Faculty of Sciences and Mathematics, University of Niš, Serbia
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Three-step random iterative sequence with Errors for asymptotically quasi-nonexpansive in the intermediate sense random operators

G.S. Saluja

Abstract

In this paper, we give a necessary and sufficient condition for strong convergence of three-step random iterative sequence with errors to a common random fixed point for a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators and also prove some strong convergence theorems using condition $(\overline{C})$ and semi-compact condition to said iteration scheme and random operators. The results presented in this paper extend and improve the corresponding results of I. Beg and M. Abbas [J. Math. Anal. Appl. 315(1) (2006), 181-201], G.S. Saluja [The Math. Stud. 77(1-4) (2008), 161-176] and many others.

1 Introduction

Random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is needed for the study of various classes of random equations. The study of random fixed point theory was initiated by the Prague school of Probabilities in the 1950s [15, 16, 30]. Common random fixed point theorems are stochastic generalization of classical common fixed point theorems. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see, e.g.,[22]) and references mentioned therein. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [11] attracted the attention of several mathematicians and gave wings to the theory. Itoh [18] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become the full fledged research area and various

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ideas associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [5, 6, 8, 10, 17, 19, 25, 26, 31]).

Papageorgiou [20, 21], Beg [3, 4] studied common random fixed points and random coincidence points of a pair of compatible random operators and proved fixed point theorems for contractive random operators in Polish spaces. Recently, Beg and Shahzad [9], Choudhury [14] and Badshah and Sayyed [2] used different iteration processes to obtain random fixed points. More recently, Beg and Abbas [7] studied common random fixed points of two asymptotically nonexpansive random operators through strong as well as weak convergence of sequence of measurable functions in the setup of uniformly convex Banach spaces. Also they construct different random iterative algorithms for asymptotically quasi-nonexpansive random operators on an arbitrary Banach space and established their convergence to random fixed point of the operators.

Recently, Saluja [23] studied three-step random iterative process with errors and he proved a strong convergence theorem to converge to a random fixed point for asymptotically quasi-nonexpansive random operator in the framework of uniformly separable convex Banach space.

The purpose of this paper is to study three-step random iterative sequence with errors and to give a necessary and sufficient condition for strong convergence of this iteration process to a common random fixed point of a finite family of asymptotically quasi-nonexpansive in the intermediate sense random operators in separable Banach spaces and also prove some strong convergence theorems for said iteration scheme and operators in uniformly separable convex Banach spaces. The results presented in this paper extend and improve the corresponding results of Beg and Abbas [6], Saluja [23] and many other known results given in the literature.

2 Preliminaries

Let (Ω, Σ) be a measurable space (Σ-sigma algebra) and let C be a nonempty subset of a Banach space X. A mapping $\xi\colon \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$, for each open subset U of X. The mapping $T\colon \Omega \times C \to C$ is a random map if and only if for each fixed $x \in C$, the mapping $T(.,x)\colon \Omega \to C$ is measurable and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega, .)\colon C \to X$ is continuous. A measurable mapping $\xi\colon \Omega \to X$ is a random fixed point of a random map $T\colon \Omega \times C \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$, for each $\omega \in \Omega$. We denote the set of random fixed points of a random map T by $RF(T)$.

Let $B(x_0, r)$ denote the spherical ball centered at x_0 with radius r, defined as the set $\{x \in X : \|x - x_0\| \leq r\}$.

We denote the nth iterate $T(\omega, T(\omega, T(\omega, \ldots, T(\omega, x) \ldots,)))$ of T by $T^n(\omega, x)$. The letter I denotes the random mapping $I\colon \Omega \times C \to C$ defined by $I(\omega, x) = x$ and $T^0 = I$.

Let C be a closed and convex subset of a separable Banach space X and the sequence of functions $\{\xi_n\}$ is pointwise convergent, that is, $\xi_n(\omega) \to q := \xi(\omega)$. Then the closedness of C implies that ξ is a mapping from Ω to C. Since C is a subset of separable Banach space X, if T is a continuous random operator then, by [[1], Lemma 8.2.3], the mapping $\omega \to T(\omega, f(\omega))$ is a measurable function for any measurable function f from Ω to C. Thus $\{\xi_n\}$ is a sequence of measurable functions. Hence $\xi\colon \Omega \to C$, being the limit of the sequence of measurable functions, is also measurable [[6], Remark 2.3].

Let $T\colon \Omega \times C \to C$ be a random operator, where C is a nonempty convex subset of a separable Banach space X.

Definition 2.1. (1) Mapping T is said to be asymptotically nonexpansive random operator if there exists a sequence of measurable mapping $h_n\colon \Omega \to [1, \infty)$ with $\lim\limits_{n\to\infty} h_n(\omega) = 1$, for each $\omega \in \Omega$, such that for $x, y \in C$, we have

$$\|T^n(\omega, x) - T^n(\omega, y)\| \quad \leq \quad h_n(\omega)\, \|x - y\|\,, \text{ for each } \omega \in \Omega. \tag{2.1}$$

(2) T is said to be asymptotically quasi-nonexpansive random operator if for each $\omega \in \Omega$, $G(\omega) = \{x \in C : x = T(\omega, x)\} \neq \phi$ and there exists a sequence of measurable mapping $h_n\colon \Omega \to [1, \infty)$ with $\lim\limits_{n\to\infty} h_n(\omega) = 1$, for each $\omega \in \Omega$, such that for $x \in C$ and $y \in G(\omega)$, the following inequality holds:

$$\|T^n(\omega, x) - y\| \quad \leq \quad h_n(\omega)\, \|x - y\|\,, \text{ for each } \omega \in \Omega. \tag{2.2}$$

(3) T is said to be asymptotically quasi-nonexpansive in the intermediate sense random operator provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x\in C,\ y\in G(\omega)} \Big(\|T^n(\omega, x) - y\| - \|x - y\| \Big) \leq 0 \text{ for each } \omega \in \Omega \tag{2.3}$$

where $G(\omega) = \{x \in C : x = T(\omega, x)\} \neq \phi$.

