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An implicit iteration process with errors for asymptotically nonexpansive mappings in the intermediate sense in Banach spaces

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Abstract. The aim of this article is to study an implicit iteration process with errors for a finite family of non-Lipschitzian asymptotically nonexpansive mappings in the intermediate sense in the setting of Banach spaces, also we establish some strong convergence theorems and a weak convergence theorem for said scheme to converge to common fixed point for non-Lipschitzian asymptotically nonexpansive mappings in the intermediate sense. The results presented in this paper extend and improve the corresponding results of [1], [3]-[8], [10]-[12], [14]-[15], [17] and many others.

1. Introduction and Preliminaries

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. Recall the following concepts.

(1) T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.1}$$

for all $x, y \in K$.

(2) T is asymptotically nonexpansive if there exists a sequence $\{a_n\}$ in $[1, \infty)$ with $a_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq a_n \|x - y\|, \tag{1.2}$$

for all $x, y \in K$ and $n \geq 1$.

(3) T is uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \tag{1.3}$$

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for all $x, y \in K$ and $n \geq 1$.

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] as a generalization of the class of nonexpansive mappings. T is said to be asymptotically nonexpansive in the intermediate sense [2] if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} \left(\left\|T^n x - T^n y\right\| - \left\|x - y\right\| \right) \leq 0. \tag{1.4}$$

From the above definitions, it follows that asymptotically nonexpansive mappings must be asymptotically nonexpansive mapping in the intermediate sense and asymptotically quasi-nonexpansive mapping. But the converse does not hold as the following example:

Example 1.1. *Let $X = \mathbb{R}$ be a normed linear space and $K = [0, 1]$. For each $x \in K$, we define*

$$T(x) = \begin{cases} kx, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n |x - y| \leq |x - y|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\begin{aligned} \limsup_{n\to\infty} \{|T^n x - T^n y| - |x - y|\} &= \limsup_{n\to\infty} \{k^n \left\|x - y\right\| - \left\|x - y\right\|\} \\ &\leq 0 \end{aligned}$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$ and for all $x, y \in K$, $n \in \mathbb{N}$. Hence T is an asymptotically nonexpansive mapping in the intermediate sense.

Example 1.2. *Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in K$, define*

$$T(x) = \begin{cases} kx\,sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive mapping in the intermediate sense but it is not asymptotically nonexpansive mapping.

We say that a Banach space E satisfies the *Opial's condition* [9] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \left\|x_n - y\right\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [9].

Let K be a nonempty closed convex subset of a Banach space E. Then $I - T$ is demiclosed at zero if, for any sequence $\{x_n\}$ in K, condition $x_n \to x$ weakly and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ implies $(I - T)x = 0$.

Let E be a Hilbert space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, \ldots, T_N\}\colon K \to K$ be N nonexpansive mappings. In 2001, Xu and Ori [18] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{n(mod\ N)} x_n, \quad n \geq 1, \tag{1.5}$$

where $x_0 \in K$ is an initial point, $\{\alpha_n\}_{n\geq 1}$ is a real sequence in $(0, 1)$ and proved the weakly convergence of the sequence $\{x_n\}$ defined by (1.5) to a common fixed point $p \in F = \bigcap_{i=1}^{N} F(T_i)$.

In 2003, Sun [14] introduced the following implicit iterative sequence $\{x_n\}$ defined by

$$x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n, \quad n \geq 1, \tag{1.6}$$

for a finite family of asymptotically quasi-nonexpansive self-mappings on a bounded closed convex subset K of a Hilbert space E with $\{\alpha_n\}$ a sequence in $(0, 1)$ and an initial point $x_0 \in K$, where $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, and proved the strong convergence of the sequence $\{x_n\}$ defined by (1.6) to a common fixed point $p \in F = \bigcap_{i=1}^{N} F(T_i)$.

In 2006, Gu [6] introduced the following implicit iterative sequence $\{x_n\}$ with errors

$$\begin{aligned} x_n &= (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + u_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T_{i(n)}^{k(n)} x_n + v_n, \quad n \geq 1, \end{aligned} \tag{1.7}$$

for a finite family of asymptotically nonexpansive mappings on a closed convex subset K of a Banach space X with $K + K \subset K$, $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0,1], $\{u_n\}$ and $\{v_n\}$ be two sequences in K, and an initial point $x_0 \in K$, where $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, and proved the weak and strong convergence of the sequence $\{x_n\}$ defined by (1.7) to a common fixed point $p \in F = \bigcap_{i=1}^{N} F(T_i)$.

