Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Functional Analysis, Approximation and Computation **3:2** (2011), 1–10

Convergence theorems for two finite families of asymptotically quasi-nonexpansive mappings in the intermediate sense in convex metric spaces

G.S. Saluja

Abstract

In this paper, we obtain the strong convergence of a kind of Ishikawa type iteration process with errors to a common fixed points for two finite families of asymptotically quasi-nonexpansive mappings in the intermediate sense in convex metric spaces. Our result improves and extends the corresponding result of [1, 2, 4, 5, 6, 7, 8, 10, 11, 13, 16].

1 Introduction and Preliminaries

Throughout this paper, we assume that X is a metric space, $F(T_i) = \{x \in X : T_i x = x\}$ be the set of all fixed points of the mappings T_i $(i = 1, 2, \dots, N)$, $D(T)$ be the domain of T. The set of common fixed points of S_i and T_i $(i = 1, 2, \dots, N)$ denoted by $F(S, T)$, that is, $F(S, T) = \cap_{i=1}^{N} S(T_i) \bigcap \cap_{i=1}^{N} F(T_i)$.

Definition 1.1. ([1]) Let $T\colon D(T) \subset E \to E$ be a mapping.

(1) The mapping T is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(Tx, Ty) \leq Ld(x, y), \qquad \forall x, y \in D(T). \tag{1.1}$$

(2) The mapping T is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \qquad \forall x, y \in D(T). \tag{1.2}$$

2010 *Mathematics Subject Classifications.* 47H05, 47H09, 47H10.

Key words and Phrases. Asymptotically quasi-nonexpansive mapping in the intermediate sense; Ishikawa-type iteration process with errors; common fixed point; strong convergence; convex metric space.

Received: July 13, 2011

Communicated by Dragan S. Djordjević

(3) The mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \quad \leq \quad d(x, p), \qquad \forall x \in D(T),\ \forall p \in F(T). \tag{1.3}$$

(4) The mapping T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \quad \leq \quad k_n d(x, y), \qquad \forall x, y \in D(T),\ \forall n \in \mathbb{N}. \tag{1.4}$$

(5) The mapping T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, p) \quad \leq \quad k_n d(x, p), \qquad \forall x \in D(T),\ \forall p \in F(T),\ \forall n \in \mathbb{N}. \tag{1.5}$$

(6) T is said to be asymptotically nonexpansive type, if

$$\limsup_{n\to\infty} \left\{ \sup_{x,\ y \in D(T)} \Big(d(T^n x, T^n y) - d(x, y) \Big) \right\} \quad \leq \quad 0. \tag{1.6}$$

(7) T is said to be asymptotically quasi-nonexpansive type, if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x \in D(T),\ p \in F(T)} \Big(d(T^n x, p) - d(x, p) \Big) \right\} \quad \leq \quad 0. \tag{1.7}$$

(8) asymptotically quasi-nonexpansive mapping in the intermediate sense [18] provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \left\{ \sup_{x \in D(T),\ p \in F(T)} \Big(d(T^n x, p) - d(x, p) \Big) \right\} \quad \leq \quad 0. \tag{1.8}$$

Remark 1.1. It is easy to see that if $F(T)$ is nonempty, then nonexpansive mapping, quasi-nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping all are the special cases of asymptotically quasi-nonexpansive type mappings.

In recent years, the problem concerning convergence of iterative sequences (and sequences with errors) for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings converging to some fixed points in Hilbert spaces or Banach spaces have been considered by many authors (see, for example [2, 3, 7, 8, 9, 10, 15]).

In 1970, Takahashi [14] introduced initially a notion of convex metric space and studied the fixed point theorems for nonexpansive mappings. The convex metric space is a more general space and each normed linear space is a special example of a convex metric space. But there are many examples of convex metric spaces which are not embedded in any normed linear space (see [14]). Later on, some authors discussed the existence of fixed points and the convergence of the iterative processes for nonexpansive mappings in convex metric spaces (see, for example, [17]).

In 2004, Chang et al. [1] extended and improved the result of [9] in convex metric space. Further in the same year, Kim et al. [6] gave the necessary and sufficient conditions for asymptotically quasi-nonexpansive mappings in convex metric spaces which generalized and improved some previous known results.

There are number of results on fixed points of asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of asymptotically quasi-nonexpansive mappings were studied by Schu [12]. Subsequently, Tian [16] gave some sufficient and necessary conditions for an Ishikawa iteration process of asymptotically quasi-nonexpansive mappings to converge to a fixed point in convex metric spaces. Recently, Wang and Liu [17] gave some results for an Ishikawa type iteration process with errors to approximate a fixed point of two uniformly quasi-Lipschitzian mappings in generalized convex metric spaces.

