Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/faac`

Functional Analysis, Approximation and Computation **3:2** (2011), 57-68

Stability of systems of bi-quadratic and additive-cubic functional equations in Fréchet spaces

M. Eshaghi Gordji, A. Javadian and J. M. Rassias

Abstract

In this paper, we achieve the generalized Hyers-Ulam stability of the system of bi-quadratic functional equations

$$\begin{cases} f(x_1+x_2,y)+f(x_1-x_2,y)=2f(x_1,y)+2f(x_2,y), \\ \\ f(x,y_1+y_2)+f(x,y_1-y_2)=2f(x,y_1)+2f(x,y_2) \end{cases}$$

and the system of additive-cubic functional equations

$$\begin{cases} f(x_1+x_2,y)=f(x_1,y)+f(x_2,y), \\ \\ f(x,2y_1+y_2)+f(x,2y_1-y_2)=2f(x,y_1+y_2) \\ \qquad\qquad +2f(x,y_1-y_2)+12f(x,y_1) \end{cases}$$

in Fréchet spaces.

1 Introduction

The stability problem of functional equations originated from the following question of Ulam [28] in 1940, concerning the stability of group homomorphisms: Let $(G_1,.)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(.,.)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\longrightarrow G_2$ satisfies the inequality $d(h(x.y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\longrightarrow G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the

2010 *Mathematics Subject Classifications.* 39B82, 39B52.
Key words and Phrases. Hyers-Ulam stability; Quadratic functional equation; Cubic functional equation; Quratic functional equation, Fréchet space.
Received: November 2, 2011
Communicated by Dragan S. Djordjević

equation. In 1941, D. H. Hyers [8] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces E and E' such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $t \to f(tx)$ is continuous in real t for each fixed $x \in E$, then T is linear. This new concept is known as Hyers-Ulam stability of functional equations (see [1,2], [4-9], [17-19] and [26]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

is related to symmetric bi-additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [1,11]). The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)) \tag{1.2}$$

The stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \longrightarrow B$, where A is normed space and B Banach space (see [27], [14-17] and [20-25]). Cholewa [2] noticed that the Theorem of Skof is still true if relevant domain A is replaced by an abelian group. In the paper [3] , Czerwik proved the stability of the equation (1.1). Grabiec [6] has generalized the above mentioned result.

The oldest cubic functional equation, and was introduced by J. M. Rassias [12-13] (in 2000-2001):

$$f(x+2y) + 3f(x) = 3f(x+y) + f(x-y) + 6f(y).$$

Jun and Kim [10] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.3}$$

and they established the general solution and the stability for the functional equation (1.3). The function $f(x) = x^3$ satisfies the functional equation (1.3), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique function $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables (see also [12, 13]).

Let X, Y and Z be vector spaces on $\mathbb{R}$ or $\mathbb{C}$. We say that a mapping $f : X \times Y \to Z$ is quartic if f satisfies one of the following systems of functional equations:

$$\begin{cases} f(x_1 + x_2, y) + f(x_1 - x_2, y) = 2f(x_1, y) + 2f(x_2, y) \\ \\ f(x, y_1 + y_2) + f(x, y_1 - y_2) = 2f(x, y_1) + 2f(x, y_2), \end{cases} \tag{1.4}$$

$$\begin{cases} f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y), \\ \\ f(x, 2y_1 + y_2) + f(x, 2y_1 - y_2) = 2f(x, y_1 + y_2) \\ \qquad\qquad +2f(x, y_1 - y_2) + 12f(x, y_1) \end{cases} \tag{1.5}$$

for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. It is easy to see that the functions $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x^2y^2, g(x, y) = xy^3$ are quartic mappings which satisfying (1.4), (1.5), respectively. As another example, let A be a normed algebra and let X be a normed left $A-$module. Define $f, g : A \times X \to X$ by $f(a, x) = \|a\|^2\|x\|^2, g(a, x) = a^3x$. One can show that f, g satisfying (1.4), (1.5), respectively.

