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Sharp weighted inequalities for multilinear commutators of some sublinear operators

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Abstract

In this paper, we prove the sharp inequalities for some multilinear commutators related to certain integral operators. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator. As application, we obtain the weighted $L^p(p>1)$ norm inequalities and $L\log L$ type estimate for the multilinear commutators.

1 Introduction

Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rochberg and Weiss (see [2]) states that the commutator $[b,T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. However, it was observed that $[b,T]$ is not bounded, in general, from $L^1(R^n)$ to $L^{1,\infty}(R^n)$. In [11], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp inequalities for some multilinear commutators related to certain integral operators. In fact, we shall establish the sharp inequalities for the multilinear commutators only under certain conditions on the size of the integral operators. The integral operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator. As the applications, we obtain the weighted norm inequalities and $L\log L$ type estimate for these multilinear commutators.

2 Notations and Results

First, let us introduce some notations(see [4][8][10][11]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable

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function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that(see [4])

$$f^{\#}(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y) - c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. For $0 < r < \infty$, we denote $f^{\#}_r$ by

$$f^{\#}_r(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is that $M(f)(x) = \sup_{x\in Q} |Q|^{-1}\int_Q |f(y)|dy$, we write that $M_p(f) = (M(f^p))^{1/p}$. For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \geq 2$.

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f

$$||f||_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi(\frac{|f(y)|}{\lambda})dy \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q} ||f||_{\Phi,Q};$$

The main Young function to be using in this paper is $\Phi(t) = exp(t^r) - 1$ and $\Psi(t) = tlog^r(t+e)$, the corresponding Φ-average and maximal functions denoted by $||\cdot||_{expL^r,Q}$, M_{expL^r} and $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$. We have the following inequality, for any $r > 0$ and $m \in N$

$$M(f) \leq M_{L(logL)^r}(f), \quad M_{L(logL)^m}(f) \sim M^{m+1}(f);$$

For $r \geq 1$, we denote that

$$||b||_{osc_{expL^r}} = \sup_Q ||b - b_Q||_{expL^r,Q},$$

the spaces Osc_{expL^r} is defined by

$$Osc_{expL^r} = \{b \in L^1_{log}(R^n) : ||b||_{osc_{expL^r}} < \infty\}.$$

It has been known that(see [11])

$$||b - b_{2^kQ}||_{expL^r,2^kQ} \leq Ck||b||_{Osc_{expL^r}}.$$

It is obvious that Osc_{expL^r} coincides with the BMO space if $r = 1$. For $r_j > 0$ and $b_j \in Osc_{expL^{r_j}}$ for $j = 1, \cdots, m$, we denote that $1/r = 1/r_1 + \cdots + 1/r_m$ and $||\tilde{b}|| = \prod_{j=1}^m ||b_j||_{Osc_{expL^{r_j}}}$. Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, denote that $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\tilde{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, denote $\tilde{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\tilde{b}_\sigma||_{Osc_{expL^{r_\sigma}}} = ||b_{\sigma(1)}||_{Osc_{expL^{r_{\sigma(1)}}}} \cdots ||b_{\sigma(j)}||_{Osc_{expL^{r_{\sigma(j)}}}}$.

We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$(see [4]).

We are going to consider some integral operators as following.

Let $b_j (j = 1, ..., m)$ be the fixed locally integral functions on R^n.

Definition 1. Let $F(x, y, t)$ be a function define on $R^n \times R^n \times [0, +\infty)$, we denote that

$$F_t(f)(x) = \int_{R^n} F(x, y, t) f(y) dy$$

and

$$F_t^{\tilde{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] F(x, y, t) f(y) dy$$

for every bounded and compactly supported function f.

Let H be the Banach space $H = \{h : ||h|| < \infty\}$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ and $F_t^{\tilde{b}}(f)(x)$ as a mapping from $[0, +\infty)$ to H. Then, the multilinear commutator related to $F_t^{\tilde{b}}$ is defined by

$$T_{\tilde{b}}(f)(x) = ||F_t^{\tilde{b}}(f)(x)||,$$

we also denote that

$$T(f)(x) = ||F_t(f)(x)||.$$

Definition 2. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int \psi(x) dx = 0$,

(1) $|\psi(x)| \le C(1 + |x|)^{-(n+1)}$,

(2) $|\psi(x + y) - \psi(x)| \le C|y|^\varepsilon (1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The Littlewood-Paley multilinear commutator is defined by

$$g_\psi^{\tilde{b}}(f)(x) = \left(\int_0^\infty |F_t^{\tilde{b}}(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^{\tilde{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] \psi_t(x - y) f(y) dy$$

and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. Set $F_t(f) = \psi_t * f$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

which is the Littlewood-Paley g function (see [13]).

