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B–Browder operators and perturbations

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Abstract. Perturbation of a Banach space operator by a commuting Riesz or non-nilpotent quasinilpotent (Banach space) operator does not preserve the upper semi B-Browder (or lower semi B-Browder, or even B-Browder) spectrum of the operator. We give a sufficient condition for invariance under perturbation by commuting nilpotent operators. Our sufficient condition implies that if either our Banach space is a Hilbert space or the conjugate operator has SVEP, then perturbations of the operator by commuting nilpotent operators preserves upper semi B-Browder spectrum. Also, if our Banach space operator T is *finitely left polaroid* then perturbation by commuting quasinilpotent operators preserves upper semi B-Browder spectrum.

1. Introduction

For an operator $T \in B(\mathcal{X})$, the algebra of bounded linear transformations on an infinite dimensional complex Banach space into itself, the ascent asc(T) (resp., the descent dsc(T)) of T is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ (resp., $T^n\mathcal{X} = T^{n+1}\mathcal{X}$). Let $\phi_+(\mathcal{X}) = \{T \in B(\mathcal{X}) : T\mathcal{X}$ is closed and $\alpha(T) = \dim(\mathrm{T}^{-1}(0)) < \infty\}$, $\phi_-(\mathcal{X}) = \{T \in B(\mathcal{X}) : T\mathcal{X}$ is closed and $\beta(T) = \dim(\mathcal{X} \setminus \mathrm{T}\mathcal{X}) < \infty\}$ and $\phi(\mathcal{X}) = \phi_+(\mathcal{X}) \cap \phi_-(\mathcal{X})$, denote respectively the upper semi-Fredholm, the lower semi-Fredholm and the Fredhom elements of $B(\mathcal{X})$. Let $\sigma(T)$ denote the spectrum of T, $\sigma_a(T)$ denote the approximate point spectrum of T and $T - \lambda = T - \lambda I$ (I the identity map of $B(\mathcal{X})$ and $\lambda \in \mathbf{C}$ some complex number). Let $\sigma_{ub}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin \phi_+(\mathcal{X})$ or asc$(T - \lambda \nless \infty\}$, $\sigma_{lb}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin \phi_-(\mathcal{X})$ or dsc$(T - \lambda \nless \infty\}$ and $\sigma_b(T) = \sigma_{ub}(T) \cup \sigma_{lb}(T)$ denote respectively the upper semi-Browder, the lower semi-Browder and the Browder spectrum of T. A well known result of Rakočević [10] says that $\sigma_{ub}(T)$, $\sigma_{lb}(T)$ and $\sigma_b(T)$ are invariant under perturbation by commuting Riesz operators.

A generalization of the concept of semi-Browder spectrum is obtained as follows. An operator $T \in B(\mathcal{X})$ is semi B-Fredholm, $T \in \phi_{sbf}(\mathcal{X})$, if there is a non-negative integer n such that $T^n\mathcal{X}$ is closed and the induced operator $T_{[n]} = T|_{T^n\mathcal{X}}$, $T_{[0]} = T$, is semi-Fredholm (in the usual sense). $T_{[m]}$ is then semi-Fredholm for all $m \geq n$, and one then defines the index of T, ind$(T) = \alpha(T) - \beta(T)$, by ind$(T) =$ ind$(T_{[n]})$. See [4, 9] for more detail. The upper semi B-Fredholm, the lower semi B-Fredholm and the B-Fredhom spectrum of T are the

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sets $\sigma_{sbf_+}(T) = \{\lambda \in \sigma(T) : T - \lambda$ is not upper semi B-Fredholm$\}$, $\sigma_{sbf_-}(T) = \{\lambda \in \sigma(T) : T - \lambda$ is not lower semi B-Fredholm$\}$ and $\sigma_{bf}(T) = \sigma_{sbf_+}(T) \cup \sigma_{sbf_-}(T)$, respectively. Let $\sigma_{ubb}(T) = \{\lambda \in \sigma(T) : \lambda \in \sigma_{sbf_+}(T)$ or $\mathrm{asc}(T - \lambda) \not< \infty\}$, $\sigma_{lbb}(T) = \{\lambda \in \sigma(T) : \lambda \in \sigma_{sbf_-}(T)$ or $\mathrm{dsc}(T - \lambda) \not< \infty\}$ and $\sigma_{bb}(T) = \sigma_{ubb}(T) \cup \sigma_{lbb}(T)$, denote respectively, the upper semi B-Browder, the lower semi B-Browder and the B-Browder spectrum of T.

