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Note on stable perturbation of bounded linear operators on Hilbert spaces

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Abstract

In this paper, we investigate the equivalence conditions of stable perturbation and characterize $(I + T^+\delta T)^{-1}$ by the range and null spaces of T and $\bar{T}$. As applications, we give the representations of the Moore–Penrose inverse of the perturbed operator under stable perturbation on Hilbert spaces and certain 2×2 operator matrices.

1 Introduction

Let H, K be Hilbert spaces and let $B(H,K)$ denote the set of all bounded linear operators from H to K. Put $B(H) = B(H,H)$. For an operator $T \in B(H,K)$, let $R(T)$ and $N(T)$ denote the range and the null space of T, respectively. Denote the adjoint operator of T by T^* (in $B(K,H)$). Let $T \in B(H,K)$. Consider an operator $X \in B(K,H)$ which satisfies following equations:

$$\begin{array}{llll} (1) & TXT = T & (2) & XTX = X \\ (3) & (TX)^* = TX & (4) & (XT)^* = XT. \end{array}$$

Then we call X is the Moore–Penrose inverse of T, denoted by T^+. If X only satisfies (1) and (2), we call X is the generalized inverse of T, denoted by T^+_{GI}. we note that T^+ is unique and T^+_{GI} is not unique. It is well–known that T has a generalized inverse or the Moore–Penrose inverse iff $R(T)$ is closed. When $R(T)$ is closed, we have $(T^+)^* = (T^*)^+$ and

$$R(T^+) = R(T^*),\ N(T^+) = N(T^*),\ R((T^+)^*) = R(T),\ N((T^+)^*) = N(T)$$

and $TT^+ = P_{R(T)}$, $T^+T = I - P_{N(T)}$, where $P_{R(T)}$ (resp. $P_{N(T)}$) is an orthogonal projection from K (resp. H) onto $R(T)$ (resp. $N(T)$) (cf. [1], [5]).

The rank–preserving perturbation of the matrix plays an important role in the perturbation analysis of least square solution, Moore–Penrose inverses and Drazin

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inverse. There are a lot of results concerning the rank–preserving perturbation of the matrix. Many of them can be found in [10, 11, 12].

The notation so–called the stable perturbation of an operator on Hilbert spaces and Banach spaces is introduced by G. Chen and the second author in [2, 3]. When Hilbert spaces or Banach spaces are of finite dimensional, the stable perturbation and the rank–preserving perturbation are the same. Later this notation is generalized to the set of Banach algebras by second author in [16] and to the set of Hilbert C^*–module by Xu, Wei and Gu in [13]. Using this notation, we give the estimation of upper bounds about the perturbation of Moore–Penrose inverses and Drazin inverses in second author' series of work [2, 3, 4, 14, 15, 16].

In this paper, we continue to study stable perturbation of an operator on Hilbert spaces. Our aim is to give the representation of the Moore–Penrose inverse of the perturbed operator under stable perturbation, that is, for δT, $T \in B(H,K)$ with $R(T)$ closed and $I + T^+\delta T$ invertible in $B(H)$, we will give the expression of $(T+\delta T)^+$ when $R(T+\delta T)\cap R(T)^\perp = \{0\}$ and explicit representations of Moore–Penrose inverses of certain operator matrices on Hilbert spaces.

2 Stable perturbation

Let $\bar{T} = T+\delta T, T \in B(H,K)$. Recall that $\bar{T}$ is called to be the stable perturbation of T if $R(\bar{T})\cap R(T)^\perp = \{0\}$. We have known that if $R(T)$ is closed, $R(\bar{T})\cap R(T)^\perp = \{0\}$ and $\|T^+\|\|\delta T\| < 1$, then

$$\|\bar{T}^+\| \le \frac{\|T^+\|}{I - \|T^+\|\|\delta T\|}, \quad \frac{\|\bar{T}^+ - T^+\|}{\|T^+\|} \le \frac{1+\sqrt{5}}{2}\frac{\|T^+\|}{I - \|T^+\|\|\delta T\|}$$