Let H be the space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$, then, for each fixed $x \in R^n$, $F_t^{\tilde{b}}(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi(f)(x) = ||F_t(f)(x)|| \text{ and } g_\psi^{\tilde{b}}(f)(x) = ||F_t^{\tilde{b}}(f)(x)||.$$

Definition 3. Let $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\Omega^{\tilde{b}}(f)(x) = \left(\int_0^\infty |F_t^{\tilde{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\tilde{b}}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] f(y)dy,$$

we denote that

$$F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz integral (see [14]).

Let H be the space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$. Then, it is clear that

$$\mu_\Omega(f)(x) = ||F_t(f)(x)|| \text{ and } \mu_\Omega^{\tilde{b}}(f)(x) = ||F_t^{\tilde{b}}(f)(x)||.$$

Definition 4. Let $B_t^\delta(f)\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$. Denote that

$$B_{\delta,t}^{\tilde{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] B_t^\delta(x-y)f(y)dy,$$

where $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz multilinear commutator is defined by

$$B_{\delta,*}^{\tilde{b}}(f)(x) = \sup_{t>0} |B_{\delta,t}^{\tilde{b}}(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|,$$

which is the Bochner-Riesz operator (see [6][7]).

Let H be the space $H = \left\{ h : ||h|| = \sup_{t>0} |h(t)| < \infty \right\}$, then it is clear that

$$B_*^\delta(f)(x) = ||B_t^\delta(f)(x)|| \text{ and } B_{\delta,*}^{\tilde{b}}(f)(x) = ||B_{\delta,t}^{\tilde{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $T_{\tilde{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-3][5][6][8-11]). Our main purpose is to establish the sharp inequalities for the multilinear commutator operators.

Now we state our main results as following.

Theorem 1. Let $\varepsilon > 0$, $r_j \geq 1$ and $b_j \in Osc_{expL^{r_j}}$ for $j = 1, \cdots, m$. Denote that $1/r = 1/r_1 + \cdots + 1/r_m$.

(**1**). Then for any $0 < p < q < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,

$$(g_\psi^{\tilde{b}}(f))_p^\#(\tilde{x}) \leq C \left(||b|| M_{L(logL)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_q(g_\psi^{\tilde{b}_{\sigma^c}}(f)(\tilde{x}) \right);$$

(**2**). If $1 < p < \infty$ and $w \in A_p$, then

$$||g_\psi^{\tilde{b}}(f)||_{L^p(w)} \leq C||\tilde{b}||||f||_{L^p(w)};$$

(**3**). If $w \in A_1$. Denote that $\Phi(t) = tlog^{1/r}(t+e)$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$w(\{x \in R^n : g_\psi^{\tilde{b}}(f)(x) > \lambda\}) \leq C \int_{R^n} \Phi\left(\frac{||\tilde{b}|||f(x)|}{\lambda}\right) w(x)dx.$$

Theorem 2. Let $0 < \gamma \leq 1$, $r_j \geq 1$ and $b_j \in Osc_{expL^{r_j}}$ for $j = 1, \cdots, m$. Denote that $1/r = 1/r_1 + \cdots + 1/r_m$.

(**1**). Then for any $0 < p < q < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,

$$(\mu_\Omega^{\tilde{b}}(f))_p^\#(\tilde{x}) \leq C \left(||b|| M_{L(logL)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_q(\mu_\Omega^{\tilde{b}_{\sigma^c}}(f)(\tilde{x}) \right);$$

(**2**). If $1 < p < \infty$ and $w \in A_p$, then

$$||\mu_\Omega^{\tilde{b}}(f)||_{L^p(w)} \leq C||\tilde{b}||||f||_{L^p(w)};$$

(**3**). If $w \in A_1$. Denote that $\Phi(t) = tlog^{1/r}(t+e)$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$w(\{x \in R^n : \mu_\Omega^{\tilde{b}}(f)(x) > \lambda\}) \leq C \int_{R^n} \Phi\left(\frac{||\tilde{b}|||f(x)|}{\lambda}\right) w(x)dx.$$

Theorem 3. Let $\delta > (n-1)/2$, $r_j \geq 1$ and $b_j \in Osc_{expL^{r_j}}$ for $j = 1, \cdots, m$. Denote that $1/r = 1/r_1 + \cdots + 1/r_m$.