In functional analysis and related areas of mathematics, Fréchet spaces, named after Maurice Fréchet, are special topological vector spaces. They are generalizations of Banach spaces (normed vector spaces which are complete with respect to the metric induced by the norm).

A topological vector space X is a Fréchet space if and only if it satisfies the following three properties:

a) it is complete as a uniform space

b) it is locally convex

c) its topology can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + a, y + a)$ for all a, x, y in X. This means that a subset U of X is open if and only if for every u in U there exists an $e > 0$ such that $\{v : d(u, v) < e\}$ is a subset of U. Note that there is no natural notion of distance between two points of a Fréchet space: many different translation-invariant metrics may induce the same topology.

The vector space $C^\infty([0, 1])$ of all infinitely often differentiable functions $f : [0, 1] \to \mathbb{R}$ becomes a Fréchet space with the seminorms $\|f\|_k = sup\{|f^{(k)}(x)| : x \in [0, 1]\}$ for every integer $k \geq 0$. Here, $f^{(k)}$ denotes the k-th derivative of f, and $f^{(0)} = f$.

More generally, if M is a compact C^∞ manifold and B is a Banach space, then the set of all infinitely-often differentiable functions $f : M \to B$ can be turned into a Fréchet space; the seminorms are given by the suprema of the norms of all partial derivatives.

The space ω of real valued sequences becomes a Fréchet space if we define the k-th semi-norm of a sequence to be the absolute value of the k-th element of the sequence. Convergence in this Fréchet space is equivalent to element-wise convergence.

Not all vector spaces with complete translation-variant metrics are Fréchet spaces. An example is L_p with $p < 1$. Of course, such spaces fail to be locally convex.

Fréchet spaces are studied because even though their topological structure is more complicated due to the lack of a norm, many important results in functional analysis, like the open mapping theorem and the Banach-Steinhaus theorem, still hold.

2 Stability of system (1.4)

In this section, we investigate the generalized Hyers–Ulam stability problem for system of functional equations (1.4).

Throughout this section, X and Y will be a real vector space and a real Fréchet space by metric d, respectively. Let $f : X \times X \to Y$ be a function then we define $\Delta_f, D_f : X \times X \times X \to \mathbb{R}$ by

$$D_f(x_1, x_2, y) = d(f(x_1 + x_2, y) + f(x_1 - x_2, y), 2f(x_1, y) + 2f(x_2, y)),$$

$$\Delta_f(x, y_1, y_2) = d(f(x, y_1 + y_2) + f(x, y_1 - y_2), 2f(x, y_1) + 2f(x, y_2))$$

for all $x, x_1, x_2, y, y_1, y_2 \in X$.

Theorem 2.1. *Let $s \in \{-1, 1\}$ be fixed. Let $\phi, \psi : X \times X \times X \to [0, \infty)$ be mappings satisfying*

$$\sum_{i=0}^{\infty} \frac{\psi(2^{s(i+s)}x, 2^{si}y, 2^{si}y) + \phi(2^{si}x, 2^{si}x, 2^{si}y)}{16^{si}} < \infty$$

for all $x, y \in X$, and

$$\lim_{n\to\infty} \frac{\psi(2^{sn}x, 2^{sn}y, 2^{sn}z) + \phi(2^{sn}x, 2^{sn}y, 2^{sn}z)}{16^{sn}} = 0$$

for all $x, y, z \in X$. If $f : X \times X \to Y$ is a mapping such that

$$D_f(x_1, x_2, y) \leq \phi(x_1, x_2, y), \tag{2.1}$$

$$\Delta_f(x, y_1, y_2) \leq \psi(x, y_1, y_2) \tag{2.2}$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique quartic mapping $T : X \times X \to Y$ satisfying (1.4) and

$$d(f(x, y), T(x, y)) \leq \frac{1}{16^{\frac{s+1}{2}}} \sum_{i=0}^{\infty} \frac{\psi(2^{s(i+s)}x, 2^{si}y, 2^{si}y)}{16^{si}} + \frac{1}{4^s} \sum_{i=0}^{\infty} \frac{\phi(2^{si}x, 2^{si}x, 2^{si}y)}{16^{si}}, \tag{2.3}$$

for all $x, y \in X$.