If $T \in B(\mathcal{X})$ is the 0 operator and $Q \in B(\mathcal{X})$ is a non-nilpotent quasinilpotent operator, then

$$\sigma(T) = \{0\} = \sigma(T + Q), \sigma_{ubb}(T) = \sigma_{lbb}(T) = \emptyset, \text{and}$$
$$\sigma_{ubb}(T) = \sigma_{lbb}(T) \neq \{0\} = \sigma_{ubb}(T + Q) = \sigma_{lbb}(T + Q).$$

Thus the invaraince under perturbation by commuting quasinilpotent (more generally, Riesz) operators does not extend from $\sigma_{ub}(T)$ ($\sigma_{lb}(T)$) to $\sigma_{ubb}(T)$ (resp., $\sigma_{lbb}(T)$) for operators $T \in B(\mathcal{X})$. *Does it extend to operators commuting with nilpotents?* Let iso $\sigma_a(T)$ denote the isolated points of $\sigma_a(T)$. We prove below that the answer to this question is in the affirmative if the operator T and its perturbation $T + N$ by a commuting nilpotent N satisfy the property, henceforth referred to as *property* $(*)$, below. Let $X = T$ or $T + N$.

"For every $\lambda \in \mathrm{iso}\,\sigma_a(X)$ such that $\mathrm{asc}(X - \lambda) = d(\lambda) < \infty$, the subspace $(X - \lambda)^{-d(\lambda)}(0) + (X - \lambda)\mathcal{X}$ if closed is complemented in $\mathcal{X}$."

As a consequence, it is proved that if $\mathcal{X} = \mathcal{H}$ is a Hilbert space then σ_{ubb}, σ_{lbb} and σ_{bb} are invariant under perturbation by commuting nilpotents, and if T^* (resp., T) has SVEP (the single-valued extension property) then $\sigma_{ubb}(T)$ (resp., σ_{lbb}) is invariant under perturbation by commuting nilpotents.

An operator $T \in B(\mathcal{X})$ is left polaroid if it is left polar at every $\lambda \in \mathrm{iso}\,\sigma_a(T)$, equivalently if for every $\lambda \in \mathrm{iso}\,\sigma_a(T)$ there exists an integer $d(\lambda) \geq 0$ such that $\mathrm{asc}(T - \lambda) = d(\lambda) < \infty$ and $(T - \lambda)^{d(\lambda)+1}\mathcal{X}$ is closed; T is finitely left polaroid if T is left polaroid and $\alpha(T - \lambda) < \infty$ at every $\lambda \in \mathrm{iso}\,\sigma_a(T)$. We prove in the following that if a $T \in B(\mathcal{X})$ is finitely left polaroid and $\sigma_a(T + R) = \sigma_a(T)$ for some commuting Riesz operator $R \in B(\mathcal{X})$, then $\sigma_{ubb}(T + R) = \sigma_{ubb}(T)$.

2. Results

$T \in B(\mathcal{X})$ has SVEP, *the single-valued extension property*, at $\lambda_0 \in \mathbf{C}$ if, for every open neighbourhood $\mathcal{U}$ of λ_0, the only analytic solution $f : \mathcal{U} \to \mathcal{X}$ of $(T - \lambda)f(\lambda) = 0$ is the function $f \equiv 0$; T has SVEP on a subset $F \subseteq \mathbf{C}$ if T has SVEP at every $\lambda_0 \in F$. Recall, [1, 7], that $\mathrm{asc}(T - \lambda) = d < \infty \iff (T - \lambda)^d\mathcal{X} \cap T^{-n}(0) = \{0\}$ for every positive integer $n \implies T$ has SVEP at λ; furthermore, if $\mathrm{asc}(T - \lambda) = d < \infty$ and $(T - \lambda)^{d+1}\mathcal{X}$ is closed, then the subspace $T^{-d}(0) + T\mathcal{X}$ is closed, $\lambda \in \mathrm{iso}\,\sigma_a(T)$ and $\lambda \notin \sigma_{ubb}(T)$ [5, Lemma 3.1]. Thus, if we let $\sigma_{ld}(T) = \{\lambda \in \sigma_a(T) : T$ is not left polar at $\lambda\}$ denote the left Drazin spectrum of T, then $\sigma_{ubb}(T) \subseteq \sigma_{ld}(T)$. Conversely, $\lambda \notin \sigma_{ubb}(T) \iff T - \lambda$ is upper semi B-Fredholm and $\mathrm{asc}(T - \lambda) = d < \infty$ for some integer $d \geq 0 \implies \lambda \notin \sigma_{ld}(T)$. Thus $\sigma_{ubb}(T) = \sigma_{ld}(T)$. We say that T is *right polar* at $\lambda \in \mathbf{C}$ if there exists an integer $d(\lambda) \geq 0$ such that $\mathrm{dsc}(T-\lambda) = d(\lambda) < \infty$ and $(T-\lambda)^{d(\lambda)}\mathcal{X}$ is closed. (Observe that T is left polar at λ if and only if T^* is right polar at λ.) Let $\sigma_s(T)$ denote the surjectivity spectrum of T, and let $\sigma_{rd}(T) = \{\lambda \in \sigma_s(T) : T$ is not right polar at $T\}$ denote the right Drazin spectrum of T. A duality argument then shows that $\sigma_{lbb}(T) = \sigma_{rd}(T)$. Let $\sigma_d(T)$ denote the Drazin spectrum of T. Then:

Lemma 2.1. *$\sigma_{ubb}(T) = \sigma_{ld}(T)$, $\sigma_{lbb}(T) = \sigma_{rd}(T)$ and $\sigma_{bb}(T) = \sigma_d(T)$.*

Let $H_0(T) = \{x \in \mathcal{X} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$ denote the quasinilpotent part of $T \in B(\mathcal{X})$. $H_0(T)$ is generally a non-closed T-invariant subspace of $\mathcal{X}$ such that $T^{-n}(0) \subseteq H_0(T)$ for all natural numbers n. The following lemma is Theorem 2.3 of [2].

Lemma 2.2. *If $T \in B(\mathcal{X})$ is left polar at λ, then there exists a non-negative integer d such that $H_0(T-\lambda) = (T-\lambda)^{-d}(0)$.*

The following lemma is a straightforward consequence of the fact that T (resp., T^*) has SVEP at $\lambda \in \sigma_s(T) \setminus \sigma_{lbb}(T)$ (resp., $\lambda \in \sigma_a(T) \setminus \sigma_{ubb}(T)$) if and only if $\mathrm{asc}(T - \lambda) < \infty$ (resp., $\mathrm{dsc}(T - \lambda) < \infty$); see [5, Lemma 3.4] and the statement that follows.

Lemma 2.3. *If T (resp., T^*) has SVEP on $\sigma_s(T) \setminus \sigma_{lbb}(T)$ (resp., $\sigma_a(T) \setminus \sigma_{ubb}(T)$), then $\sigma_{lbb}(T) = \sigma_{bb}(T)$ (resp., $\sigma_{ubb}(T) = \sigma_{bb}(T)$).*

An operator $T \in B(\mathcal{X})$ is *semi-regular* if $T\mathcal{X}$ is closed and

$$\begin{aligned} k_n(T) &= \dim((\mathrm{T}^n\mathcal{X} \cap \mathrm{T}^{-1}(0)) \setminus (\mathrm{T}^{n+1}\mathcal{X} \cap \mathrm{T}^{-1}(0))) \\ &= \dim((\mathrm{T}\mathcal{X} + \mathrm{T}^{-(n+1)}(0)) \setminus (\mathrm{T}\mathcal{X} + \mathrm{T}^{-n}(0))) = 0 \end{aligned}$$

for all non-negative integers n. Observe that T is semi-regular if and only if T^* is semi-regular. Furthermore, T semi-regular implies T^n semi-regular for all integers $n \geq 1$, and conversely if T^n is semi-regular for some integer $n \geq 1$ then T is semi-regular (see [1, 7, 8]). The operator T is *quasi-Fredholm of degreee d* for some integer $d \geq 0$ if $k_n(T) = 0$ for all $n \geq d$ and the subspaces $T^{-d}(0) + T\mathcal{X}$ and $T^{-1}(0) \cap T^d\mathcal{X}$ are closed [2, 9]. T quasi-Fredholm of degree d implies T^* quasi-Fredholm of the same degree [9, Lemma 4], and if T is left polar at λ, then $T - \lambda$ is quasi-Fredholm (of degree $d(\lambda) = \mathrm{asc}(T - \lambda)$). The following lemma appears in [9, Theorem 5].

