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An existence theorem of monotonic solutions for a nonlinear functional integral equation of convolution type

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Abstract. In this paper, we shall prove an existence theorem of monotonic solutions for a nonlinear functional integral equation of convolution type. We used Darbo fixed point theorem associated with the Hausdorff measure of noncompactness.

1. Introduction

The theory of integral equations plays an important part in the theory of nonlinear functional integral equations of convolution type arise very often in applications of integral equaions in many branches of Mathematical physics such as neutron transportation, radiation and gases kinetic theorey $(c.f.$ [5, 8] , [12]). The equations of such kind have been investigated in several papers [2] , [9], where the equation have solutions in some function spaces. Also Banaś and knap [5] discussed the solvability of the considered equations in the space of Lebesgue integrable functions by using the technique of measures of weak noncompactness and the fixed point theorem due to Emmanuel [8].

In spite of this approuch gives more general result under less restrictive assumptions than those in [2] , [9] but the weak continuity conditions for an operator is not easy to be satisfied in general.

In this paper, we try to overcome this difficulty by using Darbo fixed point theorem associated with Hausdorff measure of noncompactness which is a strong measure.

2. Notation and auxiliary facts

Throughout this paper we denote by R the field of real numbers and by $R_+$the interval $[0, \infty)$, suppose that I is an arbitrary measurable subset of R not necessary bounded . Let $L^1(I)$ denote the space of Lebesgue integrable functions on the set I with the standerd norm

$$\|x\|_{L^1(I)} = \int_I |x(t)|\,dt.$$

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The space $L^1(R_+)$ will be shortly denoted by L^1 and its norm by $\|.\|$, i.e.

$$\|x\| = \int_0^\infty |x(t)|\,dt.$$

One of the must important operators studied in nonlinear functional analysis is the so-called superposition operator [1, 14].

Assuume that a function $f(t,x) = f : I \times R \to R$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$(Fx)(t) = f(t, x(t)),\ t \in I.$$

The operator F defined in such a way is called superposition operator generated by the function f.

We have the following theorem due to Appell and Zabrejko [1].

Theorem 1. The superposition operator F maps continuously the space $L^1(I)$ into itself if and only if

$$\left|f(t,x)\right| \le a(t) + b|x|,$$

for all $t \in I$ and $x \in R$, where $a(t)$ is a function from $L^1(I)$ and b is a nonnegative constant.

Next, we will mention a desired theorem concerning the compactness in measure of a subset X of $L^1(I)$ ($cf.$ |7|).

Theorem 2. Let X be a bounded subset of $L^1(I)$ consisting of functions which are a.e. nondecreasing (or nonincreasing) on the interval I. Then X is compact in measure.

Furthermore, we recall a few fact about the convolution operator ($cf.$ |11|).

Let $k \in L^1(R)$ be a given function. Then for any function $x \in L^1$, the integral

$$(Kx)(t) = \int_0^\infty k(t-s)\,x(s)\,ds$$

exists for almost every $t \in R_+$. Moreover, the function $(Kx)(t)$ belongs to the space L^1. Thus K is a linear operator which maps the space L^1 into itself and K is also bounded since

$$\|Kx\| \le \|K\|_{L^1(R)} \|x\|,$$

for every $x \in L^1$; so it will be continuous.

Hence the norm $\|K\|$ of the convolution operator is majorized by

$$\|K\|_{L^1(R)}.$$

In the sequel, we have the following theorem due to Krzyz [10].

Theorem 3. Assume that $k(t,s) = k : R_+^2 \longrightarrow R_+$ is measurable on R_+^2 and such that the integral operator

$$(Kx)(t) = \int_0^\infty k(t,s)\,x(s)\,ds, \quad t \ge 0,$$

maps L^1 into itself. Then K transforms the set of nonincreasing functions from L^1 into itself if and only if for any $A > 0$ the following implication is true

$$t_1 \prec t_2 \implies \int_0^A k(t_1, s)\,ds \ge \int_0^A k(t_2, s)\,ds.$$

3. Measures of noncompactness

We give a short note on measures of noncompactness and fixed point theorem. Let E be an arbitrary Banach space and let X be a nonempty and bounded subset of E. Denoted by B_r the closed ball in E centered at θ and radius r.

The Hausdorff measure of noncompactness $\chi(X)$ [4] is defined as :

$$\chi(X) = \inf\{r > 0 \text{ there exists a finite subset } Y \text{ of } E \text{ such that } X \sqsubset Y + B_r\}.$$

Another measure we defined in the space L^1 [3]. For any $\varepsilon > 0$, let

$$c(X) = \lim_{c \longrightarrow 0}\left\{\sup_{x\in X}\left\{\sup\left[\int_D |x(t)|dt : D \sqsubset R_+,\ meas.\ D \leq \varepsilon\right]\right\}\right\}$$

and

$$d(X) = \lim_{a \longrightarrow \infty}\left\{\sup\left[\int_a^{\infty} |x(t)|dt : x \in X\right]\right\},$$

where meas.D denotes the Lebesgue measure of a subset D.