Proof. Putting $x_1 = x_2 = x$ in (2.1), we get

$$d(f(2x, y), 4f(x, y)) \leq \phi(x, x, y). \tag{2.4}$$

Replacing y_1, y_2 by y in (2.2), we obtain

$$d(f(x, 2y), 4f(x, y)) \leq \psi(x, y, y). \tag{2.5}$$

Replacing x by $2x$ in (2.5), yields

$$d(f(2x, 2y), 4f(2x, y)) \leq \psi(2x, y, y). \tag{2.6}$$

Combining (2.4) and (2.6), we lead to

$$d(\frac{1}{16}f(2x, 2y), f(x, y)) \leq \frac{1}{4}\phi(x, x, y) + \frac{1}{16}\psi(2x, y, y). \tag{2.7}$$

From inequality (2.7), we use iterative methods and induction on n to prove our next relation:

$$\begin{aligned} d(\frac{1}{16^{sn}}f(2^{sn}x, 2^{sn}y), f(x, y)) \leq \frac{1}{4^s}\sum_{i=0}^{n-1}\frac{1}{16^{si}}\phi(2^{si}x, 2^{si}x, 2^{si}y) \\ + \frac{1}{16^{\frac{s+1}{2}}}\sum_{i=0}^{n-1}\frac{1}{16^{si}}\psi(2^{1+si}x, 2^{si}y, 2^{si}y). \end{aligned} \tag{2.8}$$

Divide (2.8) by 16^{sm} and replace x by $2^{sm}x$ to obtain that

$$\begin{aligned} & d(\frac{1}{16^{s(m+n)}}f(2^{s(m+n)}x, 2^{s(m+n)}y), \frac{1}{16^{sm}}f(2^{sm}x, 2^{sm}y)) \\ & \leq \frac{1}{4^s}\sum_{i=0}^{n-1}\frac{1}{16^{s(i+m)}}\phi(2^{s(m+i)}x, 2^{s(m+i)}x, 2^{s(m+i)}y) \\ & + \frac{1}{16^{\frac{s+1}{2}}}\sum_{i=0}^{n-1}\frac{1}{16^{s(i+m)}}\psi(2^{1+s(m+i)}x, 2^{s(m+i)}y, 2^{s(m+i)}y). \end{aligned} \tag{2.9}$$

This shows that $\{\frac{1}{16^{sn}}f(2^{sn}x, 2^{sn}y)\}$ is a Cauchy sequence in Y by taking the limit $m \to \infty$. Since Y is a Fréchet space, it follows that the sequence $\{\frac{1}{16^{sn}}f(2^{sn}x, 2^{sn}y)\}$ converges. We define $T : X \times X \to Y$ by $T(x, y) = \lim_{n\to\infty}\frac{1}{16^{sn}}f(2^{sn}x, 2^{sn}y)$ for all $x, y \in X$. It follows from (2.1) that

$$D_T(x_1, x_2, y) = \lim_{n\to\infty}\frac{1}{16^{sn}}D_f(2^{sn}x_1, 2^{sn}x_2, 2^{sn}y) \leq \lim_{n\to\infty}\frac{1}{16^{sn}}\phi(2^{sn}x_1, 2^{sn}x_2, 2^{sn}y) = 0$$

for all $x_1, x_2, y \in X$. Also it follows from (2.2) that

$$\Delta_T(x, y_1, y_2) = \lim_{n\to\infty}\frac{1}{16^{sn}}\Delta_f(2^{sn}x, 2^{sn}y_1, 2^{sn}y_2) \leq \lim_{n\to\infty}\frac{1}{16^{sn}}\phi(2^{sn}x, 2^{sn}y_1, 2^{sn}y_2) = 0$$