(**1**). Then for any $0 < p < q < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,

$$(B^{\tilde{b}}_{\delta,*}(f))^{\#}_p(\tilde{x}) \leq C \left(||b|| M_{L(logL)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_q(B^{\tilde{b}_{\sigma^c}}_{\delta,*}(f)(\tilde{x}) \right);$$

(**2**). If $1 < p < \infty$ and $w \in A_p$, then

$$||B^{\tilde{b}}_{\delta,*}(f)||_{L^p(w)} \leq C||\tilde{b}|| ||f||_{L^p(w)};$$

(**3**). If $w \in A_1$. Denote that $\Phi(t) = tlog^{1/r}(t+e)$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,

$$w(\{x \in R^n : B^{\tilde{b}}_{\delta,*}(f)(x) > \lambda\}) \leq C \int_{R^n} \Phi\left(\frac{||\tilde{b}|| |f(x)|}{\lambda} \right) w(x)dx.$$

3 Proofs of Theorems

We first prove a general theorem.

Main Theorem. Let $r_j \geq 1$ and $b_j \in Osc_{expL^{r_j}}$ for $j = 1, \cdots, m$. Denote that $1/r = 1/r_1 + \cdots + 1/r_m$. Suppose that T is the same as in Definition 1 such that T is bounded on $L^s(w)$ for all $w \in A_s$ with $1 < s < \infty$ and weak bounded of $(L^1(w), L^1(w))$ for all $w \in A_1$. If T satisfies the following size condition:

$$||F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})f)(x) - F_t((b_1 - (b_1)_{2Q}) \cdots (b_m - (b_m)_{2Q})f)(x_0)|| \leq CM_{L(logL)^{1/r}}(f)(\tilde{x})$$

for any cube $Q = Q(x_0, d)$ with supp $f \subset (2Q)^c$ and $x, \tilde{x} \in Q = Q(x_0, d)$. Then for any $0 < p < q < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,

$$(T_{\tilde{b}}(f))^{\#}_p(\tilde{x}) \leq C \left(||b|| M_{L(logL)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_q(T_{\tilde{b}_{\sigma^c}}(f)(\tilde{x})) \right).$$

To prove the theorem, we need the following lemmas.

Lemma 1.(Kolmogorov,[4, p.485]) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p} / ||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 2.([11]) Let $r_j \geq 1$ for $j = 1, \cdots, m$, we denote that $1/r = 1/r_1 + \cdots + 1/r_m$. Then

$$\frac{1}{|Q|}\int_Q |f_1(x)\cdots f_m(x)g(x)|dx \leq ||f||_{expL^{r_1},Q}\cdots ||f||_{expL^{r_m},Q}||g||_{L(logL)^{1/r},Q}.$$

Proof of Main Theorem. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f)(x) - C_0|^p dx\right)^{1/p} \\
\leq\ & C\left(||b||M_{L(logL)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma\in C_j^m} M_q(T_{\tilde{b}_{\sigma^c}}(f))(\tilde{x})\right).
\end{aligned}$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We first consider the case $m = 1$. We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{R^n\setminus 2Q}$,

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})F_t(f)(x) - F_t((b_1 - (b_1)_{2Q})f_1)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x),$$

then

$$\begin{aligned}
&|T_{b_1}(f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
=\ & \left|||F_t^{b_1}(f)(x)|| - ||F_t(((b_1)_{2Q} - b_1)f_2)(x_0)||\right| \\
\leq\ & ||F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1)f_2)(x_0)|| \\
\leq\ & ||(b_1(y) - (b_1)_{2Q})F_t(f)(x)|| + ||F_t((b_1 - (b_1)_{2Q})f_1)(x)|| \\
& + ||F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)|| \\
=\ & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by Hölder's inequality for the exponent $1/l + 1/l' = 1$ with $1 < l < q/p$ and $q = pl$, we get