It should be pointed out that the sequence defined by (1.6) is a special case of the sequence defined by (1.7) with $u_n = v_n = 0$, $\beta_n = 0$, for all $n \geq 1$.

Recently concerning the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, e.g., Bauschke [1], Chang and Cho [3], Goebel and Kirk [4], Gornicki [5], Gu [6], Halpern [7], Lions [8], Osilike [10], Reich [12], Schu [13], Sun [14], Tan and Xu [15], Wittmann [17], Xu and Ori [18] and Zhou and Chang [19]).

The purpose of this article is to study an iterative sequence defined by (1.7) for a finite family of asymptotically nonexpansive mappings in the intermediate sense in Banach spaces and establish the strong convergence theorems for said iteration scheme and mappings.

In the sequel we need the following lemmas to prove our main results.

Lemma 1.3. *(see [16]) Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4. *(Schu [13]) Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r, \quad \lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 1.5. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{T_i\}_{i=1}^N : K \to K$ be N asymptotically nonexpansive mappings in the intermediate sense with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Put*

$$A_n = \max\Big\{0, \sup_{x,y\in K,\ n\geq 1} \Big(\|T_i^n x - T_i^n y\| - \|x-y\|\Big) : i \in I\Big\}, \tag{1.8}$$

where $n = (k-1)N + i$, $k = k(n)$, $i = i(n) \in \{1, 2, \ldots, N\} = I$, such that $\sum_{n=1}^{\infty} A_n < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in K, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:

(i) $\tau = \sup\{\alpha_n : n \geq 1\} < 1$;

(ii) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

If $\{x_n\}$ is the implicit iterative sequence defined by (1.7), then for each $p \in F = \bigcap_{i=1}^N F(T_i)$ the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Since $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, for any given $p \in F$, it follows from (1.7) and (1.8) that

$$\|x_n - p\| = \|(1-\alpha_n)x_{n-1} + \alpha_n T_n^k y_n + u_n - p\| \leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|T_n^k y_n - p\| + \|u_n\| \tag{1.9}$$

where $n = (k-1)N + i$, $k = k(n)$, $i = i(n) \in \{1, 2, \ldots, N\} = I$ and $T_n = T_{i(mod\ N)} = T_i$. This implies that

$$\begin{aligned} \|x_n - p\| &\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|T_i^k y_n - p\| + \|u_n\| = (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|T_i^k y_n - T_i^k p\| + \|u_n\| \\ &\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n[\|y_n - p\| + A_n] + \|u_n\| \\ &\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n \|y_n - p\| + A_n + \|u_n\|. \end{aligned} \tag{1.10}$$

Again it follows from (1.7) and (1.8) that

$$\begin{aligned} \|y_n - p\| &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|T_i^k x_n - p\| + \|v_n\| = (1-\beta_n)\|x_n - p\| + \beta_n \|T_i^k x_n - T_i^k p\| \\ &\quad + \|v_n\| \leq (1-\beta_n)\|x_n - p\| + \beta_n[\|x_n - p\| + A_n] + \|v_n\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|x_n - p\| + A_n + \|v_n\| \leq \|x_n - p\| + A_n + \|v_n\|. \end{aligned} \tag{1.11}$$

Substituting (1.11) into (1.10), we obtain

$$\begin{aligned}\|x_n - p\| &\leq (1-\alpha_n)\|x_{n-1}-p\| + \alpha_n\|x_n - p\| + (\alpha_n+1)A_n + \alpha_n\|v_n\| + \|u_n\| \\ &\leq (1-\alpha_n)\|x_{n-1}-p\| + \alpha_n\|x_n - p\| + 2A_n + \alpha_n\|v_n\| + \|u_n\|\end{aligned}$$

which implies that

$$(1-\alpha_n)\|x_n - p\| \leq (1-\alpha_n)\|x_{n-1}-p\| + \mu_n \tag{1.12}$$

where $\mu_n = 2A_n + \alpha_n\|v_n\| + \|u_n\|$. By the assumption $\sum_{n=1}^{\infty} A_n < \infty$, condition (ii) and boundedness of the sequences $\{\alpha_n\}$, we know that $\sum_{n=1}^{\infty}\mu_n < \infty$. From condition (i) we have $\alpha_n \leq \tau < 1$, and so