for all $x, y_1, y_2 \in X$. This means that T is bi-quadratic. It remains to show that T is unique. Suppose that there exists another bi-quadratic mapping $T' : X \times X \to Y$ which satisfies (1.4) and (2.3). Since $\frac{1}{16^{sn}} T(2^{sn}x, 2^{sn}y) = T(x, y)$, and $\frac{1}{16^{sn}} T'(2^{sn}x, 2^{sn}y) = T'(x, y)$ for all $x, y \in X$, we conclude that

$$\begin{aligned}
d(T(x,y), T'(x,y)) &= \frac{1}{16^{sn}} d(T(2^{sn}x, 2^{sn}y), T'(2^{sn}x, 2^{sn}y)) \\
&\le \frac{1}{16^{sn}} d(T(2^{sn}x, 2^{sn}y), f(2^{sn}x, 2^{sn}y)) \\
&+ d(f(2^{sn}x, 2^{sn}y), T'(2^{sn}x, 2^{sn}y)) \\
&\le 2[\frac{1}{16^{\frac{s+1}{2}}} \sum_{i=0}^{\infty} \frac{\psi(2^{s(n+i+s)}x, 2^{s(n+i)}y, 2^{s(n+i)}y)}{16^{s(n+i)}} \\
&+ \frac{1}{4^s} \sum_{i=0}^{\infty} \frac{\phi(2^{s(n+i)}x, 2^{s(n+i)}x, 2^{s(n+i)}y)}{16^{s(n+i)}}]
\end{aligned}$$

for all $x, y \in X$. By letting $n \to \infty$ in this inequality, it follows that $T(x, y) = T'(x, y)$ for all $x, y \in X$, which gives the conclusion. □

Corollary 2.2. *Let $s \in \{-1, 1\}$ be fixed. Let X be a vector space and Y be a Banach space. Suppose the mappings $\phi, \psi : X \times X \times X \to [0, \infty)$ satisfying*

$$\sum_{i=0}^{\infty} \frac{\psi(2^{s(i+s)}x, 2^{si}y, 2^{si}y) + \phi(2^{si}x, 2^{si}x, 2^{si}y)}{16^{si}} < \infty$$

for all $x, y \in X$, and

$$\lim_{n \to \infty} \frac{\psi(2^{sn}x, 2^{sn}y, 2^{sn}z) + \phi(2^{sn}x, 2^{sn}y, 2^{sn}z)}{16^{sn}} = 0$$

for all $x, y, z \in X$. If $f : X \times X \to Y$ is a mapping such that

$$\|f(x_1 + x_2, y) + f(x_1 - x_2, y) - 2f(x_1, y) + 2f(x_2, y)\| \le \phi(x_1, x_2, y),$$

$$\|f(x, y_1 + y_2) + f(x, y_1 - y_2) - 2f(x, y_1) - 2f(x, y_2)\| \le \psi(x, y_1, y_2)$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique bi-quadratic mapping $T : X \times X \to Y$ satisfying (1.4) and

$$\|f(x,y) - T(x,y)\| \le \frac{1}{16^{\frac{s+1}{2}}} \sum_{i=0}^{\infty} \frac{\psi(2^{s(i+s)}x, 2^{si}y, 2^{si}y)}{16^{si}} + \frac{1}{4^s} \sum_{i=0}^{\infty} \frac{\phi(2^{si}x, 2^{si}x, 2^{si}y)}{16^{si}}$$

for all $x, y \in X$.

Proof. It follows from theorem 2.1. by putting $d(a, b) = \|a - b\|$ for all $a, b \in Y$. □

We are going to investigate the following stability problem for system of functional equations (1.4).

