Functional Analysis,
Approximation and
Computation
4:1 (2012), 21–25

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Spectral commutativity of multioperators

Muneo Chō[a], Vladimir Müller[b]

[a]*Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan*
[b]*Institute of Mathematics AV ČR, Zitna 25, 115 67 Prague 1, Czech Republic*

Abstract. We give an example of commuting pairs $A = (A_1, A_2)$, $B = (B_1, B_2)$ of matrices such that $AB := (A_1B_1, A_2B_2)$ and $BA := (B_1A_1, B_2A_2)$ are commuting pairs but $\sigma_T(AB) \neq \sigma_T(BA)$; moreover, $\sigma_T(AB) \setminus \{(z, w) : zw = 0\} \neq \sigma_T(BA) \setminus \{(z, w) : zw = 0\}$. Further, we show that $\sigma_T(AB) = \sigma_T(BA)$ if A and B are criss-cross commuting n-tuples of operators and A is normal. This gives a positive answer to a problem studied in [2].

1 Introduction

Denote by $\mathcal{B}(X)$ the set of all bounded linear operators on a Banach space X.

It is well-known for two operators $A, B \in \mathcal{B}(X)$ that the spectra of AB and BA are almost equal,

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}. \tag{1}$$

Moreover, if $\dim X < \infty$ then

$$\sigma(AB) = \sigma(BA). \tag{2}$$

The same relation (2) is true if X is a Hilbert space and at least one of the operators A, B is normal.

Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be two n-tuples of operators on a Banach space X. We denote by AB the n-tuple

$$AB = (A_1B_1, A_2B_2, \dots, A_nB_n). \tag{3}$$

Let both AB and BA be commuting n-tuples of operators, so that the Taylor spectrum σ_T is defined. The relation between $\sigma_T(AB)$ and $\sigma_T(BA)$ was studied by a number of authors, see [5], [6], [3], [2], [1].

Apart from the commutativity of n-tuples AB and BA it was also studied the stronger relation of crisss-cross commutativity. n-Tuples $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are called criss-cross commuting, if

$$A_iB_jA_k = A_kB_jA_i, \qquad B_iA_jB_k = B_kA_jB_i \tag{4}$$

for all i, j, k. Criss-cross commuting tuples were studied in [6], [4], [2].

2010 *Mathematics Subject Classification*. 47A10.
Keywords. Criss-cross commuting; Spectral commutativity.
Received: January 6, 2012; Accepted: February 5, 2012
Communicated by Dragan S. Djordjević
The “first” author is partially supported by the Grant-in-Aid Research No. 20540192. The “second” author is partially supported by the Grant No. 201/09/0473 of GA CR.
Email addresses: chiyom01@kanagawa-u.ac.jp (Muneo Chō), muller@math.cas.cz (Vladimir Müller)

2 Results

In [5] it was shown for crisscross commuting n-tuples that the relation analogous to (1) is true:

$$\sigma_T(AB) \setminus \{(0,\dots,0)\} = \sigma_T(BA) \setminus \{(0,\dots,0)\}. \tag{5}$$

Another result was obtained in [1]. If A and B are commuting n-tuples and $A_iB_j = B_jA_i$ for all $i \neq j$ (this implies that AB and BA are commuting tuples), then the following weaker relation is true:

$$\sigma_T(AB) \setminus \{(z_1,\dots,z_n) : z_1 \cdots z_n = 0\} = \sigma_T(BA) \setminus \{(z_1,\dots,z_n) : z_1 \cdots z_n = 0\}. \tag{6}$$

In general, in this situation (5) is not true.