$$1 - \alpha_n \geq 1 - \tau > 0, \tag{1.13}$$

hence from (1.12) we have

$$\|x_n - p\| \leq \|x_{n-1}-p\| + \frac{\mu_n}{1-\alpha_n} \leq \|x_{n-1}-p\| + \frac{\mu_n}{1-\tau} = \|x_{n-1}-p\| + \theta_n \tag{1.14}$$

where $\theta_n = \frac{\mu_n}{1-\tau}$.

By assumption of the theorem and condition (ii) we have that

$$\sum_{n=1}^{\infty}\theta_n = \sum_{n=1}^{\infty}\frac{\mu_n}{1-\tau} < \infty.$$

Taking $A_n = \|x_{n-1} - p\|$ in inequality (1.14), we have

$$A_{n+1} \leq A_n + \theta_n, \quad \forall n \geq 1,$$

and satisfied all conditions in Lemma 1.3. Therefore the limit $\lim_{n\to\infty}\|x_n - p\|$ exists.
Without loss of generality we may assume that

$$\lim_{n\to\infty}\|x_n - p\| = d,$$

where $d \geq 0$ is some nonnegative number and $p \in F$. This completes the proof of Lemma 1.5. □

2. Main Results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{T_i\}_{i=1}^{N} : K \to K$ be N asymptotically nonexpansive mappings in the intermediate sense with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Put*

$$A_n = \max\Big\{0, \sup_{x,y\in K,\ n\geq 1}\Big(\|T_i^n x - T_i^n y\| - \|x-y\|\Big) : i \in I\Big\},$$

where $n = (k-1)N + i$, $k = k(n)$, $i = i(n) \in \{1, 2, \ldots, N\} = I$, such that $\sum_{n=1}^{\infty} A_n < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in K, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:

(i) $\tau = \sup\{\alpha_n : n \geq 1\} < 1$;

(iii) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Then the implicit iterative sequence $\{x_n\}$ *defined by (1.7) converges strongly to a common fixed point* $p \in F = \bigcap_{i=1}^{N} F(T_i)$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.1}$$

Proof. The necessity of condition (2.1) is obvious.

Next we prove the sufficiency of Theorem 2.1. For any given $p \in F$, it follows from equation (1.14) in Lemma 1.5 that

$$\left\|x_n - p\right\| \leq \left\|x_{n-1} - p\right\| + \theta_n \quad \forall n \geq 1, \tag{2.2}$$

where $\theta_n = \frac{\mu_n}{1-\tau}$, with $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence, we have

$$d(x_n, F) \leq d(x_{n-1}, F) + \theta_n \quad \forall n \geq 1. \tag{2.3}$$

It follows from equation (2.3) and Lemma 1.3 that the limit $\lim_{n\to\infty} d(x_n, F)$ exists. By the condition (2.1), we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. For any integer $m \geq 1$, we have from equation (2.2) that

$$\left\|x_{n+m} - p\right\| \leq \left\|x_{n+m-1} - p\right\| + \theta_{n+m-1} \leq \left\|x_{n+m-2} - p\right\| + \theta_{n+m-2} + \theta_{n+m-1} \leq \cdots \leq \left\|x_n - p\right\| + \sum_{k=n}^{n+m-1} \theta_k. \tag{2.4}$$

Since $\liminf_{n\to\infty} d(x_n, F) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F$ such that $\left\|x_{n_k} - p_{n_k}\right\| \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$\left\|x_{n_k} - p_{n_k}\right\| < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty} \theta_k < \frac{\varepsilon}{4}, \tag{2.5}$$

for all $k \geq k_\varepsilon$.