Corollary 2.3. *Let $\epsilon > 0, p < 4$, and let X, Y be a normed space a Banach space, respectively. If $f : X \times X \to Y$ is a mapping such that*

$$\begin{aligned} &Max\{\|f(x, y_1 + y_2) + f(x, y_1 - y_2) - 2f(x, y_1) - 2f(x, y_2)\| \\ &\quad , \|f(x_1 + x_2, y) + f(x_1 - x_2, y) - 2f(x_1, y) + 2f(x_2, y)\|\} \\ &\quad \leq \epsilon(Min\{\|x_1\|^p + \|x_2\|^p + \|y\|^p, \|x\|^p + \|y_1\|^p + \|y_2\|^p\}) \end{aligned}$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique quartic mapping $T : X \times X \to Y$ satisfying (1.4) and

$$\|f(x, y) - T(x, y)\| \leq \frac{\epsilon}{1 - 2^{p-4}}(\frac{1 + 2^{p-3}}{2}\|x\|^p + \frac{3}{8}\|y\|^p)$$

for all $x, y \in X$.

Proof. It follows from corollary 2.2. by putting $\phi(a, b, c) = \psi(a, b, c) = \|a\|^p + \|b\|^p + \|c\|^p$ for all $a, b, c \in X$. □

By Corollary 2.3, we solve the following Hyers-Ulam stability problem for system of functional equations (1.4).

Corollary 2.4. *Let $\epsilon > 0$, and let X, Y be a normed space a Banach space, respectively. If $f : X \times X \to Y$ is a mapping such that*

$$\begin{aligned} &Max\{\|f(x, y_1 + y_2) + f(x, y_1 - y_2) - 2f(x, y_1) - 2f(x, y_2)\| \\ &\quad , \|f(x_1 + x_2, y) - f(x_1, y) + f(x_2, y)\|\} \leq \epsilon \end{aligned}$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique bi-quadratic mapping $T : X \times X \to Y$ satisfying (1.4) and

$$\|f(x, y) - T(x, y)\| \leq \frac{\epsilon}{3}$$

for all $x, y \in X$.

3 Stability of system (1.5)

Now, we investigate the generalized Hyers–Ulam stability problem for system of functional equations (1.5). From now on, let X be a real vector space and Y be a real Fréchet space by metric d. Let $f : X \times X \to Y$ be a function then we define $\Delta_f, D_f : X \times X \times X \to \mathbb{R}$ by

$$D_f(x_1, x_2, y) = d(f(x_1 + x_2, y), f(x_1, y) + f(x_2, y)),$$

$$\Delta_f(x, y_1, y_2) = d(f(x, 2y_1 + y_2) + f(x, 2y_1 - y_2) - f(x, y_1 + y_2) - f(x, y_1 - y_2) - 12f(x, y_1))$$

for all $x, x_1, x_2, y, y_1, y_2 \in X$.

Theorem 3.1. *Let $s \in \{-1,1\}$ be fixed. Let $\phi, \psi : X\times X\times X \to [0,\infty)$ be mappings satisfying*

$$\sum_{i=0}^{\infty} \frac{\psi(2^{(1+si)}x, 2^{si}y, 0) + \phi(2^{si}x, 2^{si}x, 2^{si}y)}{16^{si}} < \infty$$

for all $x, y \in X$, and

$$\lim_{n\to\infty} \frac{\psi(2^{sn}x, 2^{sn}y, 2^{sn}z) + \phi(2^{sn}x, 2^{sn}y, 2^{sn}z)}{16^{sn}} = 0$$

for all $x, y, z \in X$. If $f : X \times X \to Y$ is a mapping such that

$$D_f(x_1, x_2, y) \le \phi(x_1, x_2, y), \tag{3.1}$$

$$\Delta_f(x, y_1, y_2) \le \psi(x, y_1, y_2) \tag{3.2}$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique quartic mapping $T : X \times X \to Y$ satisfying (1.5) and

$$d(f(x,y), T(x,y)) \le \frac{1}{2} \sum_{i=|\frac{s-1}{2}|}^{\infty} \frac{1}{16^{si}} \phi(2^{si}x, 2^{si}x, 2^{si}y) + \frac{1}{32} \sum_{i=|\frac{s-1}{2}|}^{\infty} \frac{1}{16^{si}} \psi(2^{1+si}x, 2^{si}y, 0) \tag{3.3}$$

for all $x, y \in X$.