Definition 2.2. The modified random Mann iteration scheme is a sequence of function $\{\xi_n\}$ defined by

$$\xi_{n+1}(\omega) \quad = \quad (1 - \alpha_n)\xi_n(\omega) + \alpha_n T^n(\omega, \xi_n(\omega)), \text{ for each } \omega \in \Omega, \tag{2.4}$$

where $0 \leq \alpha_n \leq 1$, $n = 1, 2, \ldots$ and $\xi_0\colon \Omega \to C$ is an arbitrary measurable mapping.

Since C is a convex set, it follows that for each n, ξ_n is a mapping from Ω to C.

Definition 2.3. The modified random Ishikawa iteration scheme is the sequences of function $\{\xi_n\}$ and $\{\eta_n\}$ defined by

$$\begin{aligned} \xi_{n+1}(\omega) &= (1-\alpha_n)\xi_n(\omega) + \alpha_n T^n(\omega, \eta_n(\omega)), \\ \eta_n(\omega) &= (1-\beta_n)\xi_n(\omega) + \beta_n T^n(\omega, \xi_n(\omega)), \text{ for each } \omega \in \Omega, \end{aligned} \tag{2.5}$$

where $0 \leq \alpha_n, \beta_n \leq 1$, $n = 1, 2, \ldots$ and $\xi_0 \colon \Omega \to C$ is an arbitrary measurable mapping. Also $\{\xi_n\}$ and $\{\eta_n\}$ are sequences of functions from Ω to C.

Very recently, Tian and Yang [29] introduced the following iteration scheme in convex metric space:

Definition 2.4. Let (E, d, W) be a convex metric space and $T_i \colon E \to E$ be a finite family of uniformly quasi-Lipschitzian mappings with $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nine sequences in $[0, 1]$ with

$$\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

For a given $x_0 \in E$, define a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n &= W(x_n, T_n^n x_n, w_n; d_n, e_n, f_n), \end{aligned} \tag{2.7}$$

where $T_n^n = T_{n(mod\ N)}^n$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are any given three sequences in E. Then $\{x_n\}$ is called the Noor-type iterative sequence with errors for a finite family of uniformly quasi-Lipschitzian mappings $\{T_i\}_{i=1}^N$. They gave a necessary and sufficient condition to approximate a common fixed point for a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces.

Motivated and inspired by Tian and Yang [29] and some others we propose the following random iterative sequence with errors:

Definition 2.5. Let $\{T_i : 1 \leq i \leq N\}$ be a family of asymptotically quasi-nonexpansive in the intermediate sense random operators from $\Omega \times C \to C$, where C is a closed, convex subset of a separable Banach space E. Let $F = \cap_{i=1}^N RF(T_i) \neq \emptyset$, where $RF(T_i)$ is the set of all random fixed points of a random operator T_i for each $i \in \{1, 2, \ldots, N\}$. Let $\xi_0 \colon \Omega \to C$ be any fixed measurable map, and $\{f_n(\omega)\}$, $\{f'_n(\omega)\}$, $\{f''_n(\omega)\}$ be bounded sequences of measurable functions from Ω to C. Define sequences of functions $\{\zeta_n(\omega)\}$, $\{\eta_n(\omega)\}$ and $\{\xi_n(\omega)\}$ as follows:

$$\begin{aligned}
\zeta_n(\omega) &= \alpha_n''\xi_n(\omega) + \beta_n''T_n^n(\omega, \xi_n(\omega)) + \gamma_n''f_n''(\omega), \\
\eta_n(\omega) &= \alpha_n'\xi_n(\omega) + \beta_n'T_n^n(\omega, \zeta_n(\omega)) + \gamma_n'f_n'(\omega), \\
\xi_{n+1}(\omega) &= \alpha_n\xi_n(\omega) + \beta_nT_n^n(\omega, \eta_n(\omega)) + \gamma_nf_n(\omega), \qquad (2.8)
\end{aligned}$$

for each $\omega \in \Omega$, $n = 0, 1, 2, \dots,$ where $T_n^n = T_{n(mod\ N)}^n$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, $\{\gamma_n'\}$, $\{\alpha_n''\}$, $\{\beta_n''\}$ and $\{\gamma_n''\}$ are nine sequences of real numbers in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$.

In the sequel we need the following lemmas to prove our main results:

Lemma 2.1.(see [28]) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three nonnegative sequences of real numbers satisfying the following conditions:

$$p_{n+1} \le (1 + q_n)p_n + r_n, \quad n \ge 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty. \qquad (2.9)$$

Then

(1) $\lim_{n\to\infty} p_n$ exists.

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Lemma 2.2. (Schu [24]) Let E be a uniformly convex Banach space and $0 < a \le t_n \le b < 1$ for all $n \ge 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying

$$\limsup_{n\to\infty} \|x_n\| \le r, \qquad \limsup_{n\to\infty} \|y_n\| \le r,$$

$$\lim_{n\to\infty} \|t_nx_n + (1 - t_n)y_n\| = r,$$

for some $r \ge 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

3 Main Results

Theorem 3.1. Let E be a real uniformly separable convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators from $\Omega \times C$ to C. Let $F = \cap_{i=1}^N RF(T_i) \neq \emptyset$. Put

$$
\begin{aligned}
G_n(\omega) \quad = \quad \max\Big\{ & \sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega,\xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \vee \\
& \sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega,\eta_n(\omega)) - \xi(\omega)\| - \|\eta_n(\omega) - \xi(\omega)\| \Big) \vee \\
& \sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega,\zeta_n(\omega)) - \xi(\omega)\| - \|\zeta_n(\omega) - \xi(\omega)\| \Big) \vee 0 \Big\}
\end{aligned} \tag{3.1}
$$

such that $\sum_{n=0}^{\infty} G_n(\omega) < \infty$. Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.8) with $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \gamma'_n < \infty$, $\sum_{n=0}^{\infty} \gamma''_n < \infty$ and $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$. Then

(a) $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ exists for all $\omega \in \Omega$.

