent conditions for \( a a_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger \) to hold.

**Theorem 2.2.** Let \( \mathcal{R} \) be a ring with involution, and let \( e, f \) be invertible Hermitian elements in \( \mathcal{R} \). If \( a, b \in \mathcal{R}_{e,f}^\dagger \), then the following conditions are equivalent:

1. \( a a_{e,f}^\dagger = b b_{e,f}^\dagger a a_{e,f}^\dagger \);
2. \( a a_{e,f}^\dagger = a a_{e,f}^\dagger b b_{e,f}^\dagger \);
3. \( a = b b_{e,f}^\dagger a \);
4. \( a_{e,f}^\dagger = a_{e,f}^\dagger b b_{e,f}^\dagger \);
5. \( a^* a_{e,f} = b b_{e,f}^\dagger a^* a_{e,f} \);
6. \( a^* a_{e,f} = a^* a_{e,f} b b_{e,f}^\dagger \);
7. \( a^* a_{e,f} = a^* a_{e,f} b b_{e,f}^\dagger \);
8. \( a a^\dagger = b b^\dagger a a^\dagger \) for all choices \( a^\dagger \in a\{1\}, b^\dagger \in b\{1\} \);
9. \( a a^\dagger = b b^\dagger a a^\dagger \) for some \( a^\dagger \in a\{1\}, b^\dagger \in b\{1\} \);
10. \( a = b b^\dagger a \) for all \( b^\dagger \in b\{1\} \);
11. \( a = b b^\dagger a \) for some \( b^\dagger \in b\{1\} \);
12. \( a a^* a_{e,f} = b b^\dagger a a^* a_{e,f} \) for all \( b^\dagger \in b\{1\} \);
13. \( a a^* a_{e,f} = b b^\dagger a a^* a_{e,f} \) for some \( b^\dagger \in b\{1\} \);
14. \( a a_{e,f}^\dagger \leq b b_{e,f}^\dagger \);
15. \( a a_{e,f}^\dagger \leq_{*,e,e} b b_{e,f}^\dagger \);
16. \( a \leq b b^\dagger a \) for all \( b^\dagger \in b\{1\} \);
17. \( a \leq b b^\dagger a \) for some \( b^\dagger \in b\{1\} \);
18. \( a R \subseteq b b_{e,f}^\dagger a R \);
19. \( a R \subseteq b R \);
20. \( R a_{e,f}^\dagger \subseteq R a_{e,f}^\dagger b b_{e,f}^\dagger \);
21. \( R a_{e,f}^\dagger \subseteq R b_{e,f}^\dagger \).
Proof. (1) ⇔ (2): Applying the involution, the equality $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ is equivalent to $(e^{-1}eaa^\dagger_{e,f})^* = (e^{-1}e bb^\dagger_{e,f}e^{-1}eaa^\dagger_{e,f})^*$ which is $eaa^\dagger_{e,f}e^{-1} = eaa^\dagger_{e,f}e^{-1} e bb^\dagger_{e,f}e^{-1}$, i.e. $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$.

(1) ⇔ (3): Multiplying (1) by $a$ from the right side we get (3), and multiplying (3) by $a^\dagger_{e,f}$ from the right side we obtain (1).

(2) ⇔ (4): This part can be verified in the same way as (1) ⇔ (3).

(3) ⇔ (5): If we multiply (3) by $a^{\ast,f,e}$ from the right side we obtain (5), and if we multiply (5) by $(a^\dagger_{e,f})^{\ast,e,f}$ from the right side, by Theorem 1.1(d), we have (3).

(2) ⇔ (6): By Theorem 1.1, multiplying (2) by $a^{\ast,f,e}$ from the left side, we obtain (6). Conversely, multiplying (6) by $(a^\dagger_{e,f})^{\ast,e,f}a^\dagger_{e,f}$ from the left side, we get (2).

(6) ⇔ (7): Multiplying (6) by $a^\dagger_{e,f}$ from the left side, we obtain (7) and multiplying (7) by $a$ from the left side, we get (6).

(1) ⇔ (8): The assumption $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ is equivalent to $1 = bb^\dagger_{e,f}aa^\dagger_{e,f} = 0$. Applying Lemma 1.2, we obtain this equivalence.

(8) ⇔ (9): By Lemma 1.2.

(8) ⇔ (10), (9) ⇔ (11): Obviously.

(10) ⇔ (12): Multiplying (10) by $a^{\ast,f,e}$ from the right side, we obtain (12). On the other hand, multiplying (12) from the right side by $(a^\dagger_{e,f})^{\ast,e,f}$, we get (10).

(11) ⇔ (13): See the previous part.

(1) ⇔ (14): We can easy verify that $(aa^\dagger_{e,f})^\dagger_{e,f} = aa^\dagger_{e,f}$. Now, for $(aa^\dagger_{e,f})^\dagger_{e,f} = (aa^\dagger_{e,f})^\dagger_{e,f}$, we have $aa^\dagger_{e,f} \leq bb^\dagger_{e,f}$ if and only if $aa^\dagger_{e,f}(aa^\dagger_{e,f})^\dagger_{e,f} = bb^\dagger_{e,f}(aa^\dagger_{e,f})^\dagger_{e,f}$ and $(aa^\dagger_{e,f})^\dagger_{e,f}aa^\dagger_{e,f} = (aa^\dagger_{e,f})^\dagger_{e,f}bb^\dagger_{e,f}$, which is equivalent to $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ and $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$.