Remark 1. Let $A = (A_1,\dots,A_n)$ and $B = (B_1,\dots,B_m)$ be two tuples of operators on a Banach space X. Another natural possibility how to define the product of A and B is to consider the nm-tuple consisting of all products

$$(A_1B_1, A_1B_2, \dots, A_1B_m, A_2B_1, \dots, A_2B_m, \dots, A_nB_m).$$

This mn-tuple is commuting if A and B are criss-cross commuting in the sense of (4). However, this mn-tuple can be expressed as $\tilde{A}\tilde{B}$ where

$$\tilde{A} = (A_1,\dots,A_1, A_2,\dots,A_2,\dots,A_n \dots, A_n)$$

and

$$\tilde{B} = (B_1, B_2, \dots, B_m, B_1, \dots, B_m, \dots, B_1, \dots, B_m).$$

Thus all problems concerning this more general type of product can be reduced to the case of $m = n$ and the product defined by (3).

The first result of this paper shows that the analogy of (2) is not true even for n-tuples of matrices. Moreover, in general (6) is not true if we assume only that A, B, AB and BA are commuting tuples.

Example 2. We give an example of commuting pairs $A = (A_1, A_2)$ and $B = (B_1, B_2)$ of matrices such that $AB = (A_1B_1, A_2B_2)$ and $BA = (B_1A_1, B_2A_2)$ are commuting pairs but $\sigma_T(AB) \neq \sigma_T(BA)$. Moreover,

$$\sigma_T(AB) \setminus \{(z,w) : zw = 0\} \neq \sigma_T(BA) \setminus \{(z,w) : zw = 0\}.$$

Define matrices A_1, A_2, B_1, B_2 by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that all the matrices A_1B_1, A_2B_2, B_1A_1 and B_2A_2 are diagonal and hence mutually commuting. Moreover, $A_1A_2 = A_2A_1 = 0$ and $B_1B_2 = B_2B_1 = 0$, and so the pairs A, B are commuting.

On the other hand, A and B are not criss-cross commuting since $0 = A_2B_1A_1 \neq A_1B_1A_2 = A_2$ and $B_1 = B_1A_2B_2 \neq B_2A_2B_1 = 0$. Also, $A_1B_2 \neq B_2A_1$ and $A_2B_1 \neq B_1A_2$.

We have

$$A_1B_1 - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } A_2B_2 - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

So $\mathrm{Ker}(A_1B_1 - I) \cap \mathrm{Ker}(A_2B_2 - I) \neq \{0\}$ and $(1,1) \in \sigma_T(AB)$.
We show that $(1,1) \notin \sigma_T(BA)$.

We have $B_1A_1 - I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $B_2A_2 - I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

So $\mathrm{Ker}(B_2A_2 - I) \cap \mathrm{Ker}(B_1A_1 - I) \neq \{0\}$.
Since for commuting tuples on finite-dimensional spaces the Taylor spectrum is equal to the joint point spectrum, see [3], we have $(1,1) \notin \sigma_T(AB)$.

In the second half of this paper we consider criss-cross commuting normal tuples. Note that if A, B are operators on a Hilbert space and A is normal then the equality (1) is true in a stronger form: $\sigma(AB) = \sigma(BA)$. The analogous question for n-tuples of operators was investigated in [2] and partial results were obtained. We show that $\sigma_T(AB) = \sigma_T(BA)$ whenever A and B are criss-cross commuting tuples and A is normal, i.e., A consists of mutually commuting normal operators. This gives a positive answer to a problem studied in [2].

We start with a version of the Fuglede-Putnam theorem.
For a set $F \subset \mathbb{C}^2 \setminus \{(0,0)\}$ write

$$\tilde{F} = \{(\mu_1, \mu_2) \in \mathbb{C}^2 : \lambda_1\mu_1 + \lambda_2\mu_2 = 0 \text{ for some } (\lambda_1, \lambda_2) \in F\}.$$

Note that if $F \subset F'$ then $\tilde{F} \subset \tilde{F}'$.
Let $A = (A_1, A_2)$ be a pair of commuting normal operators on a Hilbert space H. For a Borel subset $G \subset \mathbb{C}^2$ denote by $H_A(G)$ the spectral subspace of A corresponding to the set G.