Proof. Putting $x_1 = x_2 = x$ in (3.1), we get

$$d(f(2x, y), 2f(x, y)) \le \phi(x, x, y). \tag{3.4}$$

Replacing y_1, y_2 by $y, 0$, respectively, in (3.2), we obtain

$$d(f(x, 2y), 8f(x, y)) \le \frac{1}{2}\psi(x, y, 0). \tag{3.5}$$

Replacing x by $2x$ in (3.5), yields

$$d(f(2x, 2y), 8f(2x, y)) \le \frac{1}{2}\psi(2x, y, y). \tag{3.6}$$

Combining (3.4) and (3.6), we lead to

$$d(\frac{1}{16}f(2x, 2y), f(x, y)) \le \frac{1}{2}\phi(x, x, y) + \frac{1}{32}\psi(2x, y, 0). \tag{3.7}$$

From inequality (3.7) we use iterative methods and induction on n to prove our next relation:

$$d(\frac{1}{16^{sn}} f(2^{sn}x, 2^{sn}y), f(x, y)) \le \frac{1}{2} \sum_{i=|\frac{s-1}{2}|}^{n-1+|\frac{s-1}{2}|} \frac{1}{16^{si}} \phi(2^{si}x, 2^{si}x, 2^{si}y) + \frac{1}{32} \sum_{i=|\frac{s-1}{2}|}^{n-1+|\frac{s-1}{2}|} \frac{1}{16^{si}} \psi(2^{1+si}x, 2^{si}y, 0). \tag{3.8}$$

We divide (3.8) by 16^{sm}, and replace x by $2^{sm}x$, we obtain that

$$
\begin{aligned}
d(&\frac{1}{16^{s(n+m)}} f(2^{s(n+m)}x, 2^{s(n+m)}y), f(x,y)) \\
&\leq \frac{1}{2} \sum_{i=|\frac{s-1}{2}|}^{n-1+|\frac{s-1}{2}|} \frac{1}{16^{s(i+m)}} \phi(2^{s(i+m)}x, 2^{s(i+m)}x, 2^{s(i+m)}y) \\
&+ \frac{1}{32} \sum_{i=|\frac{s-1}{2}|}^{n-1+|\frac{s-1}{2}|} \frac{1}{16^{s(i+m)}} \psi(2^{1+s(i+m)}x, 2^{s(i+m)}y, 0). \qquad (3.9)
\end{aligned}
$$

This shows that $\{\frac{1}{16^{sn}} f(2^{sn}x, 2^{sn}y)\}$ is a Cauchy sequence in Y by taking the limit $m \to \infty$. Since Y is a Banach space, it follows that sequence $\{\frac{1}{16^{sn}} f(2^{sn}x, 2^{sn}y)\}$ converges. We define $T : X \times X \to Y$ by $T(x,y) = \lim_{n\to\infty} \frac{1}{16^{sn}} f(2^{sn}x, 2^{sn}y)$ for all $x, y \in X$. The rest of proof is similar to the proof of theorem 2.1. □