(cf. [4, 15]). The above results generalize Stewart's corresponding work on matrices. But if we need not require the condition $\|T^+\|\|\delta T\| < 1$. What is the result under the hypothesis $I + T^+\delta T$ is invertible in $B(H)$? we have the following

Proposition 2.1. *Let $\bar{T} = T+\delta T, T \in B(H,K)$ with $R(T)$ closed and $I+T^+\delta T$ invertible. Then following conditions are equivalent.*

(1) $R(\bar{T})$ *is closed and* $\bar{T}^+_{GI} = T^+(I+\delta T T^+)^{-1} = (I+T^+\delta T)^{-1}T^+$

(2) $R(\bar{T}) \cap R(T)^\perp = \{0\}$

(3) $N(\bar{T})^\perp \cap N(T) = \{0\}$

(4) $\bar{T}(I+T^+\delta T)^{-1}(I-T^+T) = 0$

(5) $(I-TT^+)(I+\delta T T^+)^{-1}\bar{T} = 0$

(6) $(I+\delta T T^+)^{-1}\bar{T}$ *maps* $N(T)$ *into* $R(T)$

(7) $(I-TT^+)\delta T(I-T^+T) = (I-TT^+)\delta T(I+T^+\delta T)^{-1}T^+\delta T(I-T^+T)$

(8) $(I+\delta T T^+)^{-1}R(\bar{T}) = R(T)$

(9) $(I+T^+\delta T)^{-1}N(T) = N(\bar{T})$

(10) $(I-T^+T)N(\bar{T}) = N(T)$.

Proof. Using similar methods appeared in [4, 14, 15, 16], we can obtain the equivalence from (1) to (7).

(9)⇒(4) is clear for $R(I-T^+T) = N(T)$.

(4)⇒(9) $\bar{T}(I+T^+\delta T)^{-1}(I-T^+T) = 0$ implies that $(I+T^+\delta T)^{-1}N(T) \subset N(\bar{T})$. Now let $\xi \in N(\bar{T})$. Then

$$(-I+T^+T)\xi + (I+T^+\delta T)\xi = 0$$

and hence $\xi = (I+T^+\delta T)^{-1}(I-T^+T)\xi$, that is, $N(\bar{T}) \subset (I+T^+\delta T)^{-1}N(T)$.

(5)⇒(8) Condition (5) indicates that $(I+\delta TT^+)^{-1}R(\bar{T}) \subset R(T)$. Since

$$(I+\delta TT^+)^{-1}\bar{T}T^+T = (I+\delta TT^+)^{-1}(I+\delta TT^+)T = T,$$

we have $(I+\delta TT^+)^{-1}R(\bar{T}) \supset R(T)$. The assertion follows.

(9)⇒(10) Note that $(I-T^+T)(I+T^+\delta T)^{-1} = I-T^+T$. Thus,

$$N(T) = (I-T^+T)N(T) = (I-T^+T)(I+T^+\delta T)^{-1}N(T) = (I-T^+T)N(\bar{T}).$$

(10)⇒(4) $(I-T^+T)N(\bar{T}) = N(T)$ implies that for any $\xi \in N(T)$, there is $\eta \in N(\bar{T})$ such that $\xi = (I-T^+T)\eta$. From $\eta \in N(\bar{T})$, we have $T\eta = -\delta T\eta$ and consequently, $\xi = (I+T^+\delta T)\eta$. Therefore, $\bar{T}(I+T^+\delta T)^{-1}\xi = 0$. □

Corollary 2.2. *Let $\bar{T} = T+\delta T$, $T \in B(H,K)$ with $R(T)$ closed and $I+T^+\delta T$ invertible. If $\dim N(\bar{T}) = \dim N(T) < +\infty$ or $\dim R(\bar{T}) = \dim R(T) < +\infty$ or $\dim N(\bar{T}^*) = \dim N(T^*) < +\infty$, then $\bar{T}$ is the stable perturbation of T.*

Proof. Let $\xi \in N(\bar{T})$. Then $\delta T\xi = -T\xi$. Thus,

$$(I+T^+\delta T)\xi = (I-T^+T)\xi \in N(T),$$

i.e., $N(\bar{T}) \subset (I+T^+\delta T)^{-1}N(T)$. We conclude that $N(\bar{T}) = (I+T^+\delta T)^{-1}N(T)$ when $\dim N(\bar{T}) = \dim N(T) < +\infty$. So, $R(\bar{T}) \cap R(T)^\perp = \{0\}$ by Proposition 2.1.