(b) $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exists, where $d(\xi_n(\omega), F) = \inf_{\xi(\omega)\in F} \|\xi_n(\omega) - \xi(\omega)\|$.

(c) $\lim_{n\to\infty} \|\xi_n(\omega) - T_l(\omega,\xi_n(\omega))\| = 0$, for each $\omega \in \Omega$ and for all $l = 1, 2, \ldots, N$.

Proof. Let $\xi(\omega) \in F$, where ξ is any measurable mapping from Ω to C. Since $\{f_n(\omega)\}$, $\{f'_n(\omega)\}$ and $\{f''_n(\omega)\}$ are bounded sequences of measurable functions from Ω to C, so we can put

$$
\begin{aligned}
M(\omega) \quad = \quad \Big\{ & \sup_{\xi(\omega)\in F,\ n\geq 0} \|f_n(\omega) - \xi(\omega)\| \vee \sup_{\xi(\omega)\in F,\ n\geq 0} \|f'_n(\omega) - \xi(\omega)\| \vee \\
& \sup_{\xi(\omega)\in F,\ n\geq 0} \|f''_n(\omega) - \xi(\omega)\| \Big\}.
\end{aligned} \tag{3.2}
$$

Using (2.8), (3.1) and (3.2), we have

$$
\begin{aligned}
\|\zeta_n(\omega) - \xi(\omega)\| &= \|\alpha''_n \xi_n(\omega) + \beta''_n T_n^n(\omega,\xi_n(\omega)) + \gamma''_n f''_n(\omega) - \xi(\omega)\|, \\
&\leq \alpha''_n \|\xi_n(\omega) - \xi(\omega)\| + \beta''_n \|T_n^n(\omega,\xi_n(\omega)) - \xi(\omega)\| \\
&\quad + \gamma''_n \left\|f''_n(\omega) - \xi_(\omega)\right\| \\
&\leq \alpha''_n \|\xi_n(\omega) - \xi(\omega)\| + \beta''_n [\|\xi_n(\omega) - \xi(\omega)\| + G_n(\omega)] \\
&\quad + \gamma''_n \left\|f''_n(\omega) - \xi_(\omega)\right\| \\
&\leq (\alpha''_n + \beta''_n) \|\xi_n(\omega) - \xi(\omega)\| + \beta''_n G_n(\omega) \\
&\quad + \gamma''_n \left\|f''_n(\omega) - \xi_(\omega)\right\| \\
&= (1 - \gamma''_n) \|\xi_n(\omega) - \xi(\omega)\| + \beta''_n G_n(\omega) \\
&\quad + \gamma''_n \left\|f''_n(\omega) - \xi_(\omega)\right\| \\
&\leq \|\xi_n(\omega) - \xi(\omega)\| + G_n(\omega) + \gamma''_n M(\omega)
\end{aligned} \tag{3.3}
$$

and

$$\begin{aligned}
\|\eta_n(\omega)-\xi(\omega)\| &= \|\alpha'_n\xi_n(\omega)+\beta'_nT^n_n(\omega,\zeta_n(\omega))+\gamma'_nf'_n(\omega)-\xi(\omega)\|,\\
&\le \alpha'_n\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n\|T^n_n(\omega,\zeta_n(\omega))-\xi(\omega)\|\\
&\quad+\gamma'_n\left\|f'_n(\omega)-\xi_(\omega)\right\|\\
&\le \alpha'_n\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n[\|\zeta_n(\omega)-\xi(\omega)\|+G_n(\omega)]\\
&\quad+\gamma'_n\left\|f'_n(\omega)-\xi_(\omega)\right\|\\
&\le \alpha'_n\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n\|\zeta_n(\omega)-\xi(\omega)\|\\
&\quad+\beta'_nG_n(\omega)+\gamma'_n\left\|f'_n(\omega)-\xi_(\omega)\right\|\\
&\le \alpha'_n\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n\|\zeta_n(\omega)-\xi(\omega)\|\\
&\quad+G_n(\omega)+\gamma'_nM(\omega) \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
\|\xi_{n+1}(\omega)-\xi(\omega)\| &= \|\alpha_n\xi_n(\omega)+\beta_nT^n_n(\omega,\eta_n(\omega))+\gamma_nf_n(\omega)-\xi(\omega)\|,\\
&\le \alpha_n\|\xi_n(\omega)-\xi(\omega)\|+\beta_n\|T^n_n(\omega,\eta_n(\omega))-\xi(\omega)\|\\
&\quad+\gamma_n\left\|f_n(\omega)-\xi_(\omega)\right\|\\
&\le \alpha_n\|\xi_n(\omega)-\xi(\omega)\|+\beta_n[\|\eta_n(\omega)-\xi(\omega)\|+G_n(\omega)]\\
&\quad+\gamma_n\left\|f_n(\omega)-\xi_(\omega)\right\|\\
&\le \alpha_n\|\xi_n(\omega)-\xi(\omega)\|+\beta_n\|\eta_n(\omega)-\xi(\omega)\|\\
&\quad+\beta_nG_n(\omega)+\gamma_n\left\|f_n(\omega)-\xi_(\omega)\right\|\\
&\le \alpha_n\|\xi_n(\omega)-\xi(\omega)\|+\beta_n\|\zeta_n(\omega)-\xi(\omega)\|\\
&\quad+G_n(\omega)+\gamma_nM(\omega) \qquad (3.5)
\end{aligned}$$