Corollary 3.2. *Let $s \in \{-1, 1\}$ be fixed. Let X be a vector space and Y be a Banach space. Suppose the mappings $\phi, \psi : X \times X \times X \to [0, \infty)$ satisfy*

$$\sum_{i=0}^{\infty} \frac{\psi(2^{(1+si)}x, 2^{si}y, 0) + \phi(2^{si}x, 2^{si}x, 2^{si}y)}{16^{si}} < \infty$$

for all $x, y \in X$, and

$$\lim_{n\to\infty} \frac{\psi(2^{sn}x, 2^{sn}y, 2^{sn}z) + \phi(2^{sn}x, 2^{sn}y, 2^{sn}z)}{16^{sn}} = 0$$

for all $x, y, z \in X$. If $f : X \times X \to Y$ is a mapping such that

$$\|f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)\| \leq \phi(x_1, x_2, y),$$

$$\|f(x, 2y_1+y_2)+f(x, 2y_1-y_2)-f(x, y_1+y_2)-f(x, y_1-y_2)-12f(x, y_1)\| \leq \psi(x, y_1, y_2)$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique additive-cubic mapping $T : X \times X \to Y$ satisfying (1.5) and

$$\|f(x,y)-T(x,y)\| \leq \frac{1}{2} \sum_{i=|\frac{s-1}{2}|}^{\infty} \frac{1}{16^{si}} \phi(2^{si}x, 2^{si}x, 2^{si}y) + \frac{1}{32} \sum_{i=|\frac{s-1}{2}|}^{\infty} \frac{1}{16^{si}} \psi(2^{1+si}x, 2^{si}y, 0)$$

for all $x, y \in X$.

Proof. It follows from theorem 3.1. by putting $d(a, b) = \|a - b\|$ for all $a, b \in Y$. □

We are going to investigate the following stability problem for system of functional equations (1.4).

Corollary 3.3. *Let $\epsilon > 0, p < 4$, and let X, Y be a normed space a Banach space, respectively. If $f : X \times X \to Y$ is a mapping such that*

$$\begin{aligned} Max\{&\|f(x_1 + x_2, y) - f(x_1, y) - f(x_2, y)\|, \|f(x, 2y_1 + y_2) + f(x, 2y_1 - y_2) \\ &- f(x, y_1 + y_2) - f(x, y_1 - y_2) - 12f(x, y_1)\|\} \\ &\leq \epsilon(Min\{\|x_1\|^p + \|x_2\|^p + \|y\|^p, \|x\|^p + \|y_1\|^p + \|y_2\|^p\}) \end{aligned}$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique quartic mapping $T : X \times X \to Y$ satisfying (1.5) and

$$\|f(x, y) - T(x, y)\| \leq \frac{\epsilon}{1 - 2^{p-4}}[(1 + 2^{p-5})\|x\|^p + \frac{17}{32}\|y\|^p]$$

for all $x, y \in X$.

Proof. It follows from corollary 3.2. by putting $\phi(a, b, c) = \psi(a, b, c) = \|a\|^p + \|b\|^p + \|c\|^p$ for all $a, b, c \in X$. □

By Corollary 3.3, we solve the following Hyers-Ulam stability problem for system of functional equations (1.5).

Corollary 3.4. *Let $\epsilon > 0$, and let X, Y be a normed space a Banach space, respectively. If $f : X \times X \to Y$ is a mapping such that*

$$\begin{aligned} Max\{&\|f(x, y_1 + y_2) + f(x, y_1 - y_2) - 2f(x, y_1) - 2f(x, y_2)\| \\ &, \|f(x_1 + x_2, y) - f(x_1, y) + f(x_2, y)\|\} \leq \epsilon \end{aligned}$$

for all $x, y, x_1, x_2, y_2, y_2 \in X$, then there exists a unique additive-cubic mapping $T : X \times X \to Y$ satisfying (1.5) and

$$\|f(x, y) - T(x, y)\| \leq \frac{4\epsilon}{9}$$

for all $x, y \in X$.

References

[1] J. Aczel and J. Dhombres, Functional Equations in Several Variables, Cambridge Univ. Press, 1989.

[2] P. W. Cholewa, Remarks on the stability of functional equations, Aequationes Math. 27 (1984) 76–86.