Inspired and motivated by the above mentioned work, in this paper, we study the Ishikawa type iteration process with errors to approximate a common fixed points for two finite families of asymptotically quasi-nonexpansive mappings in the intermediate sense, and to obtain the strong convergence theorem for such mappings in convex metric spaces.

We restate the following definitions and lemmas:

Definition 1.2. ([16]) Let (X, d) be a metric space, $K = [0, 1]$, and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in [0,1] with $a_n + b_n + c_n = 1$, $(n \geq 1)$. A mapping $W\colon X^3 \times K^3 \to X$ is said to be a convex structure on X, if for any $(x, y, z, a_n, b_n, c_n) \in X^3 \times K^3$ and $u \in X$, the following inequality holds:

$$d(W(x, y, z, a_n, b_n, c_n)), u) \leq a_n d(x, u) + b_n d(y, u) + c_n d(z, u).$$

If (X, d) is a metric space with convex structure W, (X, d) is called a *convex metric space* and denotes it by (X, d, W). Let (X, d) be a convex metric space, a nonempty subset C of X is said to be convex if

$$W(x, y, z, a_n, b_n, c_n) \in C, \quad \forall (x, y, z, a_n, b_n, c_n) \in C^3 \times K^3.$$

Definition 1.3. Let (X, d) be a metric space with a convex structure $W\colon X^3 \times K^3 \to X$. Let $S_i, T_i\colon X \to X$, $i \in \{1, 2, \dots, N\}$ be asymptotically quasi-nonexpansive mappings in the intermediate sense. Define an iterative sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= W(y_n, S_{i(n)}^{k(n)} y_n, u_n, a_n, b_n, c_n), \;\; n \geq 1 \\ y_n &= W(x_n, T_{i(n)}^{k(n)} x_n, v_n, a'_n, b'_n, c'_n) \end{aligned} \tag{1.9}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are real sequences in $(0, 1)$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $n = (k(n) - 1)N + i(n)$, $i(n) \in I = \{1, 2, \dots, N\}$. $\{u_n\}$ and $\{v_n\}$ are two sequences in X for $n = 0, 1, 2, \dots$. Then $\{x_n\}$ is called the Ishikawa type iteration process with errors for two finite families S_i and T_i of asymptotically quasi-nonexpansive mappings in the intermediate sense.

Lemma 1.1. (see [8]): Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three nonnegative sequences of real numbers satisfying the following conditions:

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \quad n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then

(1) $\lim_{n\to\infty} p_n$ exists.

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

2 Main Results

Lemma 2.1. Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of X. Let $S_i, T_i\colon C \to C$ be two finite families of asymptotically quasi-nonexpansive mappings in the intermediate sense for $i = 1, 2, \dots, N$ such that $F(S, T) = \cap_{i=1}^N S(T_i) \bigcap \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa type iteration process with errors defined by (1.9) in which $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in C with the restriction $\sum_{n=1}^{\infty}(c_n + c'_n) < \infty$. Put

$$G_n = \max\Big\{ \sup_{p\in\ F(S,T),\ n\geq 1} \big\{ d(S_{i(n)}^{k(n)} y_n, p) - d(y_n, p) \big\} \vee 0 \Big\} \tag{2.1}$$

and

$$H_n = \max\Big\{ \sup_{p\in\ F(S,T),\ n\geq 1} \big\{ d(T_{i(n)}^{k(n)} x_n, p) - d(x_n, p) \big\} \vee 0 \Big\}, \tag{2.2}$$

where $n = (k(n) - 1)N + i(n)$ and $i(n) \in I = \{1, 2, \dots, N\}$. Assume that

$\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$. Then the following conclusions hold:

(1) for all $p \in F(S,T)$ and $n \geq 1$,

$$d(x_{n+1}, p) \leq d(x_n, p) + t_n, \tag{2.3}$$

where $t_n = G_n + H_n + (c_n + c'_n)M$ and

$$M = \sup_{p\in\ F(S,T),\ n\geq 0} \Big\{ d(u_n, p) + d(v_n, p) \Big\}.$$

(2) $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(S,T)$.