Theorem 3. *Let H, K be Hilbert spaces, let $A = (A_1, A_2) \in \mathcal{B}(H)^2$ and $B = (B_1, B_2) \in \mathcal{B}(K)^2$ be commuting pairs of normal operators, let $S : H \to K$ be a bounded linear operator. Then the following statements are equivalent*:

(i) $B_1SA_1 + B_2SA_2 = 0$;
(ii) $SH_A(F) \subset K_B(\tilde{F})$ *for each closed subset* $F \subset \mathbb{C}^2 \setminus \{(0,0)\}$;
(iii) $SH_A(F) \subset K_B(\tilde{F})$ *for each* F_σ *subset* $F \subset \mathbb{C}^2 \setminus \{(0,0)\}$.

Proof. Without loss of generality we can assume that A_1, A_2, B_1, B_2 are contractions. Denote by $E_A(\cdot)$ and $E_B(\cdot)$ the spectral projections onto the subspaces $H_A(\cdot)$ and $K_B(\cdot)$, respectively.

(i)⇒(ii): Suppose on the contrary that there is a closed subset $F \subset \sigma_T(A) \setminus \{(0,0)\}$ such that $SH_A(F) \not\subset K_B(\tilde{F})$. Equivalently, $E_B\big(\sigma_T(B) \setminus \tilde{F}\big)SE_A(F) \neq 0$.
Since

$$\sigma_T(B) \setminus \tilde{F} = \bigcup_{n=1}^{\infty} \Big\{(\mu_1, \mu_2) \in \sigma_T(B) : \inf_{(\lambda_1,\lambda_2)\in F} |\lambda_1\mu_1 + \lambda_2\mu_2| \geq n^{-1}\Big\},$$

it is easy to see that there are an $\varepsilon > 0$ and a closed subset $M \subset \sigma_T(B)$ such that $E_B(M)SE_A(F) \neq 0$ and $|\lambda_1\mu_1 + \lambda_2\mu_2| \geq \varepsilon$ for all $(\lambda_1, \lambda_2) \in F$ and $(\mu_1, \mu_2) \in M$.

Choose a positive number $\delta < \varepsilon/8$. Since F and M can be covered by a finite number of balls of radius δ, there are $(\lambda_1, \lambda_2) \in F$, $(\mu_1, \mu_2) \in M$ and Borel sets F', M' such that $E_B(M')SE_A(F') \neq 0$, $M' \subset M \cap \{(z, w) : |z - \mu_1| \leq \delta, |w - \mu_2| \leq \delta\}$ and $F' \subset F \cap \{(z, w) : |z - \lambda_1| \leq \delta, |w - \lambda_2| \leq \delta\}$. Set $S' = E_B(M')SE_A(F')$.

Choose $x \in H_A(F')$ of norm one such that $\|S'x\| > \|S'\|/2$. We have

$$\|B_1S'A_1x - \lambda_1\mu_1S'x\| \leq \|(B_1 - \mu_1)S'A_1x\| + \|\mu_1S'(A_1x - \lambda_1x)\| \leq 2\delta\|S'\|,$$

and similarly, $\|B_2S'A_2x - \lambda_2\mu_2S'x\| \leq 2\delta\|S'\|$. Since

$$B_1S'A_1 + B_2S'A_2 = E_B(M')(B_1SA_1 + B_2SA_2)E_A(F') = 0,$$

we have $\|(\lambda_1\mu_1 + \lambda_2\mu_2)S'x\| \le 4\delta\|S'\|$. On the other hand,

$$\|(\lambda_1\mu_1 + \lambda_2\mu_2)S'x\| \ge \|S'x\| \cdot |\lambda_1\mu_1 + \lambda_2\mu_2| \ge \varepsilon\|S'x\| > 4\delta\|S'\|,$$

a contradiction.

(ii)⇒(iii) Let $F \subset \mathbb{C}^2 \setminus \{(0,0)\}$ be a F_σ set. We can write $F = \bigcup_{k=1}^{\infty} F_k$, where F_k are closed subsets. For each k we have

$$SH_A(F_k) \subset K_B(\tilde{F}_k) \subset K_B(\tilde{F}).$$

Since $H_A(F) = \bigvee_{k=1}^{\infty} H_A(F_k)$, we also have $SH_A(F) \subset K_B(\tilde{F})$.