substituting (3.3) into (3.4), we have

$$\begin{aligned}
\|\eta_n(\omega)-\xi(\omega)\| &\le \alpha'_n\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n\Big[\|\xi_n(\omega)-\xi(\omega)\|\\
&\quad+G_n(\omega)+\gamma''_nM(\omega)\Big]+G_n(\omega)+\gamma'_nM(\omega)\\
&\le (\alpha'_n+\beta'_n)\|\xi_n(\omega)-\xi(\omega)\|+(1+\beta'_n)G_n(\omega)\\
&\quad+\beta'_n\gamma''_nM(\omega)+\gamma'_nM(\omega)\\
&= (1-\gamma'_n)\|\xi_n(\omega)-\xi(\omega)\|+(1+\beta'_n)G_n(\omega)\\
&\quad+\beta'_n\gamma''_nM(\omega)+\gamma'_nM(\omega)\\
&\le \|\xi_n(\omega)-\xi(\omega)\|+2G_n(\omega)+\gamma''_nM(\omega)+\gamma'_nM(\omega)\\
&= \|\xi_n(\omega)-\xi(\omega)\|+2G_n(\omega)+(\gamma'_n+\gamma''_n)M(\omega) \qquad (3.6)
\end{aligned}$$

substituting (3.6) into (3.5), we have

$$\begin{aligned}
\|\xi_{n+1}(\omega)-\xi(\omega)\| &\leq \alpha_n\|\xi_n(\omega)-\xi(\omega)\|+\beta_n\Big[\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +2G_n(\omega)+(\gamma_n'+\gamma_n'')M(\omega)\Big]+G_n(\omega)+\gamma_n M(\omega) \\
&\leq (\alpha_n+\beta_n)\|\xi_n(\omega)-\xi(\omega)\|+(2\beta_n+1)G_n(\omega) \\
&\quad +\beta_n(\gamma_n'+\gamma_n'')M(\omega)+\gamma_n M(\omega) \\
&= (1-\gamma_n)\|\xi_n(\omega)-\xi(\omega)\|+(2\beta_n+1)G_n(\omega) \\
&\quad +\beta_n(\gamma_n'+\gamma_n'')M(\omega)+\gamma_n M(\omega) \\
&\leq \|\xi_n(\omega)-\xi(\omega)\|+3G_n(\omega)+\gamma_n M(\omega) \\
&\quad +(\gamma_n'+\gamma_n'')M(\omega) \\
&= \|\xi_n(\omega)-\xi(\omega)\|+3G_n(\omega)+(\gamma_n+\gamma_n'+\gamma_n'')M(\omega) \\
&= \|\xi_n(\omega)-\xi(\omega)\|+3G_n(\omega)+\theta_n M(\omega) \qquad (3.7)
\end{aligned}$$

where $\theta_n=\gamma_n+\gamma_n'+\gamma_n''$. Since by hypothesis $\sum_{n=0}^{\infty}\gamma_n<\infty$, $\sum_{n=0}^{\infty}\gamma_n'<\infty$ and $\sum_{n=0}^{\infty}\gamma_n''<\infty$, it follows that $\sum_{n=0}^{\infty}\theta_n<\infty$.

In (3.7) taking infimum over all $\xi(\omega)\in F$, for all $\omega\in\Omega$, we have

$$d(\xi_{n+1}(\omega),F) \leq d(\xi_n(\omega),F)+3G_n(\omega)+\theta_n M(\omega). \qquad (3.8)$$

Since $\sum_{n=0}^{\infty}G_n(\omega)<\infty$ and $\sum_{n=0}^{\infty}\theta_n<\infty$, it follows from Lemma 2.1 that

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\| \quad \text{and} \quad \lim_{n\to\infty}d(\xi_n(\omega),F) \quad \text{exist.}$$

Without loss of generality, we can assume that

$$\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|=d, \qquad (3.9)$$

where $d\geq 0$ is some number. Since $\{\|\xi_n(\omega)-\xi(\omega)\|\}$ is a convergent sequence and so $\{\xi_n(\omega)\}$ is a bounded sequence in C.

From (3.6), we have

$$\|\eta_n(\omega)-\xi(\omega)\| \leq \|\xi_n(\omega)-\xi(\omega)\|+2G_n(\omega)+(\gamma_n'+\gamma_n'')M(\omega).$$

Taking $\limsup_{n\to\infty}$ in both sides and using (3.9), we have

$$\begin{aligned}
\limsup_{n\to\infty}\|\eta_n(\omega)-\xi(\omega)\| &\leq \limsup_{n\to\infty}\Big[\|\xi_n(\omega)-\xi(\omega)\| \\
&\quad +2G_n(\omega)+(\gamma_n'+\gamma_n'')M(\omega)\Big] \\
&\leq d. \qquad (3.10)
\end{aligned}$$

Now, note that

$$\begin{aligned}\limsup_{n\to\infty}\|T_n^n(\omega,\eta_n(\omega))-\xi(\omega)\| &\leq \limsup_{n\to\infty}\Big[\,\|\eta_n(\omega)-\xi(\omega)\|+G_n(\omega)\Big]\\ &\leq d. \end{aligned} \tag{3.11}$$

Next consider

$$\begin{aligned}\|T_n^n(\omega,\eta_n(\omega))-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_n(\omega))\| &\leq \|T_n^n(\omega,\eta_n(\omega))-\xi(\omega)\|\\ &\quad+\gamma_n\|f_n(\omega)-\xi_n(\omega)\|\end{aligned} \tag{3.12}$$

Thus,

$$\limsup_{n\to\infty}\|T_n^n(\omega,\eta_n(\omega))-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_n(\omega))\| \leq d. \tag{3.13}$$

Also,

$$\begin{aligned}\|\xi_n(\omega)-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_n(\omega))\| &\leq \|\xi_n(\omega)-\xi(\omega)\|\\ &\quad+\gamma_n\|f_n(\omega)-\xi_n(\omega)\|\end{aligned} \tag{3.14}$$

gives that

$$\limsup_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_n(\omega))\| \leq d. \tag{3.15}$$