[3] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg 62 (1992) 59–64.

[4] Z. Gajda, On stability of additive mappings, Internat. J. Math. Math. Sci. 14(1991) 431–434.

[5] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431–436.

[6] A. Grabiec, The generalized Hyers–Ulam stability of a class of functional equations, Publ. Math. Debrecen 48 (1996) 217–235.

[7] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, Birkh$\ddot{a}$user, Basel, 1998.

[8] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. 27 (1941) 222–224.

[9] G. Isac and Th. M. Rassias, On the Hyers-Ulam stability of ψ-additive mappings, J. Approx. Theory 72(1993),131–137.

[10] K. W. Jun and and H. M. Kim, The generalized Hyers-Ulam-Russias stability of a cubic functional equation, J. Math. Anal. Appl. 274, (2002), no. 2, 267–278.

[11] Pl. Kannappan, Quadratic functional equation and inner product spaces, Results Math. 27 (1995) 368–372.

[12] J. M. Rassias, Solution of the Ulam stability problem for cubic mappings, Glas. Mat. Ser. III 36(56) (2001), no. 1, 63–72.

[13] J. M. Rassias, Solution of the Ulam problem for cubic mappings. An. Univ. Timisoara Ser. Mat.-Inform. 38 (2000), no. 1, 121–132.

[14] J. M. Rassias and M. J. Rassias, Refined Ulam stability for Euler-Lagrange type mappings in in Hilbert spaces, Intern. J. Appl. Math. Stat., 7(Fe07), 2007, 126–132.

[15] J. M. Rassias, On the stability of the Euler-Lagrange functional equation. Chinese J. Math. 20 (1992), no. 2, 185-190.

[16] J. M. Rassias, On a new approximation of approximately linear mappings by linear mappings. Discuss. Math. 7 (1985), 193-196.

[17] J. M. Rassias, On approximation of approximately linear mappings by linear mappings. Bull. Sci. Math. (2) 108 (1984), no. 4, 445-446.

[18] J. M. Rassias, Complete solution of the multi-dimensional problem of Ulam. Discuss. Math. 14 (1994), 101–107.

[19] J. M. Rassias, On the stability of a multi-dimensional Cauchy type functional equation. Geometry, analysis and mechanics, 365–375, World Sci.Publ., River Edge, NJ, 1994.

[20] J. M. Rassias, Solution of a stability problem of Ulam. Functional analysis, approximation theory and numerical analysis, 241–249, World Sci.Publ., River Edge, NJ, 1994.

[21] J. M. Rassias, Solution of a stability problem of Ulam. Discuss. Math. 12 (1992), 95–103 (1993).

[22] J. M. Rassias, Solution of a problem of Ulam. J. Approx. Theory 57 (1989), no. 3, 268–273.

[23] J. M. Rassias, On approximation of approximately linear mappings by linear mappings. J.Funct. Anal. 46 (1982), no. 1, 126–130.

[24] Th. M. Rassias (Ed.), Functional Equations and Inequalities, Kluwer Academic, Dordrecht, 2000.

[25] Th. M. Rassias, On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000) 264–284.

[26] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.

[27] F. Skof, Propriet locali e approssimazione di operatori, Rend. Sem. Mat. Fis. Milano, 53 (1983),. 113129.

[28] S. M. Ulam, Problems in Modern Mathematics, Chapter VI, science ed., Wiley, New York, 1940.

Address

M. Eshaghi Gordji:
Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran
E-mail: madjid.eshaghi@gmail.com

A. Javadian:
Department of Physics, Semnan University, P. O. Box 35195-363, Semnan, Iran
E-mail: abasjavadian@gmail.com

J. M. Rassias:
Section of Mathematics and Informatics, Pedagogical Department, National and Capodistrian University of Athens, 4, Agamemnonos St., Aghia Paraskevi, Athens 15342, Greece
E-mail: jrassias@primedu.uoa.gr