We have known that $(I+\delta TT^+)^{-1}R(\bar{T}) \supset R(T)$ in the proof of (5)⇒(8) of Proposition 2.1. So $\dim R(\bar{T}) = \dim R(T) < +\infty$ implies that $(I+\delta TT^+)^{-1}R(\bar{T}) = R(T)$ and hence $R(\bar{T}) \cap R(T)^\perp = \{0\}$ by Proposition 2.1.

We have $R(\bar{T}^*) \cap R(T^*)^\perp = \{0\}$ when $\dim N(\bar{T}^*) = \dim N(T^*) < +\infty$ by above argument. Thus, $R(\bar{T}^*)$ is closed by Proposition 2.1. Since $R(\bar{T}^*) = N(\bar{T})^\perp$ and $R(T^*) = N(T)^\perp$, it follows from Proposition 2.1 that $R(\bar{T}) \cap R(T)^\perp = \{0\}$. □

Proposition 2.3. *Let $\bar{T} = T+\delta T, T \in B(H,K)$ with $R(T)$ closed and $I+T^+\delta T$ invertible. Suppose that $R(\bar{T}) \cap R(T)^\perp = \{0\}$, then $\bar{T}^+$ exist and*

$$\|\bar{T}^+\| \le \|(I+T^+\delta T)^{-1}\|\|T^+\|, \quad \|\bar{T}^+ - T^+\| \le \frac{1+\sqrt{5}}{2}\|(I+T^+\delta T)^{-1}\|\|T^+\|^2\|\delta T\|$$

Proof. Since $I + T^+\delta T$ is invertible in $B(H)$ and $R(\bar{T}) \cap R(T)^\perp = \{0\}$, it follows from Proposition 2.1 that $R(\bar{T})$ is closed and $(I - T^+T)N(\bar{T}) = N(T)$. Note that $I + T^+\delta T$ is invertible in $B(H)$ implies that $N(T)^\perp \cap N(\bar{T}) = \{0\}$ by Proposition 2.4. So $(I - T^+T)|_{N(\bar{T})} : N(\bar{T}) \to N(T)$ is a bijective bounded linear operator. By the proof of [7] (I-theorem 6.34)

$$\begin{aligned}\|(I - T^+T) - (I - \bar{T}^+\bar{T})\| &= \|(I - (I - T^+T))(I - \bar{T}^+\bar{T})\|\\ &= \|(I - (I - \bar{T}^+\bar{T}))(I - T^+T)\|\\ &\le \delta(N(\bar{T}), N(T)) \le \|T^+\|\|\delta T\|\end{aligned}$$

or

$$\|T^+T - \bar{T}^+\bar{T}\| \le \delta(N(T), N(\bar{T}) \le \|\bar{T}^+\|\|\delta T\|$$

Thus

$$\|\bar{T}^+\bar{T} - T^+T\| \le \min\{\|T^+\|\|\delta T\|, \|\bar{T}^+\|\|\delta T\|\}.$$

Similarly, we also have

$$\|\bar{T}\bar{T}^+ - TT^+\| \le \min\{\|T^+\|\|\delta T\|, \|\bar{T}^+\|\|\delta T\|\}$$

Then using the proof of Proposition 7 in [15],we have

$$\begin{aligned}\|\bar{T}^+ - T^+\| &\le \frac{1+\sqrt{5}}{2}\|\bar{T}^+\|\|T^+\|\|\delta T\|\\ &\le \frac{1+\sqrt{5}}{2}\|(I + T^+\delta T)^{-1}\|\|T^+\|^2\|\delta T\|\end{aligned}$$

□

From Proposition 2.1 and 2.3 we see that the invertibility of $I + T^+\delta T$ is very important. How to characterize it? We have the following propositions:

Proposition 2.4. *Let $\bar{T} = T + \delta T, T \in B(H, K)$ with $R(T)$ closed.*