Lemma 2.4. *If $T \in B(\mathcal{X})$ is a quasi-Fredholm operator of some degree d, and the subspaces $T^{-d}(0) + T\mathcal{X}$ and $T^{-1}(0) \cap T^d\mathcal{X}$ are complemented in $\mathcal{X}$, then there exist closed T-invariant subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $T|_{\mathcal{X}_1}$ is d-nilpotent and $T|_{\mathcal{X}_2}$ is semi-regular.*

Let $\delta_{A,B} \in B(B(\mathcal{X}))$ denote the generalized derivation $\delta_{A,B}(X) = AX - XB$. The operators $A, B \in B(\mathcal{X})$ are said to be *quasinilpotent equivalent* if $\lim_{n\to\infty} \|\delta^n_{A,B}(I)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\delta^n_{B,A}(I)\|^{\frac{1}{n}} = 0$ [7]. Quasinilpotent equivalence does not preserve finite ascent (even for commuting A and B): consider $A = 0$ and B a non-nilpotent quasinilpotent operator. However, if $A, N \in B(\mathcal{X})$ commute and N is a nilpotent, then $A, B = A + N$ are quasinilpotent equivalent and A has finite ascent implies B has finite ascent.

Lemma 2.5. *Let $A, N \in B(\mathcal{X})$ be commuting operators such that N is m-nilpotent and $asc(A - \lambda) = p < \infty$. Then $asc(A + N - \lambda) \leq m + p - 1$.*

Proof. Assume without loss of generality that $\lambda = 0$ and that $p \geq m$. Let $x \in \mathcal{X}$. We prove that $x \in (A + N)^{-(m+p)}(0)$ if and only if $x \in (A + N)^{-(m+p-1)}(0)$. Evidently, if $\mathrm{asc}(A) = p$ and $N^m = 0$, then:

$$\begin{aligned} (A+N)^{m+p} &= \sum_{t=0}^{m+p} \binom{m+p}{t} A^{m+p-t}N^t = \sum_{t=0}^{m-1} \binom{m+p}{t} A^{m+p-t}N^t \\ &\Longrightarrow \sum_{t=0}^{m-1} \binom{m+p}{t} A^{m+p-t}N^{t+m-1}x = 0 \Longrightarrow A^{m+p}N^{m-1}x = 0 \Longrightarrow A^pN^{m-1}x = 0 \\ &\Longrightarrow \sum_{t=0}^{m-1} \binom{m+p}{t} A^{m+p-t}N^{t+m-2}x = 0 \Longrightarrow A^{m+p}N^{m-2}x = 0 \Longrightarrow A^pN^{m-2}x = 0 \\ &\cdots \\ &\Longrightarrow \sum_{t=0}^{m-1} \binom{m+p}{t} A^{m+p-t}N^{t+1}x = 0 \Longrightarrow A^{m+p}Nx = 0 \Longrightarrow A^pNx = 0 \\ &\Longrightarrow \sum_{t=0}^{m-1} \binom{m+p}{t} A^{m+p-t}N^{t}x = 0 \Longrightarrow A^{m+p}x = 0 \Longrightarrow A^px = 0. \end{aligned}$$

But then

$$(A+N)^{m+p-1}x = \sum_{t=0}^{m+p-1} \binom{m+p-1}{t} A^{m+p-1-t}N^tx = 0.$$

Hence $(A + N)^{-(m+p)}(0) \subseteq (A + N)^{-(m+p-1)}(0)$. The reverse inclusion being evident, the proof is complete. □

Observe that $\delta^m_{A+N-\lambda,A-\lambda}(I) = \delta^m_{A-\lambda,A+N-\lambda}(I) = 0$, all $\lambda \in \mathbf{C}$, for the operators A and N of Lemma 2.5; in particular, $\sigma_x(A) = \sigma_x(A+N)$, where $\sigma_x = \sigma$ or σ_a.

We prove now our main result.

Theorem 2.6. *Let $T, N \in B(\mathcal{X})$ be commuting operators such that N is nilpotent. Then:*
(a) A sufficient condition for $\sigma_x(T+N) = \sigma_x(T)$, where $\sigma_x = \sigma_{ubb}$ or σ_{lbb}, is that T and $T+N$ satisfy property $(*)$.

(b) $\sigma_{bb}(T+N) = \sigma_{bb}(T)$.