For any $m \geq 1$ and for all $n \geq n_{k_\varepsilon}$, by equation (2.5), we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \left\|x_{n+m} - p_{n_k}\right\| + \left\|x_n - p_{n_k}\right\| \leq \left\|x_{n_k} - p_{n_k}\right\| + \sum_{k=n_{k_\varepsilon}}^{\infty} \theta_k + \left\|x_{n_k} - p_{n_k}\right\| + \sum_{k=n_{k_\varepsilon}}^{\infty} \theta_k \\ &= 2\left\|x_{n_k} - p_{n_k}\right\| + 2\sum_{k=n_{k_\varepsilon}}^{\infty} \theta_k < 2 \cdot \frac{\varepsilon}{4} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{2.6}$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. By the completeness of K, we can assume that $\lim_{n\to\infty} x_n = p^*$. Since the set of fixed points of an asymptotically nonexpansive mapping in the intermediate sense is closed, hence F is closed. This implies that $p^* \in F$, and so p^* is a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $\{T_i\}_{i=1}^N : K \to K$ be N asymptotically nonexpansive mappings in the intermediate sense with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and there exists an T_l, $1 \leq l \leq N$, which is semi-compact (without loss of generality, we can assume that T_1 is semi-compact). Put*

$$A_n = \max\Big\{0, \sup_{x,y \in K,\ n \geq 1} \Big(\big\|T_i^n x - T_i^n y\big\| - \big\|x - y\big\| \Big) : i \in I\Big\},$$

where $n = (k-1)N + i$, $k = k(n)$, $i = i(n) \in \{1, 2, \ldots, N\} = I$, such that $\sum_{n=1}^{\infty} A_n < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in K, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:

(i) $\tau = \sup\{\alpha_n : n \geq 1\} < 1$;

(ii) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$;

(iii) $0 < a_1 = \inf\{\alpha_n : n \geq 1\} \leq \sup\{\alpha_n : n \geq 1\} = a_2 < 1$;

(iv) $0 \leq \delta = \sup\{\beta_n : n \geq 1\} < 1$;

(v) there exists a constant $L > 0$ such that, for any $i \in \{1, 2, \ldots, N\}$, we have

$$\big\|T_i^n x - T_i^n y\big\| \leq L \big\|x - y\big\|, \quad \forall n \geq 1,$$

for all $x, y \in K$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.7) converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. First, we prove that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, \ldots, N. \tag{2.7}$$

Let $p \in F$. Put $d = \big\|x_n - p\big\|$, where $d \geq 0$ is some nonnegative number. It follows from equation (1.7) that

$$\|x_n - p\| = \|(1 - \alpha_n)[x_{n-1} - p + u_n] + \alpha_n[T_i^k y_n - p + u_n]\| \to d, \quad n \to \infty. \tag{2.8}$$

Again since $\lim_{n\to\infty} \big\|x_n - p\big\|$ exists, so $\{x_n\}$ is a bounded sequence in K. By virtue of condition (ii) and the boundedness of sequences $\{x_n\}$ we have

$$\limsup_{n\to\infty} \big\|x_{n-1} - p + u_n\big\| \leq \limsup_{n\to\infty} \big\|x_{n-1} - p\big\| + \limsup_{n\to\infty} \|u_n\| = d, \quad p \in F. \tag{2.9}$$

It follows from equation (1.11) and condition (ii) that

$$\begin{aligned}
\limsup_{n\to\infty} \big\|T_n^k y_n - p + u_n\big\| &\leq \limsup_{n\to\infty} \big\|T_n^k y_n - p\big\| + \limsup_{n\to\infty} \|u_n\| \\
&\leq \limsup_{n\to\infty} [\big\|y_n - p\big\| + A_n] \leq \limsup_{n\to\infty} [\big\|x_n - p\big\| + 2A_n + \|v_n\|] \\
&\leq \limsup_{n\to\infty} \big\|x_n - p\big\| + 2\limsup_{n\to\infty} A_n + \limsup_{n\to\infty} \|v_n\| = d, \quad p \in F,
\end{aligned} \tag{2.10}$$

where $n = (k-1)N + i$, $k = k(n)$, $i = i(n) \in \{1, 2, \ldots, N\} = I$ and $T_n = T_{i(mod\ N)} = T_i$.