(1) *If $I + T^+\delta T$ is invertible in $B(H)$. Then $N(T)^\perp \cap N(\bar{T}) = \{0\}$ and $R(\bar{T})^\perp \cap R(T) = \{0\}$*

(2) *If $R(\bar{T}) \cap R(T)^\perp = \{0\}$ and $N(T)^\perp \cap N(\bar{T}) = \{0\}$, then $N(I + T^+\delta T) = \{0\}$;*

(3) *If $R(\bar{T})^\perp \cap R(T) = \{0\}$ and $N(T) \cap N(\bar{T})^\perp = \{0\}$, then $\overline{R(I + T^+\delta T)} = H$.*

Proof. (1) Let $x \in N(T)^\perp \cap N(\bar{T})$, then $\bar{T}x = 0$, and $T^+Tx = x$. Since

$$0 = T^+\bar{T}x = (T^+T + T^+\delta T)x = (T^+T - I + I + T^+\delta T)x = (I + T^+\delta T)x$$

we have $x = 0$.

$I + T^+\delta T$ is invertible implies that $I + (\delta T)^*(T^*)^+$ is invertible. So by above argument, we get that $N(T^*)^\perp \cap N(\bar{T}^*) = \{0\}$. Consequently, $R(\bar{T})^\perp \cap R(T) = \{0\}$ for $R(T) = N(T^*)^\perp, R(\bar{T})^\perp = N(\bar{T}^*)$.

(2) Let $x \in N(I + T^+\delta T)$, i.e., $x + T^+\delta Tx = 0$. Thus

$$0 = T^+Tx + T^+\delta Tx + (I - T^+T)x = T^+\bar{T}x + (I - T^+T)x.$$

This indicates that $T^+\bar{T}x = 0$ and $(I - T^+T)x = 0$ since $R(\bar{T}) \cap R(T)^\perp = \{0\}$ and $\bar{T}x \in N(T^+) = R(T)^\perp$. We have $\bar{T}x = 0$, i.e. $x \in N(\bar{T})$. It follows from $x = T^+Tx$ and $N(T)^\perp \cap N(\bar{T}) = \{0\}$ that $x = 0$.

Let $x \in N(I + \delta TT^+)$ and put $x_1 = TT^+x, x_2 = (I - TT^+)x \in R(T)^\perp$, then

$$(T + \delta T)T^+x_1 + x_2 = x_1 + x_2 + \delta TT^+x_1 = 0$$

and hence $\bar{T}T^+x_1 = x_2 = 0$ for $R(\bar{T})^\perp \cap R(T) = \{0\}$. Then from $T^+x_1 \in N(\bar{T}) \cap N(T)^\perp = \{0\}$ and $x_1 = TT^+x_1$, we have $x = 0$.

(3) To prove $\overline{R(I + T^+\delta T)} = H$, it need only to show that $N(I+(\delta T)^*(T^*)^+) = \{0\}$. By the above argument, we need check that

$$R(\bar{T}^*) \cap R(T^*)^\perp = \{0\},\ N(T^*)^\perp \cap N(\bar{T}^*) = \{0\}.$$

But this can be deduced from $R(\bar{T}^*) \subset N(\bar{T})^\perp$, $R(T^*)^\perp = N(T)$, $N(T^*)^\perp = R(T)$ and $N(\bar{T}^*) = R(\bar{T})^\perp$. □

Let V be a subspace in H and P_V be the orthogonal projection from H onto $\overline{V}$.

Proposition 2.5. *Let $\bar{T}, T \in B(H, K)$. Set $n(T, \bar{T}) = I - P_{N(T)} - P_{N(\bar{T})}$, $r(T, \bar{T}) = I - P_{R(T)} - P_{R(\bar{T})}$. Then*

(1) $N(n(T, \bar{T})) = \{0\}$ *iff* $N(\bar{T})^\perp \cap N(T) = \{0\}$ *and* $N(T)^\perp \cap N(\bar{T}) = \{0\}$;

(2) $N(r(T, \bar{T})) = \{0\}$ *iff* $R(\bar{T})^\perp \cap R(T) = \{0\}$ *and* $R(T)^\perp \cap R(\bar{T}) = \{0\}$.