Moreover, we note that

$$\begin{aligned} d &= \lim_{n\to\infty}\|\xi_{n+1}(\omega)-\xi(\omega)\|\\ &= \lim_{n\to\infty}\|\alpha_n\xi_n(\omega)+\beta_nT_n^n(\omega,\eta_n(\omega))+\gamma_nf_n(\omega)-\xi(\omega)\|\\ &= \lim_{n\to\infty}\|\alpha_n\xi_n(\omega)+\beta_nT_n^n(\omega,\eta_n(\omega))+\gamma_nf_n(\omega)-(1-\beta_n)\xi(\omega)-\beta_n\xi(\omega)\|\\ &= \lim_{n\to\infty}\|\beta_n[T_n^n(\omega,\eta_n(\omega))-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_n(\omega))]\\ &\qquad+(1-\beta_n)[\xi_n(\omega)-\xi(\omega)+\gamma_n(f_n(\omega)-\xi_n(\omega))]\|. \end{aligned} \tag{3.16}$$

Therefore, from (3.13) - (3.16) and Lemma 2.2, we have

$$\lim_{n\to\infty} \|T_n^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| = 0, \tag{3.17}$$

for each $\omega \in \Omega$.

Now, for each $n \geq 0$, we have

$$\begin{aligned} \|\xi_n(\omega) - \xi(\omega)\| &\leq \|T_n^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| + \|T_n^n(\omega, \eta_n(\omega)) - \xi(\omega)\| \\ &\leq \|T_n^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| + \Big(\|\eta_n(\omega) - \xi(\omega)\| + G_n(\omega) \Big) \end{aligned}$$

since $G_n(\omega) \to 0$ as $n \to \infty$ and using (3.17), we obtain that

$$d = \lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| \leq \liminf_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\|. \tag{3.18}$$

It follows that

$$d \leq \liminf_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| \leq \limsup_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| \leq d.$$

This implies that

$$\lim_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| = d, \tag{3.19}$$

for each $\omega \in \Omega$.

On the other hand, form (3.3), we note that

$$\|\zeta_n(\omega) - \xi(\omega)\| \leq \|\xi_n(\omega) - \xi(\omega)\| + G_n(\omega) + \gamma_n'' M(\omega) \tag{3.20}$$

Taking $\limsup_{n\to\infty}$ in both sides and using (3.9), we have

$$\begin{aligned} \limsup_{n\to\infty} \|\zeta_n(\omega) - \xi(\omega)\| &\leq \limsup_{n\to\infty} \Big[\|\xi_n(\omega) - \xi(\omega)\| \\ &\quad + G_n(\omega) + \gamma_n'' M(\omega) \Big] \\ &\leq d, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\|T_n^n(\omega,\zeta_n(\omega))-\xi(\omega)\| &\leq \limsup_{n\to\infty}\Big[\|\zeta_n(\omega)-\xi(\omega)\|+G_n(\omega)\Big]\\ &\leq d.\end{aligned} \tag{3.22}$$

Next, we consider

$$\begin{aligned}\|T_n^n(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\| &\leq \|T_n^n(\omega,\zeta_n(\omega))-\xi(\omega)\|\\ &\quad+\gamma_n'\|f_n'(\omega)-\xi_n(\omega)\|\end{aligned} \tag{3.23}$$

Thus,

$$\limsup_{n\to\infty}\|T_n^n(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\| \leq d. \tag{3.24}$$

Also,

$$\begin{aligned}\|\xi_n(\omega)-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\| &\leq \|\xi_n(\omega)-\xi(\omega)\|\\ &\quad+\gamma_n'\|f_n'(\omega)-\xi_n(\omega)\|\end{aligned} \tag{3.25}$$

gives that

$$\limsup_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\| \leq d. \tag{3.26}$$

Since $\lim_{n\to\infty}\|\eta_n(\omega)-\xi(\omega)\|=d$, we obtain

$$\begin{aligned}d &= \lim_{n\to\infty}\|\eta_n(\omega)-\xi(\omega)\|\\ &= \lim_{n\to\infty}\|\alpha_n'\xi_n(\omega)+\beta_n'T_n^n(\omega,\zeta_n(\omega))+\gamma_n'f_n'(\omega)-\xi(\omega)\|\\ &= \lim_{n\to\infty}\|\alpha_n'\xi_n(\omega)+\beta_n'T_n^n(\omega,\zeta_n(\omega))+\gamma_n'f_n'(\omega)-(1-\beta_n')\xi(\omega)-\beta_n'\xi(\omega)\|\\ &= \lim_{n\to\infty}\|\beta_n'[T_n^n(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))]\\ &\qquad+(1-\beta_n')[\xi_n(\omega)-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))]\|.\end{aligned} \tag{3.27}$$

Therefore, from (3.24) - (3.27) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|T_n^n(\omega,\zeta_n(\omega))-\xi_n(\omega)\|=0, \tag{3.28}$$

for each $\omega \in \Omega$.

Similarly, by using the same arguments as in proof above, we have

$$\lim_{n\to\infty} \|T_n^n(\omega, \xi_n(\omega)) - \xi_n(\omega)\| = 0, \tag{3.29}$$

for each $\omega \in \Omega$.