Proof. (a) Let $\lambda \notin \sigma_{ubb}(T)$. Then $\lambda \in \text{iso}\,\sigma_a(T)$, which since $\sigma_a(T) = \sigma_a(T+N)$ implies $\lambda \in \text{iso}\,\sigma_a(T+N)$. Furthermore, $\text{asc}(T-\lambda) = d < \infty$ ($\Longrightarrow \text{asc}(T+N-\lambda) < \infty$, see Lemma 2.5), there exists an integer $p \geq 0$ such that $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ (by Lemma 2.2), and the subspaces $(T-\lambda)^{-1}(0) \cap (T-\lambda)^d\mathcal{X}$ and $(T-\lambda)^{-d}(0) + (T-\lambda)\mathcal{X}$ are closed. Hence, since T satisfies property $(*)$, it follows from Lemma 2.4 that there exist T-invariant closed subspaces E_1 and E_2 of $\mathcal{X}$ such that $\mathcal{X} = E_1 \oplus E_2$, $T_1 - \lambda = (T-\lambda)|_{E_1}$ is nilpotent and $T_2 - \lambda = (T-\lambda)|_{E_2}$ is semi-regular (indeed, bounded below since $\text{asc}(T-\lambda) = d \Longrightarrow (T-\lambda)^{-n}(0) \cap (T-\lambda)^d\mathcal{X} = \{0\}$ for every natural number n). The commutativity of T and N implies $N = N|_{E_1} \oplus N|_{E_2} = N_1 \oplus N_2$, where N_i commutes with T_i $(i = 1, 2)$, $T_1 + N_1 - \lambda = (T+N-\lambda)|_{E_1}$ is nilpotent and $T_2 + N_2 - \lambda = (T+N-\lambda)|_{E_2}$ is bounded below. But then $\lambda \notin \sigma_{ubb}(T+N)$; hence $\sigma_{ubb}(T+N) \subseteq \sigma_{ubb}(T)$. Since $T+N$ also satisfies property $(*)$, the reverse inclusion is a consequence of $\sigma_{ubb}(T) = \sigma_{ubb}((T+N)-N)$.

Now let $\lambda \notin \sigma_{lbb}(T)$. Then $\lambda \notin \sigma_{ubb}(T^*)$ $(= \sigma_{ld}(T^*))$. Thus $T^* - \lambda I^*$ is quasi-Fredholm $\Longleftrightarrow T - \lambda$ is quasi-Fredholm, and this in view of property $(*)$ implies that the closed subspaces $T^{*-1}(0) \cap T^{*d}\mathcal{X}$ $(= (T\mathcal{X} + T^{-d}(0))^{\perp})$ and $T^*\mathcal{X} + T^{*-d}(0)$ are complemented in $\mathcal{X}^*$. Consequently, see above, $\lambda \notin \sigma_{ubb}(T^* + N^*) = \sigma_{lbb}(T+N) \Longrightarrow \sigma_{lbb}(T+N) \subseteq \sigma_{lbb}(T)$. The reverse inclusion being evident, the proof is complete.

(b) The proof in this case is a straightforward consequence of the argument of part (a), for in this case $\text{asc}(X-\lambda) = \text{dsc}(X-\lambda) < \infty$ for all $\lambda \notin \sigma_{bb}(X)$, $X = T$ or $T+N$, implies both T and $T+N$ satisfy property $(*)$. Indeed, there exist closed T-invariant subspaces E_1 and E_2 of $\mathcal{X}$ such that $\mathcal{X} = E_1 \oplus E_2$, $(X-\lambda)|_{E_1}$ is nilpotent and $(X-\lambda)|_{E_2}$ is Fredholm at every $\lambda \notin \sigma_{bb}(X)$ [9, Theorem 7]. □

Remark 2.7. It is not clear if Theorem 2.6(a) holds without the hypothesis that $T+N$ satisfies property $(*)$. Observe, however, that if T is left polaroid, then $\lambda \notin \sigma_{ubb}(T+N) \Longrightarrow \lambda \in \text{iso}\,\sigma_a(T) \Longrightarrow \lambda \notin \sigma_{ubb}(T)$. Hence, if $T, N \in B(\mathcal{X})$ are commuting operators such that N is nilpotent and T is a left polaroid operator which satisfies property $(*)$, then $\sigma_{ubb}(T) = \sigma_{ubb}(T+N)$.

It is quite straightforward to see that if $T \in B(\mathcal{X})$ (resp., $T^* \in B(\mathcal{X}^*)$) has SVEP at a point λ, and $N \in B(\mathcal{X})$ is a nilpotent which commutes with T, then $T+N$ (resp., $T^* + N^*$) has SVEP at λ. The following corollary, [3, Theorem 4.3], is immediate from this observation, Lemma 2.3 and Theorem 2.6(b).