Proof. We only prove (1) since the proof of (2) is the same as of (1).

If $N(n(T, \bar{T})) = \{0\}$, then $\forall x \in N(\bar{T})^\perp \cap N(T)$,

$$(I - P_{N(T)} - P_{N(\bar{T})})x = P_{N(T)}x - P_{N(\bar{T})^\perp}x = 0 \Rightarrow x = 0$$

$\forall y \in N(T)^\perp \cap N(\bar{T})$,

$$(I - P_{N(T)} - P_{N(\bar{T})})y = P_{N(T)^\perp}y - P_{N(\bar{T})}y = 0 \Rightarrow y = 0$$

Thus we have $N(\bar{T})^\perp \cap N(T) = \{0\}, N(T)^\perp \cap N(\bar{T}) = \{0\}$.

Conversely, if $N(\bar{T})^\perp \cap N(T) = \{0\}, N(T)^\perp \cap N(\bar{T}) = \{0\}$, then

$$(I - P_{N(T)} - P_{N(\bar{T})})x = 0 \Rightarrow P_{N(\bar{T})^\perp}x = P_{N(T)}x \in N(\bar{T})^\perp \cap N(T) = \{0\}$$

hence

$$P_{N(\bar{T})^\perp}x = P_{N(T)}x = 0 \Rightarrow x \in N(T)^\perp \cap N(\bar{T}) = \{0\}.$$

So $N(n(T, \bar{T})) = \{0\}$. □

Remark 2.6. *If H, K are finite dimensional Hilbert spaces, then*

$$N(\bar{T})^{\perp} \cap N(T) = \{0\}, \;\; N(T)^{\perp} \cap N(\bar{T}) = \{0\}$$

if and only if $I - P_{N(T)} - P_{N(\bar{T})}$ is invertible.

Corollary 2.7. *Let $\bar{T} = T + \delta T, T \in B(H, K)$ with $R(T)$ closed. Assume that δT is a compact operator. If both $n(T, \bar{T})$ and $r(T, \bar{T})$ are injective, then $I + T^{+}\delta T$ is invertible in $B(H)$.*

Proof. By proposition 2.4 and proposition 2.5, $N(n(T, \bar{T})) = \{0\}$ and $N(r(T, \bar{T})) = \{0\}$ imply that $N(I + T^{+}\delta T) = \{0\}$ and $\overline{R(I + T^{+}\delta T)} = H$. Since δT is a compact operator, it follows from [8, Theorem 4.23] that $R(I + T^{+}\delta T)$ is closed. Thus, $I + T^{+}\delta T$ is invertible in $B(H)$. □

Corollary 2.8. *Let H, K be finite dimensional Hilbert spaces and let $\bar{T} = T + \delta T$, $T \in B(H, K)$. Then $I + T^{+}\delta T$ is invertible and* $\mathrm{rank}\,(\bar{T}) = \mathrm{rank}\,(T)$ *iff*

$$R(\bar{T}) \cap R(T)^{\perp} = \{0\}, \;\; N(T)^{\perp} \cap N(\bar{T}) = \{0\}.$$

Proof. If $I + T^{+}\delta T$ is invertible and $\mathrm{rank}\,(\bar{T}) = \mathrm{rank}\,(T)$, then $R(\bar{T}) \cap R(T)^{\perp} = \{0\}$ by Corollary 2.2. By Proposition 2.4 (1), that $I + T^{+}\delta T$ is invertible means that $N(T)^{\perp} \cap N(\bar{T}) = \{0\}$.

Conversely, assume that

$$R(\bar{T}) \cap R(T)^{\perp} = \{0\}, \;\; N(T)^{\perp} \cap N(\bar{T}) = \{0\}.$$

Then by Proposition 2.4 (2), $N(I + T^{+}\delta T) = \{0\}$ and so that $I + T^{+}\delta T$ is invertible since H is finite dimensional. Thus, $(I + T^{+}\delta T)^{-1} N(T) = N(\bar{T})$ by Proposition 2.1 and consequently, $\mathrm{rank}\,(\bar{T}) = \mathrm{rank}\,(T)$. □