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q (A(x))^p dx\right)^{1/p} = \left(\frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}|^p |T(f)(x)|^p dx\right)^{1/p} \\
\leq\ & \left(\frac{C}{|2Q|}\int_{2Q} |b_1(x) - (b_1)_{2Q}|^{pl'}\right)^{1/pl'} \left(\frac{1}{|Q|}\int_Q |T(f)(x)|^{pl} dx\right)^{1/pl} \\
\leq\ & C||b_1||_{Osc_{expL^r}} M_{pl}(T(f))(\tilde{x}) \\
\leq\ & C||b_1||_{Osc_{expL^r}} M_q(T(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, by Lemma 1 and the weak type (1,1) of T, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q (B(x))^p dx\right)^{1/p} = \left(\frac{1}{|Q|}\int_Q |T((b_1-(b_1)_{2Q})f_1)(x)|^p dx\right)^{1/p} \\
\leq\ & C|2Q|^{-1}\frac{||T((b_1-(b_1)_{2Q})f_1)||_{L^p}}{|2Q|^{1/p-1}} \leq C|2Q|^{-1}||T((b_1-(b_1)_{2Q})f\chi_{2Q})||_{WL^1} \\
\leq\ & C|2Q|^{-1}\int_{2Q}|b_1(x)-(b_1)_{2Q}||f(x)|dx \leq C||b_1-(b_1)_{2Q}||_{\exp L^r,2Q}||f||_{L(logL)^{1/r},2Q} \\
\leq\ & C||b_1||_{Osc_{expL^r}} M_{L(logL)^{1/r}}(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, using the size condition of T, we have

$$\left(\frac{1}{|Q|}\int_Q (C(x))^p dx\right)^{1/p} \leq CM_{L(logL)^{1/r}}(f)(\tilde{x}).$$

Now, we consider the case $m \geq 2$. We write, for $b = (b_1, ..., b_m)$,

$$\begin{aligned}
&F_t^{\tilde{b}}(f)(x) = \int_{R^n}\left[\prod_{j=1}^m (b_j(x)-b_j(y))\right]F(x,y,t)f(y)dy \\
=\ & \int_{R^n}\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q}))F(x,y,t)f(y)dy \\
=\ & \sum_{j=0}^m \sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma \int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}F(x,y,t)f(y)dy \\
=\ & (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x) \\
& +(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma \int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}F(x,y,t)f(y)dy \\
=\ & (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x) \\
& +(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x),
\end{aligned}$$

thus

$$\begin{aligned}
& |T_{\tilde{b}}(f)(x) - T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)| \\
\leq\ & ||F_{\tilde{b}}(f)(x) - F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)|| \\
\leq\ & ||(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)|| \\
& + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x)|| \\
& + ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|| \\
& + ||F_t(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x) - F_t(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(x_0)|| \\
=\ & I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}$$

For $I_1(x)$ and $I_2(x)$, similar to the proof of the Case $m=1$, we get

$$\left(\frac{1}{|Q|}\int_Q (I_1(x))^p dx\right)^{1/p} \leq CM_q(T(f))(\tilde{x})$$

and

$$\left(\frac{1}{|Q|}\int_Q (I_2(x))^p dx\right)^{1/p} \leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} M_{L(logL)^{1/r}}(f)(\tilde{x}).$$

For I_3, by the weak type (1,1) of T and Lemma 2, we obtain

$$\begin{aligned}
& \left(\frac{1}{|Q|}\int_Q (I_3(x))^p dx\right)^{1/p} \\
\leq\ & \frac{C}{|2Q|}\int_{2Q} |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||f(x)|dx \\
\leq\ & C||b_1-(b_1)_{2Q}||_{\exp L^{r_1},2Q}\cdots||b_m-(b_m)_{2Q}||_{\exp L^{r_m},2Q}||f||_{L(logL)^{1/r},2Q} \\
\leq\ & C||b||M_{L(logL)^{1/r}}(f)(\tilde{x}).
\end{aligned}$$

For I_4, using the size condition of T, we have

$$\left(\frac{1}{|Q|}\int_Q (I_4(x))^p dx\right)^{1/p} \leq CM_{L(logL)^{1/r}}(f)(\tilde{x}).$$

This completes the proof of the main theorem.