Proof. (1) Let $p \in F(S,T)$. It follows from (1.9), (2.1) and (2.2) that

$$\begin{aligned} d(x_{n+1}, p) &= d(W(y_n, S_{i(n)}^{k(n)} y_n, u_n; a_n, b_n, c_n), p) \\ &\leq a_n d(y_n, p) + b_n d(S_{i(n)}^{k(n)} y_n, p) + c_n d(u_n, p) \\ &\leq a_n d(y_n, p) + b_n [d(y_n, p) + G_n] + c_n d(u_n, p) \\ &\leq (a_n + b_n) d(y_n, p) + b_n G_n + c_n d(u_n, p) \\ &= (1 - c_n) d(y_n, p) + b_n G_n + c_n d(u_n, p) \\ &\leq d(y_n, p) + G_n + c_n d(u_n, p), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_{i(n)}^{k(n)} x_n, v_n; a'_n, b'_n, c'_n), p) \\ &\leq a'_n d(x_n, p) + b'_n d(T_{i(n)}^{k(n)} x_n, p) + c'_n d(v_n, p) \\ &\leq a'_n d(x_n, p) + b'_n [d(x_n, p) + H_n] + c'_n d(v_n, p) \\ &\leq (a'_n + b'_n) d(x_n, p) + b'_n H_n + c'_n d(v_n, p) \\ &= (1 - c'_n) d(x_n, p) + b'_n H_n + c'_n d(v_n, p) \\ &\leq d(x_n, p) + H_n + c'_n d(v_n, p). \end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_n, p) + H_n + c'_n d(v_n, p) + G_n + c_n d(u_n, p) \\ &\leq d(x_n, p) + G_n + H_n + c_n d(u_n, p) + c'_n d(v_n, p) \\ &\leq d(x_n, p) + G_n + H_n + (c_n + c'_n) M \\ &= d(x_n, p) + t_n, \end{aligned} \tag{2.6}$$

where $t_n = G_n + H_n + (c_n + c'_n)M$ and

$$M = \sup_{p\in\ F(S,T),\ n\geq 0} \Big\{d(u_n, p) + d(v_n, p)\Big\}.$$

This completes the proof of part (1).

(2) From part (1) above, we have

$$d(x_{n+1}, p) \quad \leq \quad d(x_n, p) + t_n,$$

where $t_n = G_n+H_n+(c_n+c'_n)M$. Since by assumptions $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} H_n < \infty$ and $\sum_{n=1}^{\infty}(c_n + c'_n) < \infty$, it follows that $\sum_{n=1}^{\infty} t_n < \infty$. Thus from Lemma 1.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists. This completes the proof of part (2).

Theorem 2.1. Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of X. Let $S_i, T_i \colon C \to C$ be two finite families of asymptotically quasi-nonexpansive mappings in the intermediate sense for $i = 1, 2, \dots, N$ such that $F(S,T) = \cap_{i=1}^{N} S(T_i) \bigcap \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa type iteration process with errors defined by (1.9) and $\{u_n\}$, $\{v_n\}$ be two bounded sequences in C. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be six sequences in $[0,1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $\forall n \geq 1$, with the restriction $\sum_{n=1}^{\infty}(c_n + c'_n) < \infty$. Put

$$G_n \quad = \quad \max\Big\{ \sup_{p\in\ F(S,T),\ n\geq 1} \big\{d(S_{i(n)}^{k(n)} y_n, p) - d(y_n, p)\big\} \vee 0\Big\}$$

and

$$H_n \quad = \quad \max\Big\{ \sup_{p\in\ F(S,T),\ n\geq 1} \big\{d(T_{i(n)}^{k(n)} x_n, p) - d(x_n, p)\big\} \vee 0\Big\},$$

where $n = (k(n) - 1)N + i(n)$ and $i(n) \in I = \{1, 2, \dots, N\}$. Assume that $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$. Then the sequence $\{x_n\}$ converges to a common fixed point p in $F(S,T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$, where $d(x, F(S,T)) = \inf_{p\in F(S,T)} d(x, p)$.

Proof. The necessity is obvious. Thus we will only prove the sufficiency. By Lemma 2.1(2), $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(S,T)$. Hence (2.6) and Lemma 1.1 guarantee that $\lim_{n\to\infty} d(x_n, F(S,T))$ exists and by the hypothesis $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$, we have $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$.

Let $\varepsilon > 0$ be given, since $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$, there exists a positive integer n_1 such that $d(x_n, F(S,T)) < \frac{\varepsilon}{3}$ as $n \geq n_1$. Thus, there exists $q \in F(S,T)$ such that for above $\varepsilon > 0$ there exists an integer $n_2 \geq n_1$ such that as $n \geq n_2$, $d(x_n, q) < \frac{\varepsilon}{2}$.