Therefore from condition (iii), (2.8) - (2.10), and Lemma 1.4 we know that

$$\lim_{n\to\infty} \left\|T_n^k y_n - x_{n-1}\right\| = 0. \tag{2.11}$$

From (1.7), (2.11) and condition (ii), we have

$$\|x_n - x_{n-1}\| = \left\|\alpha_n[T_i^k y_n - x_{n-1}] + u_n\right\| \leq \alpha_n \left\|T_i^k y_n - x_{n-1}\right\| + \|u_n\| \to 0, \quad \text{as} \quad n \to \infty, \tag{2.12}$$

which implies that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0, \tag{2.13}$$

and so

$$\lim_{n\to\infty} \left\|x_n - x_{n+j}\right\| = 0 \quad \forall j = 1, 2, \ldots, N. \tag{2.14}$$

On the other hand, we have

$$\left\|x_n - T_n^k x_n\right\| \leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - T_n^k y_n\right\| + \left\|T_n^k y_n - T_n^k x_n\right\|, \tag{2.15}$$

where $n = (k-1)N + i$, $k = k(n)$ and $i = i(n)$.

Now, we consider the third term of the right hand side of (2.15). From (1.7), (1.8) and the condition (iv) we have

$$\begin{aligned}\left\|T_n^k y_n - T_n^k x_n\right\| &\leq \left\|y_n - x_n\right\| + A_n \leq \left\|(1-\beta_n)x_n + \beta_n T_n^k x_n + v_n - x_n\right\| + A_n \\ &\leq \beta_n \left\|T_n^k x_n - x_n\right\| + \|v_n\| + A_n \leq \delta \left\|T_n^k x_n - x_n\right\| + \|v_n\| + A_n.\end{aligned} \tag{2.16}$$

Substituting (2.16) into (2.15), we obtain that

$$(1-\delta) \cdot \left\|x_n - T_n^k x_n\right\| \leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - T_n^k y_n\right\| + \|v_n\| + A_n. \tag{2.17}$$

Hence, by virtue of the condition (ii), (2.11) and (2.13), we have

$$(1-\delta).\limsup_{n\to\infty} \left\|x_n - T_n^k x_n\right\| \leq 0. \tag{2.18}$$

From the condition (iv), $0 \leq \delta < 1$, hence from (2.18) we have

$$\lim_{n\to\infty} \left\|x_n - T_n^k x_n\right\| = 0. \tag{2.19}$$

Now, we prove that (2.7) holds. In fact, since for each $n > N$, $n = (n-N)(mod\ N)$ and $n = (k(n)-1)N + i(n)$, hence $n - N = ((k(n)-1)-1)N + i(n) = (k(n-N)-1)N + i(n-N)$, that is,

$$k(n-N) = k(n) - 1 \quad \text{and} \quad i(n-N) = i(n).$$

From (2.14), (2.19) and condition (v) that

$$\begin{aligned}\|x_n - T_n x_n\| &\leq \left\|x_n - T_n^k x_n\right\| + \left\|T_n^k x_n - T_n x_n\right\| \leq \left\|x_n - T_n^k x_n\right\| + L\left\|T_n^{k-1} x_n - x_n\right\| \\ &\leq \left\|x_n - T_n^k x_n\right\| + L\Big\{\left\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\right\| + \left\|T_{n-N}^{k-1} x_{n-N} - x_{n-N}\right\| + \|x_{n-N} - x_n\|\Big\} \\ &\leq \left\|x_n - T_n^k x_n\right\| + L^2\|x_n - x_{n-N}\| + L\left\|T_{n-N}^{k-1} x_{n-N} - x_{n-N}\right\| + L\|x_{n-N} - x_n\| \\ &= \left\|x_n - T_n^k x_n\right\| + L(L+1)\|x_n - x_{n-N}\| + L\left\|T_{n-N}^{k-1} x_{n-N} - x_{n-N}\right\| \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0, \tag{2.20}$$

and so, from condition (v), (2.13) and (2.20), it follows that, for any $j = 1, 2, \ldots, N$,

$$\begin{aligned}\left\|x_n - T_{n+j} x_n\right\| &\leq \left\|x_n - x_{n+j}\right\| + \left\|x_{n+j} - T_{n+j} x_{n+j}\right\| + \left\|T_{n+j} x_{n+j} - T_{n+j} x_n\right\| \\ &\leq (1+L)\left\|x_n - x_{n+j}\right\| + \left\|x_{n+j} - T_{n+j} x_{n+j}\right\| \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\|x_n - T_{n+j} x_n\right\| = 0, \tag{2.21}$$

for all $j = 1, 2, \ldots, N$.