To prove Theorem 1, 2 and 3, it suffices to verify that $g_\psi^{\tilde{b}}$, $\mu_\Omega^{\tilde{b}}$ and $B_{\delta,*}^{\tilde{b}}$ satisfy the size condition in Main Theorem.

Suppose supp$f\subset(2Q)^c$ and $x\in Q=Q(x_0,d)$. Note that $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$.

For $g_\psi^{\tilde{b}}$, by the condition of ψ, we obtain

$$
\begin{aligned}
&||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)\\
&-F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x_0)||\\
&\le \left[\int_0^\infty\left(\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}|\cdots|b_m(y)-(b_m)_{2Q}||f(y)||\psi_t(x-y)\right.\right.\\
&\left.\left.-\psi_t(x_0-y)|dy\right)^2\frac{dt}{t}\right]^{1/2}\\
&\le C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}|\cdots|b_m(y)-(b_m)_{2Q}||f(y)|\\
&\left(\int_0^\infty\frac{|x_0-x|^{2\varepsilon}tdt}{(t+|x_0-y|)^{2(n+1+\varepsilon)}}\right)^{1/2}dy\\
&\le C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}|\cdots|b_m(y)-(b_m)_{2Q}||f(y)|\frac{|x_0-x|^\varepsilon}{|x_0-y|^{n+\varepsilon}}dy\\
&\le C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^\varepsilon|x_0-y|^{-(n+\varepsilon)}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\\
&\le C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\\
&\le C\sum_{k=1}^\infty 2^{-k\varepsilon}\prod_{j=1}^m||b_j-(b_j)_{2Q}||_{expL^{r_j},2^{k+1}Q}||f||_{L(logL)^{1/r},2^{k+1}Q}\\
&\le C\sum_{k=1}^\infty k^m2^{-k\varepsilon}\prod_{j=1}^m||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x})\\
&\le C\prod_{j=1}^m||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x}).
\end{aligned}
$$

For $\mu_\Omega^{\tilde{b}}$, we write

$$
\begin{aligned}
& ||F_t(\prod_{j=1}^m (b_j - (b_j)_{2Q})f)(x) - F_t(\prod_{j=1}^m (b_j - (b_j)_{2Q})f)(x_0)|| \\
= \;& (\int_0^\infty | \int_{|x-y|\le t} \frac{\Omega(x-y)f(y)}{|x-y|^{n-1}} \left[\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})\right] dy \\
& - \int_{|x_0-y|\le t} \frac{\Omega(x_0-y)f(y)}{|x_0-y|^{n-1}} \left[\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})\right] dy|^2 \frac{dt}{t^3})^{1/2} \\
\le \;& \left(\int_0^\infty \left[\int_{|x-y|\le t, |x_0-y|>t} \frac{|\Omega(x-y)||f(y)|}{|x-y|^{n-1}} \left|\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})\right| dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
& + \left(\int_0^\infty \left[\int_{|x-y|>t, |x_0-y|\le t} \frac{|\Omega(x_0-y)||f(y)|}{|x_0-y|^{n-1}} \left|\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})\right| dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
& + \left(\int_0^\infty \left[\int_{|x-y|\le t, |x_0-y|\le t} \left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right| \left|\prod_{j=1}^m (b_j(y) - (b_j)_{2Q})\right|\right.\right. \\
& \left.\left. |f(y)| dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
\equiv \;& J_1 + J_2 + J_3,
\end{aligned}
$$

then

$$\begin{aligned}
J_1 &\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\left(\int_{|x-y|\leq t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy\\
&\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy\\
&\leq C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\\
&\leq C\sum_{k=1}^\infty 2^{-k/2}\prod_{j=1}^m||b_j-(b_j)_{2Q}||_{expL^{r_j},2^{k+1}Q}||f||_{L(logL)^{1/r},2^{k+1}Q}\\
&\leq C\sum_{k=1}^\infty k^m2^{-k/2}\prod_{j=1}^m||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x})\\
&\leq C\prod_{j=1}^m||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x}).
\end{aligned}$$

Similarly, we have $J_2\leq C\prod_{j=1}^m||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x})$.