(1) ⇔ (15): Since $(aa^\dagger_{e,f})^{\ast,e,f} = e^{-1}(e^{-1}eaa^\dagger_{e,f})^*, e = aa^\dagger_{e,f}$, we show this equivalence in the same way as (1) ⇔ (14).

(10) ⇒ (16): For $a^\dagger = a^\dagger_{e,f}$, we already proved this part.

(16) ⇒ (17): Obviously.

(17) ⇒ (11): Suppose that $a \leq b^\dagger$ for some $b^\dagger \in b\{1\}$. Then, for some $a^\dagger$, we have $aa^\dagger = bb^\dagger a$, so $a = bb^\dagger a$.

(3) ⇒ (18) ⇒ (19): Obviously.

(19) ⇒ (3): The hypothesis $a\mathcal{R} \subseteq b\mathcal{R}$ gives $a = bx$, for some $x \in \mathcal{R}$. Therefore, $a = bb^\dagger_{e,f}(bx) = bb^\dagger_{e,f}a$.

(4) ⇒ (20) ⇒ (21) ⇒ (4): Similarly as (3) ⇒ (18) ⇒ (19) ⇒ (3).
(4) $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$ and $\forall b^- \in b\{1\} \ w = aa^*,f,e + 1 - bb^- \in R^{-1}$;

(5) $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$ and $\exists b^- \in b\{1\} \ w = aa^*,f,e + 1 - bb^- \in R^{-1}$;

(6) $aa^\dagger_{e,f}bb^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$, $u = aa^\dagger_{e,f} + 1 - bb^\dagger_{e,f} \in R^{-1}$ and $l = bb^\dagger_{e,f} + 1 - aa^\dagger_{e,f} \in R^{-1}$;

(7) $aa^\dagger_{e,f}bb^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$, $v = aa^*,f,e + 1 - bb^\dagger_{e,f} \in R^{-1}$ and $k = bb^*,f,e + 1 - aa^\dagger_{e,f} \in R^{-1}$.

Proof. (1) $\Rightarrow$ (2): It is easy to check.

(2) $\Leftrightarrow$ (3): Using Theorem 2.2, $(aa^\dagger_{e,f} + 1 - bb^\dagger_{e,f})(aa^*,f,e + 1 - aa^\dagger_{e,f}) = aa^*,f,e + 1 - bb^\dagger_{e,f}$. By Lemma 1.1, $aa^*,f,e + 1 - aa^\dagger_{e,f} \in R^{-1}$ and then $u \in R^{-1} \Leftrightarrow v \in R^{-1}$.

(3) $\Rightarrow$ (1): Observe that, by Theorem 2.2, $vaa^\dagger_{e,f} = aa^*,f,e = vbb^\dagger_{e,f}$. Since $v \in R^{-1}$, we have $aa^\dagger_{e,f} = bb^\dagger_{e,f}$.

(3) $\Rightarrow$ (4): By Theorem 2.2, we have $aa^*,f,e = bb^\dagger_{e,f}aa^*,f,e = bb^\dagger_{e,f}aa^*,f,e bb^\dagger_{e,f}$. Now, by [8, Proposition 3], $v = aa^*,f,e + 1 - bb^\dagger_{e,f} = bb^\dagger_{e,f}aa^*,f,e bb^\dagger_{e,f} + 1 - bb^\dagger_{e,f} \in R^{-1}$, if and only if $bb^\dagger_{e,f}aa^*,f,e bb^- + 1 - bb^- \in R^{-1}$, $\forall b^- \in b\{1\}$, i.e. $1 - (bb^-)(aa^*,f,e + 1) \in R^{-1}$ for all $b^- \in b\{1\}$, which is equivalent to $1 - bb^- (bb^-aa^*,f,e + 1) = w \in R^{-1}$, $\forall b^- \in b\{1\}$.

(4) $\Rightarrow$ (3) and (5): Obvioulsy.

(5) $\Rightarrow$ (4): From $w = aa^*,f,e + 1 - bb^- = 1 - bb^- (-aa^*,f,e + 1) \in R^{-1}$, we deduce that $1 - (-aa^*,f,e + 1)bb^- = bb^- aa^*,f,e bb^- + 1 - bb^- \in R^{-1}$. Then, by [8, Proposition 3], $bb^- aa^*,f,e bb^- + 1 - bb^- = 1 - (-aa^*,f,e + 1)bb^- \in R^{-1}$, for all $b^- \in \{1\}$, which gives $1 - bb^- (-aa^*,f,e + 1) = bb^- aa^*,f,e + 1 - bb^- = aa^*,f,e + 1 - bb^- \in R^{-1}$.

(1) $\Rightarrow$ (6): Obvioulsy.

(6) $\Rightarrow$ (1): Since, by $aa^\dagger_{e,f}bb^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$, $bb^\dagger_{e,f}u = bb^\dagger_{e,f}aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}u$ and $u \in R^{-1}$, then $bb^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ and $l \in R^{-1}$ give $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$. Thus, $aa^\dagger_{e,f} = bb^\dagger_{e,f}$.

(7) $\Rightarrow$ (3): The equality $aa^\dagger_{e,f}bb^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ implies $aa^\dagger_{e,f}k = aa^\dagger_{e,f}bb^*,f,e = bb^\dagger_{e,f}aa^\dagger_{e,f}k$. Because $k \in R^{-1}$, then $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ and the condition (3) holds.

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