Now for any $n, m \geq n_2$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, q) + d(x_m, q) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in a nonempty closed convex subset C of a complete convex metric space X, therefore it converges to a point, say $p \in C$. Now, we have to show that p is a common fixed point of S_i and T_i for $i = 1, 2, \ldots, N$. Indeed, we know that the set $F(S,T) = \cap_{i=1}^{N} S(T_i) \bigcap \cap_{i=1}^{N} F(T_i)$ is closed. From the continuity of $d(x, F(S,T)) = 0$ with $\lim_{n\to\infty} d(x_n, F(S,T)) = 0$ and $\lim_{n\to\infty} x_n = p$, we get

$$d(p, F(S,T)) = 0,$$

and so $p \in F(S,T)$, that is, p is a common fixed point of S_i and T_i for $i = 1, 2, \ldots, N$. This completes the proof.

If we take $S = T$ in Theorem 2.1, then it reduces to the following:

Theorem 2.2. Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of X. Let $T_i \colon C \to C$ be a finite family of asymptotically quasi-nonexpansive mappings in the intermediate sense for $i = 1, 2, \ldots, N$ such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa type iteration process with errors defined as follows:

$$\begin{aligned} x_{n+1} &= W(y_n, T_{i(n)}^{k(n)} y_n, u_n, a_n, b_n, c_n), \ \ n \geq 1 \\ y_n &= W(x_n, T_{i(n)}^{k(n)} x_n, v_n, a'_n, b'_n, c'_n) \end{aligned} \tag{2.7}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are real sequences in $(0,1)$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in C with the restriction $\sum_{n=1}^{\infty}(c_n + c'_n) < \infty$. Put

$$G_n = \max\Big\{ \sup_{p\in\ F,\ n\geq 1} \big\{ d(T_{i(n)}^{k(n)} y_n, p) - d(y_n, p) \big\} \vee 0 \Big\}$$

and

$$H_n = \max\Big\{ \sup_{p\in\ F,\ n\geq 1} \big\{ d(T_{i(n)}^{k(n)} x_n, p) - d(x_n, p) \big\} \vee 0 \Big\},$$

where $n = (k(n) - 1)N + i(n)$ and $i(n) \in I = \{1, 2, \ldots, N\}$. Assume that $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$. Then the sequence $\{x_n\}$ converges to a common fixed point p in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$.

If we take $S_i = T_i = T$ for $i = 1, 2, \ldots, N$ in Theorem 2.1, then it reduces to the following:

Theorem 2.3. Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of X. Let $T\colon C \to C$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa type iteration process with errors defined as follows:

$$\begin{aligned} x_{n+1} &= W(y_n, T^n y_n, u_n, a_n, b_n, c_n), \ \ n \geq 1 \\ y_n &= W(x_n, T^n x_n, v_n, a'_n, b'_n, c'_n) \end{aligned} \tag{2.8}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are real sequences in $(0, 1)$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in C with the restriction $\sum_{n=1}^{\infty}(c_n + c'_n) < \infty$. Put

$$G_n = \max\Big\{\sup_{p\in\ F(T),\ n\geq 1}\big\{d(T^n y_n, p) - d(y_n, p)\big\} \vee 0\Big\}$$

and

$$H_n = \max\Big\{\sup_{p\in\ F(T),\ n\geq 1}\big\{d(T^n x_n, p) - d(x_n, p)\big\} \vee 0\Big\}.$$

Assume that $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$. Then the sequence $\{x_n\}$ converges to a fixed point p in $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$.

Remark 2.1. Theorem 2.1 extends and improves the corresponding result of [1, 2, 4, 5, 6, 7, 8, 10, 11, 13, 16] to the case of more general class of quasi-nonexpansive, asymptotically nonexpansive, asymptotically quasi-nonexpansive, asymptotically quasi-nonexpansive type mappings and iteration scheme considered in this paper.

3 Applications

In this section, we apply Theorem 2.1 and Theorem 2.2 to obtain some convergence theorems for schemes (1.9) and (2.7) respectively.

Theorem 3.1. Let X, C, $F(S, T)$, S_i, T_i and $\{x_n\}$ be same as in Theorem 2.1. Suppose that there exists a map T_j which satisfies the following conditions: (1) $\liminf_{n\to\infty} d(x_n, T_j x_n) = 0$; (2) there exists a function $\varphi\colon [0, \infty) \to [0, \infty)$which is right continuous at 0, $\varphi(0) = 0$ and $\varphi(d(x_n, T_j x_n)) \geq d(x_n, F(S, T))$ for all n. Then the sequence $\{x_n\}$ converges to a common fixed point in $F(S, T)$.