Moreover, since

$$\|\xi_{n+1}(\omega) - \xi_n(\omega)\| \leq \beta_n \|T_n^n(\omega, \eta_n(\omega)) - \xi_n(\omega)\| + \gamma_n \|f_n(\omega) - \xi_n(\omega)\| \tag{3.30}$$

hence from (3.17) and using $\gamma_n \to 0$ as $n \to \infty$, we obtain

$$\|\xi_{n+1}(\omega) - \xi_n(\omega)\| = 0, \tag{3.31}$$

for each $\omega \in \Omega$, and so,

$$\|\xi_n(\omega) - \xi_{n+j}(\omega)\| = 0, \quad \forall j = 1, 2, \ldots, N. \tag{3.32}$$

On the other hand, we have

$$\begin{aligned}
\left\|\xi_n(\omega) - T_n^{n+1}(\omega, \xi_n(\omega))\right\| &\leq \|\xi_n(\omega) - \xi_{n+1}(\omega)\| + \left\|\xi_{n+1}(\omega) - T_n^{n+1}(\omega, \xi_{n+1}(\omega))\right\| \\
&\quad + \left\|T_n^{n+1}(\omega, \xi_{n+1}(\omega)) - T_n^{n+1}(\omega, \xi_n(\omega))\right\| \\
&\leq \|\xi_n(\omega) - \xi_{n+1}(\omega)\| + \left\|\xi_{n+1}(\omega) - T_n^{n+1}(\omega, \xi_{n+1}(\omega))\right\| \\
&\quad + L \|\xi_{n+1}(\omega) - \xi_n(\omega)\| \\
&= (1 + L) \|\xi_n(\omega) - \xi_{n+1}(\omega)\| \\
&\quad + \left\|\xi_{n+1}(\omega) - T_n^{n+1}(\omega, \xi_{n+1}(\omega))\right\|.
\end{aligned} \tag{3.33}$$

Using (3.29) and (3.31), we have

$$\left\|\xi_n(\omega) - T_n^{n+1}(\omega, \xi_n(\omega))\right\| \to 0, \text{ as } n \to \infty, \tag{3.34}$$

for each $\omega \in \Omega$.

Thus using (3.29) and (3.34), we obtain

$$\begin{aligned}
\|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| &\leq \left\|\xi_n(\omega) - T_n^{n+1}(\omega, \xi_n(\omega))\right\| + \left\|T_n^{n+1}(\omega, \xi_n(\omega)) - T_n(\omega, \xi_n(\omega))\right\| \\
&\leq \left\|\xi_n(\omega) - T_n^{n+1}(\omega, \xi_n(\omega))\right\| + L \left\|T_n^{n}(\omega, \xi_n(\omega)) - \xi_n(\omega)\right\| \\
&\to 0, \text{ as } n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_n(\omega, \xi_n(\omega))\| = 0, \tag{3.35}$$

for each $\omega \in \Omega$. Consequently, from (3.32) and (3.37), it follows that, for any $l = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\|\xi_n(\omega) - T_{n+l}(\omega, \xi_n(\omega))\| &\leq \|\xi_n(\omega) - \xi_{n+l}(\omega)\| + \|\xi_{n+l}(\omega) - T_{n+l}(\omega, \xi_{n+l}(\omega))\| \\
&\quad + \|T_{n+l}(\omega, \xi_{n+l}(\omega)) - T_{n+l}(\omega, \xi_n(\omega))\| \\
&\leq \|\xi_n(\omega) - \xi_{n+l}(\omega)\| + \|\xi_{n+l}(\omega) - T_{n+l}(\omega, \xi_{n+l}(\omega))\| \\
&\quad + L \|\xi_{n+l}(\omega) - \xi_n(\omega)\| \\
&\quad \to 0 \text{ as } n \to \infty,
\end{aligned}$$

for each $\omega \in \Omega$, which implies that

$$\bigcup_{j=1}^{N} \Big\{ \|\xi_n(\omega) - T_{n+j}(\omega, \xi_n(\omega))\| \Big\}_{n=1}^{\infty} \to 0, \text{ as } n \to \infty. \tag{3.36}$$

Since for each $l = 1, 2, \ldots, N$, $\Big\{ \|\xi_n(\omega) - T_l(\omega, \xi_n(\omega))\| \Big\}_{n=1}^{\infty}$ is a subsequence of $\bigcup_{j=1}^{N} \Big\{ \|\xi_n(\omega) - T_{n+j}(\omega, \xi_n(\omega))\| \Big\}_{n=1}^{\infty}$, we have

$$\lim_{n\to\infty} \|\xi_n(\omega) - T_l(\omega, \xi_n(\omega))\| = 0, \tag{3.37}$$

for each $\omega \in \Omega$ and for all $l = 1, 2, \ldots, N$. This completes the proof.

Theorem 3.2. Let E be a real uniformly separable convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators as in Theorem 3.1. Let $F = \cap_{i=1}^N RF(T_i) \neq \emptyset$. Put

$$\begin{aligned}
G_n(\omega) = \max\Big\{ &\sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \vee \\
&\sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega, \eta_n(\omega)) - \xi(\omega)\| - \|\eta_n(\omega) - \xi(\omega)\| \Big) \vee \\
&\sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega, \zeta_n(\omega)) - \xi(\omega)\| - \|\zeta_n(\omega) - \xi(\omega)\| \Big) \vee 0 \Big\}
\end{aligned}$$

such that $\sum_{n=0}^{\infty} G_n(\omega) < \infty$. Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.8) with $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \gamma'_n < \infty$, $\sum_{n=0}^{\infty} \gamma''_n < \infty$ and $\{\alpha_n\}$ be a sequence as in Theorem 3.1. Then the sequence $\{\xi_n(\omega)\}$ converges to a common random fixed point of the random operators $\{T_i : i = 1, 2, \ldots, N\}$ if and only if $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$.