3 Some representations of the Moore–Penrose inverses of operators

Lemma 3.1. *Let $A \in B(H, K)$. Suppose that there is $B \in B(K, H)$ such that $ABA = B$ and $BAB = B$. Then $A^{+} = -(I - P - P^{*})^{-1} B (I - Q - Q^{*})^{-1}$, where $P = I - BA$ and $Q = AB$.*

Proof. It is easy to check that P and Q are idempotent operators with $R(P) = N(A)$ and $R(Q) = R(A)$. By [4, Lemma 3], the orthogonal projections from H onto $N(A)$ and K onto $R(A)$ are

$$O(P) = -P(I - P - P^{*})^{-1}, \quad O(Q) = -Q(I - Q - Q^{*})^{-1}$$

respectively. Moreover, by [4, Lemma 4],

$$\begin{aligned} A^{+} = (I - O(P))BO(Q) &= -\,(I + P(I - P - P^{*})^{-1})BQ(I - Q - Q^{*})^{-1} \\ &= -\,(I - P^{*})(I - P - P^{*})^{-1} B (I - Q - Q^{*})^{-1} \\ &= (P + P^{*} - I)^{-1}(I - P) B (I - Q - Q^{*})^{-1} \\ &= (P + P^{*} - I)^{-1} B (I - Q - Q^{*})^{-1}. \end{aligned}$$

$\square$

Let $T \in B(H,K)$ with $R(T)$ closed. Put

$$\begin{aligned} K_1 &= (TT^+)K, & K_2 &= (I - TT^+)K,\\ H_1 &= (T^+T)H, & H_2 &= (I - T^+T)H,\\ \delta_1 &= TT^+\delta T\,T^+T, & \delta_2 &= TT^+\delta T(I - T^+T),\\ \delta_3 &= (I - TT^+)\delta T\,T^+T, & \delta_4 &= (I - TT^+)\delta T(I - T^+T). \end{aligned}$$

Then $\delta_1,\ \delta_2,\ \delta_3,\ \delta_4$ can be regarded as operators in $B(H_1,K_1)$, $B(H_2,K_1)$, $B(H_1,K_2)$ and $B(H_2,K_2)$ respectively, and δT can be expressed as $\delta T = \begin{pmatrix}\delta_1 & \delta_2\\ \delta_3 & \delta_4\end{pmatrix}$. Put $T_1 = T|_{H_1}$. Then $T_1 \in B(H_1,K_1)$ with $T_1^{-1} \in B(K_1,H_1)$. Moreover,

$$T = \begin{pmatrix}T_1 & 0\\ 0 & 0\end{pmatrix}, \quad T^+ = \begin{pmatrix}T_1^{-1} & 0\\ 0 & 0\end{pmatrix}.$$

For convenience, let I_1 (resp. I_2) denote the identity operator on H_1 and K_1 (resp. H_2 and K_2). Thus, from

$$I + T^+\delta T = \begin{pmatrix}I_1 + T_1^{-1}\delta_1 & 0\\ 0 & I_2\end{pmatrix},$$

we get that $I + T^+\delta T$ is invertible in $B(H)$ iff $I_1 + T_1^{-1}\delta_1$ is invertible in $B(H_1)$.

Theorem 3.2. *Let $\bar{T} = T + \delta T$, $T \in B(H,K)$ with $R(T)$ closed and $I + T^+\delta T$ invertible in $B(H)$. Suppose that $R(\bar{T}) \cap R(T)^\perp = \{0\}$. Then $R(\bar{T})$ is closed and $\bar{T}^+$ has the form $\bar{T}^+ = \begin{pmatrix}\bar{T}_1 & \bar{T}_2\\ \bar{T}_3 & \bar{T}_4\end{pmatrix}$, where $\Delta_1 = \delta_3(I_1 + T_1^{-1}\delta_1)^{-1}T_1^{-1}$, $\Delta_2 = (I_1 + T_1^{-1}\delta_1)^{-1}T_1^{-1}\delta_2$ and*

