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Functional Analysis, Approximation and Computation **3:2** (2011), 69–77

# Commutativity to within scalars on Banach space

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**Abstract**

We investigate the operator equation $AB = \lambda BA$ for normal operators on Banach space. In particular, if it holds non trivially on uniformly convex Banach spaces, then $|\lambda| = 1$.

## 1. Introduction

Operators $A$ and $B$ are [8] said to $\lambda$-commute if

1.1 $$AB = \lambda BA,$$

"non trivially" provided $AB \neq 0$. This [7],[9],[15] has been studied for bounded linear operators $A$, $B$ on a complex Hilbert space, and is relevant to quantum mechanical observables and their spectra. In particular, when the operators $A$ and $B$ are Hermitian or normal, the value of $\lambda$ is restricted: if $A$ and $B$ are normal, then necessarily $|\lambda| = 1$. This is shown [7],[15] using the Fuglede-Putnam theorem, and in [8] without.

In the present paper, our aim is to extend this to Banach spaces, with "Hermitian" and "normal" according to Bonsall and Duncan [5],[6].

Let $T \in B(\mathcal{X})$ be a bounded linear operator on a complex Banach space $\mathcal{X}$. Writing $\mathcal{X}^\dagger$ for the dual space of $\mathcal{X}$, the (spatial) numerical range $W(T)$ of $T$ is defined by

1.2 $$W(T) = W_{\mathcal{X}}(T) = \{\, f(Tx) \,:\, (x, f) \in \Pi(\mathcal{X})\},$$

where

1.3 $$\Pi(\mathcal{X}) = \{\, (x, f) \in \mathcal{X} \times \mathcal{X}^\dagger \,:\, f(x) = \|f\| = \|x\| = 1\}.$$

This research is partially supported by Grant-in-Aid Research No. 20540192.
This research is partially supported by Grant-in-Aid Research No. 20540178.
2010 *Mathematics Subject Classifications.* 47A10, 47B07.
*Key words and Phrases.* Banach space; hermitian operator; normal operator.
Received: November 26, 2011
Communicated by Dragan S. Djordjević

In general $W(T)$ is a subset of the "algebraic" numerical range $V(T) = V_{\mathcal{A}}(T)$ where $\mathcal{A} = B(\mathcal{X})$. An operator $T \in B(\mathcal{X})$ is said to be *Hermitian*, written $T \in \text{Re } B(\mathcal{X})$, provided it has real numerical range:

1.4 $$W(T) \subseteq \mathbf{R}.$$

We distinguish the "Palmer subspace" of $B(\mathcal{X})$,

1.5 $$\text{Reim } B(\mathcal{X}) = \{H + iK : H, K \in \text{Re } B(\mathcal{X})\},$$

and recall ([5] Lemma 5.7) that $H$ and $K$ are uniquely determined by $T = H + iK$. We are therefore in a position to define an *involution* on the Palmer subspace, writing

1.6 $$(H + iK)^* = H - iK \ (H, K \in \text{Re } B(\mathcal{X})).$$

Evidently $T \in \text{Reim } B(\mathcal{X})$ is Hermitian iff $T^* = T$. We shall also refer to $H$ and $K$ as the "real and imaginary parts" of $T = H + iK$. Now $T = H + iK$ can be called *normal* if its real and imaginary parts commute:

1.7 $$HK = KH \ ; \ \textit{equivalently } T^*T = TT^*.$$

We remark that, in general on a Banach space, products of commuting Hermitian operators need not [1] be Hermitian: however ([5] Lemma 5.4)

1.8 $$S, T \in \text{Re } B(\mathcal{X}) \Longrightarrow i(ST - TS) \in \text{Re } B(\mathcal{X}).$$

This means that the Palmer subspace is also [12] a Lie algebra. By the Hahn-Banach Theorem, it is clear that if $T \neq 0$, then $W(T) \neq \{0\}$, and indeed the spectrum is always ([5] Theorem 2.6) a subset of the closure of the numerical range. Conversely ([5] Theorem 5.14) if $T$ is normal, then the convex hull of the spectrum coincides with the closure of the numerical range:

1.9 $$\text{cvx } \sigma(T) = \text{cl } W(T),$$

and hence there is implication

1.10 $$\sigma(T) = \{0\} \Longrightarrow T = 0.$$

We also have ([13] Theorem 4.7) that if $T \in B(\mathcal{X})$ is normal then its spectrum and approximate point spectrum coincide:

1.11 $$T \textit{ bounded below } \Longrightarrow T \textit{ invertible}.$$

The "Fuglede-Putnam theorem" extends to Banach space normality: if $A, B \in B(\mathcal{X})$ are normal and $X \in B(\mathcal{X})$ is arbitrary, then there is implication

1.12 $$AX = XB \Longrightarrow A^*X = XB^*.$$

This is because if $T \in B(\mathcal{X})$ is normal then ([10] Lemma 3) $T^{*-1}(0) \subseteq T^{-1}(0)$, while if $a, b \in \mathcal{A}$ are normal in a Banach algebra $\mathcal{A}$ then so is $L_a - R_b \in B(\mathcal{A})$. In turn this guarantees that commuting sums of normal operators are normal.