Proof. If for some $\xi \in F$, $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = 0$ for each $\omega \in \Omega$, then obviously $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$, then we have

$$\lim_{n\to\infty} d(\xi_n(\omega), F) = 0, \quad \text{for each } \omega \in \Omega.$$

Thus for any $\varepsilon > 0$ there exists a positive integer N_1 such that for $n \geq N_1$,

$$d(\xi_n(\omega), F) < \frac{\varepsilon}{6}, \quad \text{for each } \omega \in \Omega. \tag{3.38}$$

Again since $\sum_{n=0}^{\infty} G_n(\omega) < \infty$ and $\sum_{n=0}^{\infty} \theta_n < \infty$ imply that there exist positive integers N_2 and N_3 such that

$$\sum_{j=n}^{\infty} G_j(\omega) < \frac{\varepsilon}{18}, \quad \forall n \geq N_2 \tag{3.39}$$

and

$$\sum_{j=n}^{\infty} \theta_j < \frac{\varepsilon}{6M(\omega)}, \quad \forall n \geq N_3 \tag{3.40}$$

Let $N = \max\{N_1, N_2, N_3\}$. It follows from (3.7), that

$$\|\xi_{n+1}(\omega) - \xi(\omega)\| \;\; \leq \;\; \|\xi_n(\omega) - \xi(\omega)\| + 3G_n(\omega) + \theta_n M(\omega). \tag{3.41}$$

Now, for each $m, n \geq N$ and each $\omega \in \Omega$, we have

$$\begin{aligned}
\|\xi_n(\omega)-\xi_m(\omega)\| &\leq \|\xi_n(\omega)-\xi(\omega)\|+\|\xi_m(\omega)-\xi(\omega)\| \\
&\leq \|\xi_N(\omega)-\xi(\omega)\|+3\sum_{j=N+1}^{n}G_j(\omega)+M(\omega)\sum_{j=N+1}^{n}\theta_j \\
&\quad +\|\xi_N(\omega)-\xi(\omega)\|+3\sum_{j=N+1}^{n}G_j(\omega)+M(\omega)\sum_{j=N+1}^{n}\theta_j \\
&= 2\|\xi_N(\omega)-\xi(\omega)\|+6\sum_{j=N+1}^{n}G_j(\omega)+2M(\omega)\sum_{j=N+1}^{n}\theta_j \\
&< 2.\frac{\varepsilon}{6}+6.\frac{\varepsilon}{18}+2M(\omega).\frac{\varepsilon}{6M(\omega)} \\
&< \varepsilon. \qquad (3.42)
\end{aligned}$$

This implies that $\{\xi_n(\omega)\}$ is a Cauchy sequence for each $\omega\in\Omega$. Therefore $\xi_n(\omega)\to p(\omega)$ for each $\omega\in\Omega$, and $p\colon\Omega\to C$, being the limit of the sequence of measurable function, is also measurable. Now, $\lim_{n\to\infty}d(\xi_n(\omega),F)=0$ for each $\omega\in\Omega$, and the set F is closed, we have $p(\omega)\in F$, that is, p is a common random fixed point of the random operators $\{T_i : i=1,2,\ldots,N\}$. This completes the proof.

Recall that the following:

A mapping $T\colon C\to C$ where C is a subset of a Banach space E with $F(T)\neq\emptyset$ is said to satisfy *condition* (A) [27] if there exists a nondecreasing function $f\colon[0,\infty)\to[0,\infty)$ with $f(0)=0$, $f(r)>0$ for all $r\in(0,\infty)$ such that for all $x\in C$

$$\|x-Tx\|\geq f(d(x,F(T))),$$

where $d(x,F(T))=\inf\{\|x-p\| : p\in F(T)\}$.

A family $\{T_i\}_{i=1}^{N}$ of N self-mappings of C with $\mathcal{F}=\cap_{i=1}^{N}F(T_i)\neq\emptyset$ is said to satisfy

(1) *condition* (B) on C ([13]) if there is a nondecreasing function $f\colon[0,1]\to[0,1]$ with $f(0)=0$, $f(r)>0$ for all $r\in(0,\infty)$ and all $x\in C$ such that

$$\max_{1\leq l\leq N}\{\|x-T_lx\|\}\geq f(d(x,\mathcal{F}));$$

(2) *condition* $(\overline{C})$ on C ([12]) if there is a nondecreasing function $f\colon[0,1]\to[0,1]$ with $f(0)=0$, $f(r)>0$ for all $r\in(0,\infty)$ and all $x\in C$ such that

$$\|x-T_lx\|\geq f(d(x,\mathcal{F}));$$

for at least one T_l, $l = 1, 2, \dots, N$; or in other words at least one of the $T_l's$ satisfies *condition*(A).

Condition (B) reduces to condition (A) when all but one of the $T_i's$ are identities. Also conditions (B) and $(\overline{C})$ are equivalent (see [12]).

A random operator $T\colon \Omega \times C \to C$ is said to satisfy condition (A), condition (B), condition $(\overline{C})$, if the map $T(\omega, .)\colon C \to C$ is so, for each $\omega \in \Omega$.

Let $T\colon \Omega \times C \to C$ be a random map. Then T is said to be

(i) completely continuous random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\{\xi_n(\omega)\}$ is bounded for each $\omega \in \Omega$ then $T(\omega, \xi_n(\omega))$ has convergent subsequence for each $\omega \in \Omega$.

(ii) demicompact random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\{\xi_n(\omega) - T(\omega, \xi_n(\omega))\}$ converges, there exists a subsequence say $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges strongly to some $\xi(\omega)$ for each $\omega \in \Omega$, where ξ is a measurable mapping from Ω to C.

(iii) semi-compact random operator if for a sequence of measurable mappings ξ_n from $\Omega \to C$ such that $\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| \to 0$, for every $\omega \in \Omega$, there exists a subsequence say $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges strongly to some $\xi(\omega)$ for each $\omega \in \Omega$, where ξ is a measurable mapping from Ω to C.

Senter and Dotson [27] established a relation between *condition* (A) and *demicompactness*. They actually showed that the *condition* (A) is weaker than demicompactness for a nonexpansive mapping.

Every compact operator is demicompact. Since every completely continuous mapping $T\colon C \to C$ is continuous and demicompact, so it satisfies *condition* (A).

Therefore to study strong convergence of $\{x_n\}$ defined by (2.8) we use *condition* $(\overline{C})$ instead of the complete continuity of the mappings $T_1, T_2, \dots, T_N$.