Without loss of generality, we can assume that $n_k = i(mod\ N)$ for all k and some $i \in \{1, 2, \ldots, N\}$. For any fixed $l \in \{1, 2, \ldots, N\}$, we can find a $j \in \{1, 2, \ldots, N\}$, independent of k, such that $i + j = l(mod\ N)$, and so $n_k + j = l(mod\ N)$ for all k. Hence, from (2.21), we have

$$\lim_{n_k\to\infty} \left\|x_{n_k} - T_l x_{n_k}\right\| = 0, \tag{2.22}$$

for all $l = 1, 2, \ldots, N$.

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \tag{2.23}$$

for all $l = 1, 2, \ldots, N$.

That is, (2.7) holds.

Now, we prove that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

For any given $p \in F = \bigcap_{i=1}^N F(T_i)$, by the same method as given in proving Lemma 1.5 and (2.23), we can prove that

$$\lim_{n\to\infty} \left\|x_n - p\right\| = d, \tag{2.24}$$

where $d \geq 0$ is some nonnegative number, and

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \tag{2.25}$$

for all $l = 1, 2, \dots, N$.

Especially, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{2.26}$$

By the assumption of the theorem, T_1 is semi-compact, therefore it follows from (2.26) that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in K$. Hence from (2.25) we have that

$$\|x^* - T_l x^*\| = \lim_{n_i\to\infty} \left\|x_{n_i} - T_l x_{n_i}\right\| = 0,$$

for all $l = 1, 2, \dots, N$, which implies that

$$x^* \in F = \bigcap_{i=1}^{N} F(T_i).$$

Take $p = x^*$ in (2.24), similarly we can prove that

$$\lim_{n\to\infty} \|x_n - x^*\| = d_1,$$

where $d_1 \geq 0$ is some nonnegative number. From $x_{n_i} \to x^*$ we know that $d_1 = 0$, i.e., $x_n \to x^*$. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let E be a real Banach space satisfying Opial's condition and K be a weakly compact subset of E with $K + K \subset K$. Let $T_i \colon K \to K$ be N asymptotically nonexpansive mappings in the intermediate sense with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Put*

$$A_n \quad = \quad \max\Big\{0, \sup_{x,y\in K,\ n\geq 1} \Big(\left\|T_i^n x - T_i^n y\right\| - \left\|x - y\right\|\Big) : i \in I\Big\},$$

where $n = (k-1)N+i$, $k = k(n)$, $i = i(n) \in \{1, 2, \dots, N\} = I$, such that $\sum_{n=1}^\infty A_n < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in K, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ with the restrictions $\tau = \sup\{\alpha_n : n \geq 1\} < 1$, $\sum_{n=1}^\infty \|u_n\| < \infty$ and $\sum_{n=1}^\infty \|v_n\| < \infty$. Suppose that $\{T_i : i \in I\}$ has a common fixed point, $I - T_i$ for all $i \in I = \{1, 2, \dots, N\}$ is demiclosed at zero and $\{x_n\}$ is an approximating common fixed point sequence for T_i, that is, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for all $i \in I = \{1, 2, \dots, N\}$. Then the implicit iterative sequence $\{x_n\}$ defined by (1.7) converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.

Proof. First, we show that $\omega_w(x_n) \subset F$. Let $x_{n_k} \to x$ weakly. By assumption, we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$. Since $I - T_i$, for all $i \in I$ is demiclosed at zero, $x \in F$. By Opial's condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$. This completes the proof. □

Remark 2.4. *Our results improve and extend the corresponding results of of Chang and Cho [3] to the case of more general class of asymptotically nonexpansive mapping considered in this paper.*

Remark 2.5. *Our results also improve and extend the corresponding results of [1, 4, 5, 7, 8, 10, 12, 14, 15, 17] to the case of more general class of spaces, mappings and iteration schemes considered in this paper.*

Remark 2.6. *Our results also extend the corresponding results of Gu [6] to the case of more general class of asymptotically nonexpansive mapping considered in this paper.*

Remark 2.7. *Theorem 2.2 extends and improves Theorem 2.3 of Plubtieng and Wangkeeree [11] to the case of implicit iteration process with errors for a finite family of mappings considered in this paper.*

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