## 2. Hermitians

The only way that Hermitian operators can non trivially $\lambda$-commute is that $\lambda$ is real:

**Theorem 1.** *Suppose $A, B \in B(\mathcal{X})$ $\lambda$ commute on the Banach space $\mathcal{X}$, in the sense* (1.1)*: then there is implication*

2.1 $$\sigma(AB) \neq \{0\} \Longrightarrow |\lambda| = 1.$$

*If either $A$ or $B$ is Hermitian, then*

2.2 $$AB \neq 0 \Longrightarrow \lambda \in \mathbf{R},$$

*and if both $A$ and $B$ are Hermitian then*

2.3 $$AB \neq 0 \Longrightarrow \lambda \in \{1, -1\},$$

*Proof.* If the product $AB$ is not a quasinilpotent, $\sigma(AB) \neq \{0\}$, then it has positive spectral radius $r(AB)$, so that

$$0 < r(BA) = r(AB) = |\lambda| r(BA),$$

giving $|\lambda| = 1$ . If for example $A$ is Hermitian then both $A$ and $\lambda A$ are normal, and hence by Fuglede-Putnam (1.12)

$$A^* B = \overline{\lambda} B A^*.$$

Since $A^* = A$, we have

$$\lambda BA = AB = \overline{\lambda} BA \neq 0,$$

giving $\lambda = \overline{\lambda}$. The argument is the same if $B$ is Hermitian. If in addition $\sigma(AB) \neq \{0\}$ then both (2.1) and (2.2) hold, forcing $\lambda = \pm 1$. Suppose finally $A$ and $B$ are both Hermitian, with $\sigma(AB) = \{0\}$; then, recalling (1.8) and (1.10),

2.4 $$i(\lambda - 1)BA = i(AB - BA) = 0,$$

since $i(AB - BA)$ is both Hermitian and quasinilpotent. *Box*

From (2.4) it also follows that $AB = BA$: thus if $A$ and $B$ are both Hermitian there is implication

2.5 $$AB = -BA \neq 0 \Longrightarrow \sigma(AB) \neq \{0\}.$$

### 3. Uniformly convex spaces

We would like to extend the essence of Theorem 1 to normal operators; we can succeed if the Banach space $\mathcal{X}$ is *uniformly convex* in the sense that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

3.1 $$\|x\| = \|y\| = 1 \textit{ and } \|x - y\| \geq \varepsilon \implies \|(x+y)/2\| \leq 1 - \delta.$$

We need ([11] Definition 1.9.2) a process of *enlargement*, $\mathcal{X} \mapsto \mathbf{Q}(\mathcal{X})$, in which $\mathcal{X}$ is isometrically embedded in a larger space

3.2 $$\mathbf{Q}(\mathcal{X}) = \ell_\infty(\mathcal{X})/c_0(\mathcal{X}).$$

A description in terms of "Banach limits" is given by de Barra [3] and Mattila [13]. If $T : \mathcal{X} \to \mathcal{Y}$ is bounded then there is induced in an obvious way $\mathbf{Q}(T) : \mathbf{Q}(\mathcal{X}) \to \mathbf{Q}(\mathcal{Y})$, and the mapping $T \mapsto \mathbf{Q}(T)$ is linear, multiplicative and isometric. The most important feature of the functor $\mathbf{Q}$ is ([11] Theorem 3.3.5) implication

3.3 $$\mathbf{Q}(T) \textit{ one one} \Longrightarrow T \textit{ bounded below} \Longrightarrow \mathbf{Q}(T) \textit{ bounded below.}$$

It is also true that the spectrum is preserved: when $\mathcal{Y} = \mathcal{X}$

3.4 $$\sigma\mathbf{Q}(T) = \sigma(T),$$

as is [2],[13] the closed convex hull of the numerical range:

3.5 $$W\mathbf{Q}(T) = \mathrm{cl}\ \mathrm{cvx}\ W(T).$$

It follows that

3.6 $$T \textit{ Hermitian, or normal} \implies \mathbf{Q}(T) \textit{ Hermitian, or normal.}$$

Combined with (3.3) and (1.11), this shows that if $T \in B(\mathcal{X})$ is normal then