Corollary 2.8. *Let $T, N \in B(\mathcal{X})$ be commuting operators such that N is nilpotent. If T (resp., T^*) has SVEP on $\sigma_s(T) \setminus \sigma_{lbb}(T)$ (resp., $\sigma_a(T) \setminus \sigma_{ubb}(T)$), then $\sigma_{lbb}(T) = \sigma_{lbb}(T+N)$ (resp., $\sigma_{ubb}(T) = \sigma_{ubb}(T+N)$).*

A more satisfactory result is obtained in the case in which $\mathcal{X} = \mathcal{H}$ is a Hilbert space, for every closed subspace of a Hilbert space is complemented. We have [3, Theorem 4.4]:

Corollary 2.9. *Let $T, N \in B(\mathcal{H})$ be commuting operators such that N is nilpotent. Then $\sigma_x(T+N) = \sigma_x(T)$, where $\sigma_x = \sigma_{ubb}$ or σ_{lbb}.*

As seen above, the invariance under perturbation by commuting non-nilpotent quasinilpotent operators does not extend from σ_{ub} (etc.) to σ_{ubb} (etc.). However, if an operator $T \in B(\mathcal{X})$ is finitely left polaroid, then we have the following.

Theorem 2.10. *Let $T, Q \in B(\mathcal{X})$ be commuting operators such that T is finitely left polaroid and Q is quasinilpotent. Then $\sigma_{ubb}(T) = \sigma_{ub}(T) = \sigma_{ub}(T+Q) = \sigma_{ubb}(T+Q)$.*

Proof. Evidently, $\lim_{n\to\infty} \|\delta^n_{T,T+Q}(I)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\delta^n_{T+Q,T}(I)\|^{\frac{1}{n}} = 0$, which implies that T and $T+Q$ are quasinilpotent equivalent, hence have the same approximate point spectrum. Let $\lambda \notin \sigma_{ubb}(T) = \sigma_{ld}(T)$. Then T is left polar at λ (implies $\mathrm{asc}(T-\lambda) = d < \infty$ and $(T-\lambda)^{d+1}\mathcal{X}$ is closed for some non-negative integer d) and $\alpha(T-\lambda) < \infty$. Apply [6, Theorem 3.8] to conclude $(T-\lambda)\mathcal{X}$ is closed, $\mathrm{asc}(A-\lambda) < \infty$ and $\alpha(T-\lambda) < \infty$. ([6, Theorem 3.8] implies in particular that for an operator $T \in B(\mathcal{X})$, if $\alpha(T) < \infty$ and $\mathrm{asc}(T) < \infty$, then $T^n\mathcal{X}$ is closed for an integer $n > 1$ if and only if $T\mathcal{X}$ is closed.) Hence $\lambda \notin \sigma_{ub}(T)$. Since $\sigma_{ubb}(T) \subseteq \sigma_{ub}(T)$, and since σ_{ub} is invariant under perturbation by commuting quasinilpotents, we conclude $\sigma_{ubb}(T) = \sigma_{ub}(T) = \sigma_{ub}(T+Q) \supseteq \sigma_{ubb}(T+Q)$. Now let $\lambda \notin \sigma_{ubb}(T+Q) = \sigma_{ld}(T+Q)$. Then $\lambda \in \mathrm{iso}\,\sigma_a(T+Q) = \mathrm{iso}\,\sigma_a(T)$, and this, since T is finitely left polaroid, implies $\lambda \notin \sigma_{ub}(T)$ (see above). Hence $\sigma_{ub}(T) \subseteq \sigma_{ubb}(T+Q)$, which completes the proof. □

A duality argument proves the following corollary.

Corollary 2.11. *Let $T, Q \in B(\mathcal{X})$ be commuting operators such that T is finitely right polaroid and Q is quasinilpotent. Then $\sigma_{lbb}(T) = \sigma_{lb}(T) = \sigma_{lb}(T+Q) = \sigma_{lbb}(T+Q)$.*

Perturbation by commuting Riesz operators does not preserve the approximate point spectrum. However, if $\sigma_a(T) = \sigma_a(T+R)$ for a Riesz operator $R \in B(\mathcal{X})$ which commutes with a finitely left polaroid operator $T \in B(\mathcal{X})$, then (the argument of the proof of Theorem 2.10 implies that) $\sigma_{ubb}(T) = \sigma_{ub}(T) = \sigma_{ub}(T+R) = \sigma_{ubb}(T+R)$.

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