(iii)⇒(i): Let $\varepsilon > 0$. Let $(C_i)_{i=1}^{\infty}$ be nonempty disjoint F_σ sets with diameters $< \varepsilon$ such that $\bigcup_i C_i = \mathbb{C} \setminus \{0\}$. For each i fix $\lambda_i \in C_i$. Thus $C_i \subset \{z \in \mathbb{C} : |z - \lambda_i| < \varepsilon\}$. Set $F_0 = \{(0,0)\}$, $F_0' = \{(0,w) : w \in \mathbb{C} \setminus \{0\}\}$, $F_0'' = \{(w,0) : w \in \mathbb{C} \setminus \{0\}\}$ and, for $i \in \mathbb{N}$, $F_i = \{(z, cz) : z \ne 0, c \in C_i\}$.

Clearly $(B_1SA_1 + B_2SA_2)_{|H_A(F_0)} = 0$ since both A_1 and A_2 act on $H_A(F_0)$ as the zero operator.

Let $x \in H_A(F_0')$. We have $(B_1SA_1 + B_2SA_2)x = B_2SA_2x$, where $A_2x \in H_A(F_0')$ and $SA_2x \in K_B(\tilde{F_0'}) = K_B(\{(z,0) : z \in \mathbb{C}\}$. So $B_2SA_2x = 0$ and $(B_1SA_1 + B_2SA_2)_{|H_A(F_0')} = 0$.

Similarly it is possible to show that $(B_1SA_1 + B_2SA_2)_{|H_A(F_0'')} = 0$.

Clearly we have $\|(A_2 - \lambda_iA_1)_{|H_A(F_i)}\| < \varepsilon$ and $\|(B_1 + \lambda_iB_2)_{|K_B(\tilde{F}_i)}\| < \varepsilon$ for each $i \ge 1$. For $x \in H_A(F_i)$ we have

$$\|B_1SA_1x + B_2SA_2x\| = \|B_1SA_1x + \lambda_iB_2SA_1x - \lambda_iB_2SA_1x + B_2SA_2x\|$$

$$\le \|(B_1 + \lambda_iB_2)SA_1x\| + \|B_2S(A_2 - \lambda_iA_1)x\| < 2\varepsilon\|S\| \cdot \|x\|.$$

Thus $\|(B_1SA_1 + B_2SA_2)_{|H_A(F_i)}\| \le 2\varepsilon\|S\|$ for all i. For $x \in H_A(F_i)$ we have $SA_1x \in K_B(\tilde{F}_i)$ and

$$B_1SA_1x = B_1E_B\big(\tilde{F}_i \setminus \{(0,0)\}\big)SA_1x + B_1E_B\big(\{(0,0)\}\big)SA_1x \in K_B\big(\tilde{F}_i \setminus \{(0,0)\}\big).$$

Similarly $B_2SA_2x \in K_B\big(\tilde{F}_i \setminus \{(0,0)\}\big)$ and we have

$$(B_1SA_1 + B_2SA_2)H_A(F_j) \subset K_B(\tilde{F}_j \setminus \{(0,0)\}).$$

Since the sets $F_j \setminus \{(0,0)\}$ are mutually disjoint, the spaces $K_B(F_j \setminus \{(0,0)\})$ are orthogonal. Thus $\|B_1SA_1 + B_2SA_2\| \le 2\varepsilon\|S\|$.

Since ε was arbitrary, we have $B_1SA_1 + B_2SA_2 = 0$. □

Remark 4. Let A_1, A_2, B_1, B_2, S satisfy the conditions of the previous theorem. Since the spectral subspaces of A and A^* coincide and satisfy $H_A(F) = H_{A^*}(\bar{F})$ where $\bar{F} = \{\bar{z} : z \in F\}$, and similar relations hold for B and B^*, Theorem 3 implies the following general form of the Fuglede-Putnam theorem, see [7], [8]: if $B_1SA_1 + B_2SA_2 = 0$ then $B_1^*SA_1^* + B_2^*SA_2^* = 0$.