$$\begin{aligned} \bar{T}_1 &=(I_1 + \Delta_2\Delta_2^*)^{-1}(I_1 + T_1^{-1}\delta_1)^{-1}T_1^{-1}(I_1 + \Delta_1^*\Delta_1)^{-1}\\ \bar{T}_2 &=(I_1 + \Delta_2\Delta_2^*)^{-1}(I_1 + T_1^{-1}\delta_1)^{-1}T_1^{-1}\Delta_1^*(I_2 + \Delta_1\Delta_1^*)^{-1}\\ \bar{T}_3 &=(I_2 + \Delta_2^*\Delta_2)^{-1}\Delta_2^*(I_1 + T_1^{-1}\delta_1)^{-1}T_1^{-1}(I_1 + \Delta_1^*\Delta_1)^{-1}\\ \bar{T}_4 &=(I_2 + \Delta_2^*\Delta_2)^{-1}\Delta_2^*(I_1 + T_1^{-1}\delta_1)^{-1}T_1^{-1}\Delta_1^*(I_2 + \Delta_1\Delta_1^*)^{-1}. \end{aligned}$$

Proof. By Proposition 2.1, $R(\bar{T}) \cap R(T)^\perp = \{0\}$ implies that $R(\bar{T})$ is closed and

$$\bar{T}^+_{GI} = (I + T^+\delta T)^{-1}T^+ = \begin{pmatrix}(I_1 + T_1^{-1}\delta_1)^{-1}T_1^{-1} & 0\\ 0 & 0\end{pmatrix}.$$

Set $\bar{Q} = \bar{T}\bar{T}^+_{GI} = \begin{pmatrix}I_1 & 0\\ \Delta_1 & 0\end{pmatrix}$, $\bar{P} = I - \bar{T}^+_{GI}\bar{T} = \begin{pmatrix}0 & -\Delta_2\\ 0 & 0\end{pmatrix}$. Then

$$(I - \bar{Q} - \bar{Q}^*)^{-1} = -\begin{pmatrix}I_1 & \Delta_1^*\\ \Delta_2 & -I_2\end{pmatrix}^{-1} = -\begin{pmatrix}(I_1 + \Delta_1^*\Delta_1)^{-1} & \Delta_1^*(I_2 + \Delta_1\Delta_1^*)^{-1}\\ (I_2 + \Delta_1\Delta_1^*)^{-1}\Delta_1 & -(I_2 + \Delta_1\Delta_1^*)^{-1}\end{pmatrix},$$

$$(\bar{P} + \bar{P}^* - I)^{-1} = -\begin{pmatrix}I_1 & \Delta_2\\ \Delta_2^* & -I_2\end{pmatrix}^{-1} = -\begin{pmatrix}(I_1 + \Delta_2\Delta_2^*)^{-1} & \Delta_2(I_2 + \Delta_2^*\Delta_2)^{-1}\\ (I_2 + \Delta_2^*\Delta_2)^{-1}\Delta_2^* & -(I_2 + \Delta_2^*\Delta_2)^{-1}\end{pmatrix}.$$

Thus, Using the expression $\bar{T}^+ = (\bar{P}+\bar{P}^*-I)^{-1}\bar{T}^+_{GI}(I-\bar{Q}-\bar{Q}^*)^{-1}$ given in Lemma 3.1, we can get the result. $\square$

Theorem 3.3. *Let* $\bar{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be operator matrix from* $H = H_1 \oplus H_2$ *to* $K = K_1 \oplus K_2$ *with* $R(A)$ *and* $R(D)$ *closed, where* $A \in B(H_1, K_1)$, $B \in B(H_2, K_1)$, $C \in B(H_1, K_2)$, $D \in B(H_2, K_2)$. *If* $I - D^+CA^+B$ *is invertible and* $(I - AA^+)B = 0$, $(I - DD^+)C = 0$, *then* $\bar{T}^+ = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, *here*