Proof. Conditions (1) and (2) yield that

$$\begin{aligned} \liminf_{n\to\infty} d(x_n, F(S, T)) &\leq \liminf_{n\to\infty} \varphi(d(x_n, T_j x_n)) \\ &= \varphi(\liminf_{n\to\infty} d(x_n, T_j x_n)) = \varphi(0) = 0, \end{aligned}$$

that is, $\liminf_{n\to\infty} d(x_n, F(S,T)) = 0$. By Theorem 2.1, $\{x_n\}$ converges to a common fixed point in $F(S,T)$.

Theorem 3.2. Let X, C, F, T_i and $\{x_n\}$ be same as in Theorem 2.2. Suppose that there exists a map T_j which satisfies the following conditions:(1) $\liminf_{n\to\infty} d(x_n, T_j x_n) = 0$; (2) there exists a function φ: $[0,\infty) \to [0,\infty)$which is right continuous at 0, $\varphi(0) = 0$ and $\varphi(d(x_n, T_j x_n)) \geq d(x_n, F)$ for all n. Then the sequence $\{x_n\}$ converges to a common fixed point in F.

Proof. Conditions (1) and (2) yield that

$$\begin{aligned}\liminf_{n\to\infty} d(x_n, F) &\leq \liminf_{n\to\infty} \varphi(d(x_n, T_j x_n)) \\ &= \varphi(\liminf_{n\to\infty} d(x_n, T_j x_n)) = \varphi(0) = 0,\end{aligned}$$

that is, $\liminf_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.2, $\{x_n\}$ converges to a common fixed point in F.

Remark 3.1. Theorem 3.2 of Shahzad and Udomene [13], is a special case of Theorem 3.1 by suitably choosing the spaces, the iterative scheme and the mappings.

Remark 3.2. Theorem 3.2 of Khan et al. [4], Theorem 3.2 of Khan and Ahmed [5] are special cases of Theorem 3.2 by suitably choosing the spaces, the iterative scheme and the mappings.

References

[1] S.S. Chang, J.K. Kim and D.S. Jin, *Iterative sequences with errors for asymptotically quasi-nonexpansive type mappings in convex metric spaces*, Archives of Inequality and Applications 2(2004), 365-374.

[2] M.K. Ghosh and L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings*, J. Math. Anal. Appl. 207(1997), 96-103.

[3] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35(1972), 171-174.

[4] A.R. Khan, A.A. Domlo and H. Fukhar-ud-din, *Common fixed points of Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 341(2008), 1-11.

[5] A.R. Khan and M.A. Ahmed, *Convergence of a general iterative scheme for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces and applications*, Comput. Math. Appl. 59(2010), 2990-2995.

[6] J.K. Kim, K.H. Kim and K.S. Kim, *Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces*, Nonlinear Anal. Convex Anal. RIMS Vol. 1365(2004), pp. 156-165.

[7] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. 259(2001), 1-7.

[8] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl. 259(2001), 18-24.

[9] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member of uniformly convex Banach spaces*, J. Math. Anal. Appl. 266(2002), 468-471.

[10] W.V. Petryshyn and T.E. Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl. 43(1973), 459-497.

[11] G.S. Saluja, *Convergence of fixed point of asymptotically quasi-nonexpansive type mappings in convex metric spaces*, J. Nonlinear Sci. Appl. 1(3) (2008), 132-144.

[12] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158(1991), 407-413.

[13] N. Shahzad and A. Udomene, *Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces*, Fixed Point Theory and Applications, Vol.2006, Article ID 18909, Pages 1-10.

[14] W. Takahashi, *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep. 22(1970), 142-149.

[15] K.K. Tan and H.K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 122(1994), 733-739.

[16] Y.X. Tian, *Convergence of an Ishikawa type iterative scheme for asymptotically quasi-nonexpansive mappings*, Compt. Math. Appl. 49(2005), 1905-1912.

[17] C. Wang and L.W. Liu, *Convergence theorems of fixed points of uniformly quasi-Lipschizian mappings in convex metric spaces*, Nonlinear Anal. TMA 70(2009), 2067-2071.

[18] Y. Yao and Y.C. Liou, *New iterative schemes for asymptotically quasi-nonexpansive mappings*, Journal of Inequalities and Applications, Volume 2010, Article ID 934692, 9 pages, doi:10.1155/2010/934692.

Address

G. S. Saluja:
Department of Mathematics & Information Technology,
Govt. Nagarjuna P.G. College of Science,
Raipur - 492010 (C.G.), India.

E-mail: saluja_1963@rediffmail.com, saluja1963@gmail.com