3.7 $$\mathbf{Q}(T) \textit{ one one} \Longleftrightarrow T \textit{ invertible.}$$

Uniformly convexity in the sense (3.1) is ([3] Theorem 4) preserved under enlargement:

3.8 $$\mathcal{X} \textit{ uniformly convex} \implies \mathbf{Q}(\mathcal{X}) \textit{ uniformly convex.}$$

We shall describe an operator $T \in B(\mathcal{X})$ as *almost simply polar* if

3.9 $$\mathcal{X} = \overline{R}(T) \oplus N(T),$$

where $R(T)$ and $N(T)$ are the range and kernel of $T$, and $\overline{R}(T)$ the closure of the range. If $T \in B(\mathcal{X})$ is almost simply polar and also has closed range then it is "simply polar" or "group invertible". Mattila ([13] Theorem 4.4) has shown that if $\mathcal{X}$ is uniformly convex and if $T \in B(\mathcal{X})$ is normal then (3.9) holds.

**Theorem 2.** *If $A, B \in B(\mathcal{X})$ are both normal, and $\lambda$-commute, and if $A$ is almost simply polar on $\mathcal{X}$, then $\overline{R}(A)$ and $N(A)$ are invariant under $B$ and $B^*$, and the restrictions of $B$ to the null space and the closure of the range of $A$ are also normal.*
*Proof.* We may write $P = P^2 \in B(\mathcal{X})$ for the projection for which

3.10 $$P(\mathcal{X}) = \overline{R}(A)\ ,\ P^{-1}(0) = N(A).$$

Obviously $P$ commutes with $A$, and hence, if $A$ and $B$ $\lambda$-commute, also with $B$. By Fuglede-Putnam (1.12) it follows that $P$ commutes with $B^*$. Hence $\overline{R}(A)$ is invariant for $B$ and $B^*$. $\triangle$

We reach our main result:
**Theorem 3.** *Suppose that either $\mathcal{X}$ or its dual $\mathcal{X}^\dagger$ is uniformly convex: then if $A, B \in B(\mathcal{X})$ are both normal, and $\lambda$-commute, there is implication*

3.11 $$AB \neq 0 \Longrightarrow |\lambda| = 1.$$

*Proof.* If $\mathcal{X}$ is uniformly convex we consider two cases: either $A \in B(\mathcal{X})$ is invertible, or not. If $A$ is invertible then we can argue $B = \lambda A^{-1}BA$ and hence, remembering (1.10),

$$0 < r(B) = |\lambda| r(A^{-1}BA) = |\lambda| r(B).$$

If $A$ is not invertible then $0 \in \mathbf{C}$ is in its spectrum, which coincides with its approximate point spectrum, and also $A$ is almost simply polar in the sense of (3.9). The same is true of the enlargement $\mathbf{Q}(A)$, which is also normal, and by uniform convexity almost simply polar: but now $0 \in \mathbf{C}$ is (3.7) actually an eigenvalue of $\mathbf{Q}(A)$. Restricted to $\overline{R}\mathbf{Q}(A)$, both $\mathbf{Q}(A)$ and $\mathbf{Q}(B)$ are normal and $\lambda$-commute. Since in addition the restriction of $\mathbf{Q}(A)$ is one one it is (3.7) invertible. We are therefore back in the first case, giving $|\lambda| = 1$.

If instead the dual space $\mathcal{X}^\dagger$ is uniformly convex and if $A, B \in B(\mathcal{X})$ are normal and $\lambda$-commute then the same is true of $A^\dagger, B^\dagger \in B(\mathcal{X}^\dagger)$ $\triangle$

## 4. A converse

We conclude with a sort of converse, valid for arbitrary Banach spaces $\mathcal{X}$. Begin with the remark that if $A = H + iK$ and $B = E + iF$ can be expressed as linear combinations of Hermitian operators then

4.1 $$\{AB, B^*A^*\} \subseteq \text{Reim } B(\mathcal{X}) \Longleftrightarrow \{AB + B^*A^*, i(AB - B^*A^*)\} \subseteq \text{Reim } B(\mathcal{X}),$$

and

4.2 $$\{AB, B^*A^*\} \subseteq \text{Re } B(\mathcal{X}) \Longleftrightarrow \{AB + B^*A^*, i(AB - B^*A^*)\} \subseteq \text{Re } B(\mathcal{X}),$$

If in addition $AB$ and $B^*A^*$ commute,

4.3 $$(AB)(B^*A^*) = (B^*A^*)(AB),$$

then also

4.4 $$AB, B^*A^* \text{ normal} \iff AB + B^*A^*, i(AB - B^*A^*) \text{ normal}.$$