$$\begin{aligned}
\Delta &= (I - D^+CA^+B)^{-1} \\
\Theta &= (P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1} \\
P_{11} &= I - (I + A^+B\Delta D^+C)(I - A^+A) - (I - A^+A)(I + A^+B\Delta D^+C)^* \\
P_{12} &= A^+B\Delta(I - D^+D) + (I - A^+A)(\Delta D^+C)^* \\
P_{21} &= \Delta D^+C(I - A^+A) + (I - D^+D)(A^+B\Delta)^* \\
P_{22} &= I - \Delta(I - D^+D) - (I - D^+D)\Delta^* \\
T_{11} &= \Theta(A^+ + A^+B\Delta D^+CA^+) + \Theta P_{12}P_{22}^{-1}\Delta D^+CA^+ \\
T_{12} &= -\Theta A^+B\Delta D^+ - \Theta P_{12}P_{22}^{-1}\Delta D^+ \\
T_{21} &= -P_{22}^{-1}P_{21}\Theta(A^+ + A^+B\Delta D^+CA^+) - (P_{22}^{-1} + P_{22}^{-1}P_{21}\Theta P_{12}P_{22}^{-1})\Delta D^+CA^+ \\
T_{22} &= P_{22}^{-1}P_{21}\Theta A^+B\Delta D^+ + (P_{22}^{-1} + P_{22}^{-1}P_{21}\Theta P_{12}P_{22}^{-1})\Delta D^+
\end{aligned}$$

Proof. Let

$$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \delta T = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

then $\bar{T} = T + \delta T$. Clearly,

$$T^+ = \begin{pmatrix} A^+ & 0 \\ 0 & D^+ \end{pmatrix}, \quad I + T^+\delta T = \begin{pmatrix} I & A^+B \\ D^+C & I \end{pmatrix}$$

Since

$$\begin{pmatrix} I & 0 \\ -D^+C & I \end{pmatrix}\begin{pmatrix} I & A^+B \\ D^+C & I \end{pmatrix} = \begin{pmatrix} I & A^+B \\ 0 & I - D^+CA^+B \end{pmatrix}$$

then $I + T^+\delta T$ is invertible when $I - D^+CA^+B$ is invertible. It is easy to check that $R(\bar{T}) \cap R(T)^\perp = R(\bar{T}) \cap N(T^+) = \{0\}$ when $(I - AA^+)B = 0$, $(I - DD^+)C = 0$.

So, by Proposition 2.1,

$$\bar{T}^+_{GI} = (I + T^+\delta T)^{-1}T^+ = \begin{pmatrix} A^+ + A^+B\Delta D^+CA^+ & -A^+B\Delta D^+ \\ -\Delta D^+CA^+ & \Delta D^+ \end{pmatrix},$$

where $\Delta = (I - D^+CA^+B)^{-1}$. Let $\bar{Q} = \bar{T}\bar{T}^+_{GI}$ and $\bar{P} = I - \bar{T}^+_{GI}\bar{T}$. Then

$$\begin{aligned}
(I - Q - Q^*)^{-1} &= \begin{pmatrix} I - 2AA^+ & 0 \\ 0 & I - 2DD^+ \end{pmatrix} \\
P &= \begin{pmatrix} (I + A^+B\Delta D^+C)(I - A^+A) & A^+B\Delta(D^+D - I) \\ \Delta D^+C(A^+A - I) & \Delta(I - D^+D) \end{pmatrix} \\
(I - P - P^*)^{-1} &= \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} I & 0 \\ -P_{22}^{-1}P_{21} & I \end{pmatrix} \begin{pmatrix} (P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1} & 0 \\ 0 & P_{22}^{-1} \end{pmatrix} \begin{pmatrix} I & -P_{12}P_{22}^{-1} \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \Theta & -\Theta P_{12}P_{22}^{-1} \\ -P_{22}^{-1}P_{21}\Theta & P_{22}^{-1} + P_{22}^{-1}P_{21}\Theta P_{12}P_{22}^{-1} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\Theta &= (P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1} \\
P_{11} &= I - (I + A^+B\Delta D^+C)(I - A^+A) - (I - A^+A)(I + A^+B\Delta D^+C)^* \\
P_{12} &= A^+B\Delta(I - D^+D) + (I - A^+A)(\Delta D^+C)^* \\
P_{21} &= \Delta D^+C(I - A^+A) + (I - D^+D)(A^+B\Delta)^* \\
P_{22} &= I - \Delta(I - D^+D) - (I - D^+D)\Delta^*.
\end{aligned}$$

Thus, by using Lemma 3.1, we can obtain the assertion. □

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