Theorem 5. *Let $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$ be criss-cross commuting tuples, let A be normal (i.e, $A_1, \dots, A_n$ are commuting normal operators). Then $\sigma_T(AB) = \sigma_T(BA)$.*

Proof. If $0 \in \sigma_T(A)$ then both AB and BA are Taylor singular by [2], Theorem 2.1. Thus we may assume that A is Taylor regular. For $j = 1, \dots, n$ write

$$M_j = \{(z_1, \dots, z_n) \in \sigma_T(A) : |z_j| > |z_i| \quad (i < j) \text{ and } |z_j| \ge |z_i| \quad (i > j)\}.$$

Let H_j be the corresponding spectral subspaces $H_j = H_A(M_j)$. Clearly $H = \bigoplus_{j=1}^{n} H_j$ and $A_iH_j \subset H_j \quad (i, j = 1, \dots, n)$. Set $c_j = \min\{|z_j| : (z_1, \dots, z_n) \in M_j\}$. Then $c_j > 0$ and $A_{j|H_j}$ is invertible for each $j = 1, \dots, n$.

Fix $k, i, j, 1 \le k, i, j \le n, i \ne j$. We have $A_iB_kA_j - A_jB_kA_i = 0$. By Theorem 3 for the pairs $(A_i, A_j), (A_j, -A_i)$ we have

$$B_kH_A\Big(\{(z_1,\dots,z_n) : |z_i| \le |z_j|, |z_j| \ge c_j/2\}\Big) \subset H_A\Big(\{(z_1,\dots,z_n) : |z_i| \le |z_j|\}\Big)$$

and

$$B_kH_A\Big(\{(z_1,\dots,z_n) : |z_i| < |z_j|\}\Big) \subset H_A\Big(\{(z_1,\dots,z_n) : |z_i| < |z_j| \text{ or } z_i = z_j = 0\}\Big).$$

Hence the spaces $H_j \quad (j = 1,\dots,n)$ are invariant with respect to the operators B_k for all k, and therefore also to all products A_kB_k, B_kA_k. Thus

$$\sigma_T(AB) = \bigcup_{j=1}^{n} \sigma_T(AB|_{H_j}) \qquad \text{and} \qquad \sigma_T(BA) = \bigcup_{j=1}^{n} \sigma_T(BA|_{H_j}).$$

Since $A_j|_{H_j}$ is invertible for all j, by [2], Theorem 3.3 we have $\sigma_T(AB|_{H_j}) = \sigma_T(BA|_{H_j})$. Hence $\sigma_T(AB) = \sigma_T(BA)$. □

Acknowledgment. The paper was written during the second author's stay at the Kanagawa University. The author would like to thank for perfect working conditions and warm hospitality there.

References

[1] C. Benhida, E.H. Zerouali, On Taylor and other joint spectra for commuting n-tuples of operators, J. Math. Anal. Appl. **326** (2007), 521–532.
[2] M. Chō, R.E. Curto, T. Huruya, n-Tuples of operators satisfying $\sigma_T(AB) = \sigma_T(BA)$, Lin. Alg. Appl. **341** (2002), 291–298.
[3] M. Chō, M. Takaguchi, Joint spectra of matrices, Sci. Rep. Hirosaki Univ. **26** (1979), 15–19.
[4] R. Harte, On criss-cross commutativity, J. Operator Theory **37** (1997), 303–309.
[5] S. Li, On the commuting properties of Taylor's spectrum, Chinese Sci. Bull. **37** (1992), 1849–1852.
[6] S. Li, Taylor spectral invariance for crisscross commuting pairs on Banach spaces, Proc. Amer. Math. Soc. **124** (1996), 2069–2071.
[7] C.R. Putnam, Normal operators and strong limit approximations, Indiana Univ. Math. J. **32** (1983), 377–379.
[8] G. Weiss, The Fuglede commutativity theorem modulo the Hilbert-Schmidt class and generating functions for matrix operators II, J. Operator Theory **5** (1981), 3–16.