Theorem 3.3. Let E be a real uniformly separable convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators as in Theorem 3.1 and satisfying *condition* $(\overline{C})$. Let $F = \cap_{i=1}^N RF(T_i) \neq \emptyset$. Put

$$\begin{aligned} G_n(\omega) \;\; = \;\; \max\Big\{ & \sup_{\xi(\omega)\in F,\;\; n\geq 0} \Big(\|T_n^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \vee \\ & \sup_{\xi(\omega)\in F,\;\; n\geq 0} \Big(\|T_n^n(\omega, \eta_n(\omega)) - \xi(\omega)\| - \|\eta_n(\omega) - \xi(\omega)\| \Big) \vee \\ & \sup_{\xi(\omega)\in F,\;\; n\geq 0} \Big(\|T_n^n(\omega, \zeta_n(\omega)) - \xi(\omega)\| - \|\zeta_n(\omega) - \xi(\omega)\| \Big) \vee 0 \Big\} \end{aligned}$$

such that $\sum_{n=0}^{\infty} G_n(\omega) < \infty$. Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.8) with $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \gamma'_n < \infty$, $\sum_{n=0}^{\infty} \gamma''_n < \infty$ and $\{\alpha_n\}$ be a sequence as in Theorem 3.1. Then the sequence $\{\xi_n(\omega)\}$ converges to a common random fixed point of random operators $\{T_i : i = 1, 2, \ldots, N\}$.

Proof. By Theorem 3.1, we know that $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ and $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exist. Let one of $T'_i s$, say T_s, $s \in \{1, 2, \ldots, N\}$ satisfy condition (A), also by Theorem 3.1, we have $\lim_{n\to\infty} \|\xi_n(\omega) - T_s(\omega, \xi_n(\omega))\| = 0$, so we have $\lim_{n\to\infty} f(d(\xi_n(\omega), F)) = 0$, for each $\omega \in \Omega$. By the property of f and the fact that $\lim_{n\to\infty} d(\xi_n(\omega), F)$ exists, we have $\lim_{n\to\infty} d(\xi_n(\omega), F) = 0$ for each $\omega \in \Omega$. By Theorem 3.2, we obtain $\{\xi_n\}$ converges strongly to a common random fixed point in F. This completes the proof.

Theorem 3.4. Let E be a real uniformly separable convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly L-Lipschitzian asymptotically quasi-nonexpansive in the intermediate sense random operators as in Theorem 3.1 such that one of the mapping in $\{T_1, T_2, \ldots, T_N\}$ is semi-compact. Let $F = \cap_{i=1}^N RF(T_i) \neq \emptyset$. Put

$$\begin{aligned} G_n(\omega) \;=\; \max\Big\{ & \sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega, \xi_n(\omega)) - \xi(\omega)\| - \|\xi_n(\omega) - \xi(\omega)\| \Big) \vee \\ & \sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega, \eta_n(\omega)) - \xi(\omega)\| - \|\eta_n(\omega) - \xi(\omega)\| \Big) \vee \\ & \sup_{\xi(\omega)\in F,\ n\geq 0} \Big(\|T_n^n(\omega, \zeta_n(\omega)) - \xi(\omega)\| - \|\zeta_n(\omega) - \xi(\omega)\| \Big) \vee 0 \Big\} \end{aligned}$$

such that $\sum_{n=0}^{\infty} G_n(\omega) < \infty$. Let $\{\xi_n(\omega)\}$ be the sequence defined by (2.8) with $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\sum_{n=0}^{\infty} \gamma'_n < \infty$, $\sum_{n=0}^{\infty} \gamma''_n < \infty$ and $\{\alpha_n\}$ be a sequence as in Theorem 3.1. Then the sequence $\{\xi_n(\omega)\}$ converges to a common random fixed point of random operators $\{T_i : i = 1, 2, \ldots, N\}$.

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \ldots, N\}$. By Theorem 3.1, we have $\lim_{n\to\infty} \|\xi_n(\omega) - T_{i_0}(\omega, \xi_n(\omega))\| = 0$. So there exists a subsequence $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ such that $\lim_{n_j\to\infty} \xi_{n_j}(\omega) = \xi_0(\omega)$ for each $\omega \in \Omega$. Obviously ξ_0 is measurable mapping from $\Omega \to C$. Now again by Theorem 3.1 we have

$$\lim_{n_j\to\infty} \left\|\xi_{n_j}(\omega) - T_l(\omega, \xi_{n_j}(\omega))\right\| = 0, \tag{3.43}$$

for each $\omega \in \Omega$ and for all $l \in \{1, 2, \ldots, N\}$. So $\|\xi_0(\omega) - T_l(\omega, \xi_0(\omega))\| = 0$ for all $l \in \{1, 2, \ldots, N\}$, which implies that $\xi_0(\omega) \in F$, also $\liminf_{n\to\infty} d(\xi_n(\omega), F) = 0$. Hence, by Theorem 3.2, we obtain $\{\xi_n(\omega)\}$ converges strongly to a common random fixed point in F. This completes the proof.

Remark 3.1. Our results extend and improve the corresponding results of Beg and Abbas [6] to the case of more general class of asymptotically nonexpansive random operator and three-step random iterative sequence with errors for a finite family of random operators considered in this paper.

Remark 3.2. Our results also extend and improve the corresponding results of Saluja [23] to the case of more general class of asymptotically quasi-nonexpansive random operator and three-step random iterative sequence with errors for a finite family of random operators considered in this paper.

Remark 3.3. Theorem 3.2 is a stochastic version of Theorem 2.2 of Tian and Yang [29] for more general class of uniformly quasi-Lipschitzian mappings.

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Address

G. S. Saluja:
Department of Mathematics and Information Technology,
Govt. Nagarjuna P.G. College of Science,
Raipur - 492010 (C.G.), India.

E-mail: `saluja_1963@rediffmail.com, saluja1963@gmail.com`