Put

$$\gamma(X) = c(X) + d(X).$$

Then we have the follwing theorem which connects between the two measures $\chi(X)$ and $\gamma(X)$ [3].

Theorem 4. Let X be a nonempty, bounded and compact in measure subset of L^1, then

$$\chi(X) \leq \gamma(X) \leq 2\chi(X).$$

As an application of measures of noncompactness, we recall the fixed point theorem due to Darbo [6] .

Theorem 5. Let Q be a nonempty , bounded closed and convex subset of E and let $H : Q \longrightarrow Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ , i.e. there exists $k \in [0, 1]$ such that

$$\mu(HX) \leq k\mu(X),$$

for every nonempty subset X of Q. Then H has at least one fixed point in the set Q.

4. Main results

This section is devoted to the study of the following nonlinear integral equation of convolution type

$$x(t) = f_1\left(t, \int_0^{\infty} k(t-s)\, f_2\Big(s, x\big(\phi(s)\big)\Big)\, ds\right), \quad t \geq 0 \ (1).$$

For further purposes the operator

$$(Hx)(t) = f_1\left(t, \int_0^{\infty} k(t-s)\, f_2\Big(s, x\big(\phi(s)\big)\Big)\, ds\right)$$

will be often written as the product

$$H x = F K f_2\Big(s, x\,\big(\phi(s)\big)\Big)$$

of the convolution operator

$$(K x)(t) = \int_0^\infty k(t-s)\,x(s)\,ds$$

and the superposition operator

$$(F x)(t) = f\,(t,\,x\,(t)).$$

Thus equation (1) becomes

$$x = H x = F K f_2\Big(s, x\,\big(\phi\big)\Big). \quad (2)$$

We shall treat the equation (1) under the following assumptions which are listed below.

(i) The function $f_i : R_+ \times R \longrightarrow R$,$i = 1,2$ satisfies Carathéodory conditions and there are tow functions $a_i \in L^1$, $i = 1,2$ and tow nonnegative constants b_i, $i = 1,2$ such that

$$\big|f_i\,(t,x)\big| \le a_i(t) + b_i|x|, \quad i = 1,2,$$

for all $t \in R_+$ and $x \in R$. Moreover, $f_i\,(t,x) \ge 0$, $i = 1,2$ for $x \ge 0$ and f_i is assumed to be nonincreasing in the first variable and nondecreasing in the second one;

(ii) The function $k : R \longrightarrow R_+$ belongs to the space $L^1(R)$ and for any $A > 0$ and for all $t_1, t_2 \in R_+$, the following condition is satisfied

$$t_1 < t_2 \implies \int_0^A k(t_1 - s)\,ds \ge \int_0^A k(t_2 - s)\,ds\,;$$

(iii) The function $\phi : R_+ \longrightarrow R_+$ is increasing absolutely continuous and there is a constant $M > 0$ with the property $\acute{\phi}. \ge M$ for almost all $t \in R_+$,

$$(iv)\ b_1 b_2 \|K\| M^{-1} < 1.$$

Then we can prove the following theorem.

Theorem 6. Let the assumptions (i) $\longrightarrow$ (iv) be satisfied , then the equation (1) has at least one solution $x \in L^1$ being a.e. nonincreasing on R_+.

Proof. First of all observe that for a given $x \in L^1$ the function $H x$ belongs to L^1, which is a consequence of the assumptions

$$(i) \longrightarrow (iii)\ .$$

Additionally, using (2) we get

$$\|H x\| = \int_0^\infty \left| f_1\Big(t, \int_0^\infty k\,(t-s)\ f_2\big(s, x\,(\phi(s))\big)\,ds\Big)\right|$$

Then

$$\|Hx\| = \left\| FKf_2\left(s, x\left(\phi(s)\right)\right)\right\|$$

$$\leq \int_0^\infty \left[a_1(t) + b_1 \left| \int_0^\infty k(t-s)\ f_2\left(s, x\left(\phi(s)\right)\right) ds\right|\right] dt$$

$$\leq \|a_1(t)\| + b_1 \left\|K f_2 x\left(\phi\right)\right\|,$$

where F_i , $i = 1,2$ *are* the superposition operators generated by f_i , $i = 1,2$ i.e. we have

$$\|Hx\| \quad \leq \|a_1\| + b_1 \|K\| \int_0^\infty \left[a_2(t) + b_2 \left| x\left(\phi(t)\right)\right|\right] dt$$

$$\leq \|a_1\| + b_1 \|K\| \|a_2\| + b_1 b_2 M^{-1} \int_0^\infty |x(u)|\, du,$$

where $u = \phi(t)$ i.e.