If $(A, B) = (H + iK, E + iF)$ with Hermitian $H, K, E, F$ then these conditions can be expressed in terms of real and imaginary parts: $AB + B^*A^*$ and $AB - B^*A^*$ will be in Reim $B(\mathcal{X})$, or Hermitian, or normal, iff the same is true of all four operators $(HK + KH) - (EF + FE), i(HF - FH) + i(KE - EK), (HF + FH) + (KE + EK), -i(HE - EH) + i(KF - FK)$. Now by (1.8) two of these are automatically Hermitian: thus

4.5 $$\begin{gathered}(H + iK)(E + iF), (E - iF)(H - iK) \text{ Hermitian} \iff \\ (HK + KH) - (EF + FE), (HF + FH) + (KE + EK) \text{ Hermitian},\end{gathered}$$

and, in the presence of (4.3),

4.6 $$\begin{gathered}(H + iK)(E + iF), (E - iF)(H - iK) \text{ normal} \iff \\ (HK + KH) - (EF + FE), (HF + FH) + (KE + EK) \text{ normal}.\end{gathered}$$

**Theorem 4.** *If $A, B$ are normal, and non trivially $\lambda$-commute, and if $\lambda \neq 1$, then the following are equivalent*:

4.7 $$AB \text{ is normal} \; ;$$

4.8 $$\sigma(AB) \neq \{0\} \; ;$$

4.9 $$|\lambda| = 1.$$

*Proof.* With no restriction on $\lambda$, implication (4.7)$\Longrightarrow$(4.8)$\Longrightarrow$(4.9) is (1.10) and (2.1) respectively; we prove that if $\lambda \neq 1$ then (4.9)$\Longrightarrow$(4.7). By normality and Fuglede-Putnam

$$A^*B = \overline{\lambda}BA^* \ , \ AB^* = \overline{\lambda}B^*A \text{ and } A^*B^* = \lambda B^*A^*,$$

and hence

$$ABB^*A^* = AB^*BA^* = \overline{\lambda}B^*AA^*B/\overline{\lambda} = B^*AA^*B = B^*A^*AB.$$

Thus we have commutativity (4.3), and it will be sufficient, for (4.6), to show that $(HK + KH) - (EF + FE)$ and $(HF + FH) + (KE + EK)$ are Hermitian. Since $AB = \lambda BA$,

$$HE - KF + i(HF + KE) = \lambda\big(EH - FK + i(EK + FH)\big),$$

and since $A^*B^* = \lambda B^*A^*$

$$HE - KF - i(HF + KE) = \lambda\big(EH - FK - i(EK + FH)\big).$$

Adding and subtracting,

$$HE - KF = \lambda(EH - FK)\ ,\ HF + KE = \lambda(EK + FH).$$

It follows

$$i(\lambda - 1)(EH - FK) = i(HE - EH) - i(KF - FK) \in \mathrm{Re}\ B(\mathcal{X})$$

and

$$i(\lambda - 1)(EK + FH) = i(HF - FH) - i(KE - EK) \in \mathrm{Re}\ B(\mathcal{X})$$

are both Hermitian. But now

4.10 $$|\lambda| = 1 \neq \lambda \Longrightarrow (\lambda + 1)/i(\lambda - 1) \in \mathbf{R}\ ,$$

and hence indeed

$$AB + B^*A^* = (HE + EH) - (KF + FK) = (\lambda + 1)(EH - FK) \in \mathrm{Re}\ B(\mathcal{X})$$

and

$$i(AB - B^*A^*) = (HF + FH) + (KE + EK) = (\lambda + 1)(EK + FH) \in \mathrm{Re}\ B(\mathcal{X})$$

are both Hermitian $\triangle$

The example of Anderson and Foias ([5] Example 5.8), together with a theorem of Palmer, shows that implication (3.12)$\Longrightarrow$(3.10) will fail with $\lambda = 1$. Indeed if $0 \neq P = P^2 \neq I \in B(\mathcal{Y})$ for a Hilbert space $\mathcal{Y}$ then ([5] Example 5.9)

4.11 $A = B = L_P - R_P$ *is Hermitian in* $B(\mathcal{X}) = B(B(\mathcal{Y}))$ *but* $AB$ *is not.*

While we have been unable to settle whether or not the product $L_P R_P$, and the square $(L_P - R_P)^2$, are normal, or even in the space Reim $B(\mathcal{X})$, the implication (ii)$\Longrightarrow$(iii) of Theorem 6.3 of [5] guarantees the existence of Hermitian $C$ for which $C^2$ is not in the Palmer space.

**Acknowledgment.** The authors would like to express their thanks to Prof. T. Yamazaki for useful discussion of Theorem 4.

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