We now estimate J_3. By the following inequality (see [14]):

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right|\leq C\left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right),$$

we gain

$$
\begin{aligned}
& J_3 \leq C\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)||x-x_0|}{|x_0-y|^n}\left(\int_{|x_0-y|\leq t, |x-y|\leq t}\frac{dt}{t^3}\right)^{1/2} dy \\
& +C\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)||x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\left(\int_{|x_0-y|\leq t, |x-y|\leq t}\frac{dt}{t^3}\right)^{1/2} dy \\
\leq\ & C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma}}\right)|f(y)|dy \\
\leq\ & C\sum_{k=1}^\infty (2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right||f(y)|dy \\
\leq\ & C\sum_{k=1}^\infty (2^{-k}+2^{-k\gamma})\prod_{j=1}^m ||b_j-(b_j)_{2Q}||_{expL^{r_j},2^{k+1}Q}||f||_{L(logL)^{1/r},2^{k+1}Q} \\
\leq\ & C\sum_{k=1}^\infty k^m(2^{-k}+2^{-k\gamma})\prod_{j=1}^m ||b_j||_{Osc_{expL^{r_j}}} M_{L(logL)^{1/r}}(f)(\tilde{x}) \\
\leq\ & C\prod_{j=1}^m ||b_j||_{Osc_{expL^{r_j}}} M_{L(logL)^{1/r}}(f)(\tilde{x}).
\end{aligned}
$$

For $B^{\tilde{b}}_{\delta,*}$, we consider the following two cases:

Case 1. $0<t\leq d$. In this case, notice that (see [7])

$$|B^\delta(z)|\leq c(1+|z|)^{-(\delta+(n+1)/2)},$$

we have, for $x \in Q$,

$$\begin{aligned}
&||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)\\
&-F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x_0)||\\
&= C\sup_{0<t\le d} t^{-n}\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||f(y)|\\
&(1+|x-y|/t)^{-(\delta+(n+1)/2)}dy\\
&\le C\sup_{0<t\le d}(t/d)^{\delta-(n-1)/2}\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\\
&\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\right)\\
&\le C\sum_{k=1}^{\infty}2^{k((n-1)/2-\delta)}\prod_{j=1}^{m}||b_j-(b_j)_{2Q}||_{expL^{r_j},2^{k+1}Q}||f||_{L(logL)^{1/r},2^{k+1}Q}\\
&\le C\sum_{k=1}^{\infty}k^m 2^{k((n-1)/2-\delta)}\prod_{j=1}^{m}||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x})\\
&\le C\prod_{j=1}^{m}||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x});
\end{aligned}$$

Case 2. $t > d$. In this case, we choose δ_0 such that $(n-1)/2 < \delta_0 < \min(\delta, (n+1)/2)$, notice that (see [7])

$$|(\partial/\partial z)B^{\delta}(z)| \le C(1+|z|)^{-(\delta+(n+1)/2)},$$

similar to the proof of Case 1, we obtain

$$
\begin{aligned}
&||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)\\
&-F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x_0)||\\
&\le C\sup_{t>d} t^{-n}\\
&\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right||f(y)||B^\delta((x-y)/t)-B^\delta((x_0-y)/t)|dy\\
&\le C\sup_{t>d} t^{-n-1}\\
&\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta+(n+1)/2)}dy\\
&\le C\sup_{t>d} t^{-n-1}\\
&\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right||f(y)||x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy\\
&\le C\sup_{t>d}(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta_0)}\\
&\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right||f(y)|dy\right)\\
&\le C\sum_{k=1}^\infty 2^{k((n-1)/2-\delta_0)}\prod_{j=1}^m ||b_j-(b_j)_{2Q}||_{expL^{r_j},2^{k+1}Q}||f||_{L(logL)^{1/r},2^{k+1}Q}\\
&\le C\sum_{k=1}^\infty k^m 2^{k((n-1)/2-\delta_0)}\prod_{j=1}^m ||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x})\\
&\le C\prod_{j=1}^m ||b_j||_{Osc_{expL^{r_j}}}M_{L(logL)^{1/r}}(f)(\tilde{x}).
\end{aligned}
$$

These yields the desired results.

By (1) and the boundedness of g_ψ, μ_Ω, $B_{\delta,*}$ and $M_{L(logL)^{1/r}}$, we may obtain the conclusions (2)(3) of Theorem 1, 2 and 3.

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