$$\|Hx\| \leq$$
$$\|a_1\| + b_1 \|K\| \|a_2\| + b_1 b_2 \|K\|\ M^{-1}\ \|x\|.$$

From this estimate and (iv) we infer that the operator H maps the ball B_r into itself , where

$$r = \frac{\|a_1\| + b_1 \left\|a_2\right\| \|K\|}{1 - b_1 b_2 \|K\| M^{-1}}., \quad r > 0.$$

Further, let Q_r stand for the subset of B_r consisting of all functions which are a.e. positive and nonincreasing on R_+. Note that Q_r is nonempty, bounded, closed and convex subset of L^1. Moreover, in view of theorem 2 the set Q_r is compact in measure.

Next, take $x \in Q_r$, we deduce that $x\left(\phi\right)$ is a.e. nonnegative and nonincreasing on R_+ and consequently $Kx\left(\phi\right)$ is also of the same type in virtue of the assumption (i) and Theorem 3.

Further, the assumption (i) permits us to the deduce that $Hx = FKf_2 x\left(\phi\right)$ is also a.e. positive and nonincreasing on R_+ .

This fact, together with the assertion $H : B_r \longrightarrow B_r$ gives that H is a self - mapping of the set Q_r.

In the sequel, we show that the operator H is construction with respect to Hausdorff measure of noncompactness, for this let X be a nonempty subset of Q_r and $\varepsilon > 0$. Then, for an arbitrary $x \in X$ and for a subset $D \subset R_+$ with *meas* $D \leq \varepsilon$, we obtain

$$\int_D |(Hx)(t)|\ dt \ \leq \int_D \left[a_1(t) + b_1 \left| \int_0^\infty k(t-s)\ f_2\left(s, x\left(\phi(s)\right)\right) ds\right|\right] dt$$

$$\leq \|a_1\|_{L^1(D)} + b_1 \|K\|_D \int_D \left[a_2(t) + b_2 \left| x\left(\phi(t)\right)\right|\right] dt,$$

Where $\|K\|_D$ denotes the norm of the operator

$$K : L^1(D) \longrightarrow L^1(D).$$

Consequently, we get

$$\int_D |(Hx)(t)|\ dt \leq \|a_1\|_{L^1(D)} + b_1 \|K\|_D \|a_2\|_{L^1_{(D)}} +$$

$$+b_1 b_2 \| K \|_D M^{-1} \int_{\phi(D)} | x(u) | \, du.$$

Since

$$\lim_{c \longrightarrow 0} \left\{ \sup \left[\int_D a_i(t) \, dt : D \sqsubset R_+, \ meas.\ D \leq \varepsilon \right] \right\} = 0$$

and ϕ is absolutly continuous , then we get

$$c(HX) \leq b_1 b_2 \| K \|_D M^{-1} c(X) \quad (6)$$

where the quantity $c(X)$ was defined in Section 2.

Furthermore, for fixed $T > 0$ we have

$$\int_T^\infty | (Hx) | (t) \, dt \leq \int_T^\infty \left[a_1(t) + b_1 \left| \int_0^\infty k(t-s) f_2\left(s, x\left(\phi(s)\right)\right) ds \right| \right] dt$$

$$\leq \int_T^\infty a_1(t) \, dt + b_1 \|K\| \int_T^\infty a_2(t) \, dt +$$

$$+b_1 b_2 \| K \| M^{-1} \int_T^\infty \left| x\left(\phi(t)\right) \right| \phi'(t) \, dt$$

$$\leq \int_T^\infty a_1(t) \, dt + b_1 \|K\| \int_T^\infty a_2(t) \, dt +$$

$$+ b_1 b_2 \| K \| M^{-1} \int_{\phi(T)}^\infty |x(u)| \, du ,$$

where $\phi(t) \to \infty$, as $t \to \infty$.

Since

$$\lim_{T \longrightarrow \infty} \int_T^\infty a_i(t) \, dt = 0 , \qquad i = 1, 2,$$

then as $T \to \infty$, we get

$$d(HX) \leq b_1 b_2 \|K\| M^{-1} d(X) \qquad (7).$$

Combining (6) and (7) we get

$$\gamma(HX) \leq b_1 b_2 \|K\| M^{-1} \gamma(X).$$

Since $X \subset Q_r$ is compact in measure, then by using Theorem (4) we deduce that

$$\chi(HX) \leq b_1 b_2 \|K\| M^{-1} \chi(X).$$

By using all properties of the operator H and the set Q_r as well as (*iv*) we can apply theorem (5) to get a fixed point for H which is the solution (1) .

The proof is completed .

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