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Tensor product of n-isometries II

B. P. Duggal[a]

[a] *8 Redwood Grove , Northfield Avenue, London W5 4SZ, England, United Kingdom*

Abstract. A Hilbert space operator $A \in B(\mathcal{H})$ is an m-isometry for some natural number m if

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}A^{*m-i}A^{m-i} = 0;$$

A is a strict m-isometry if A is not a k-isometry for every integer $1 \leq k < m$. Let $A, B \in B(\mathcal{H})$, and let m, n be some natural numbers. If A (resp., B) is a strict m-isometry, then: (i) the tensor product $A \otimes B$ is an $(m + n - 1)$-isometry if and only if B (resp., A) is an n-isometry; (ii) $A \otimes B$ is a strict $(m + n - 1)$-isometry if and only if B (resp., A) is a strict n-isometry. This generalizes some results of Botelho, Jamison and Zheng [5, Section 4].

To Harrison Henry Duggal on his birthday

1. Introduction

An operator $A \in B(\mathcal{H})$, the algebra of operators (equivalently, bounded linear transformations) on a complex infinite dimensional Hilbert space $\mathcal{H}$ into itself, is an m-isometry for some integer $m \geq 1$ if

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}A^{*m-i}A^{m-i} = 0;$$

A is a strict m-isometry if A is not an $(m - 1)$-isometry. Evidently, an m-isometric operator is k-isometric for all integers $k \geq m$; hence if an $A \in B(\mathcal{H})$ is a strict m-isometry, then it is not a k-isometry for all integers $1 \leq k < m$. The class of m-isometric operators is a generalization of the class of isometric operators, and a detailed study of m-isometric operators has been carried out by Agler and Stankus in a series of papers [1–3]. For $A, B \in B(\mathcal{H})$, let $L_A R_B$ denote the (length one) elementary operator of left multiplication by A and right multiplication by B. A characterization of $L_A R_B|_{C_2(\mathcal{H})}$, where $C_2(\mathcal{H})$ is the Hilbert-Schmidt class, which are either 2-isometries or 3-isometries, and a sufficient condition for $L_A R_B$ to be an m-isometry, has been carried out by Botelho and Jamison [4]. More recently, Botelho *et.al.* [5] have proved that if A (resp.,

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Email address: bpduggal@yahoo.co.uk (B. P. Duggal)

B^*) is a strict m-isometry for $m = 2$ or 3, then: (i) $L_A R_B$ is an $(n + m - 1)$-isometry if and only if B^* (resp., A) is an n-isometry; (ii) $L_A R_B$ is a strict $(n + m - 1)$-isometry if and only if B^* (resp., A) is a strict n-isometry. Generalizing [4, Proposition 4.1], see also [4, Remark 4.1], the author proved in [7, Theorem 2.10] that if $A \in B(\mathcal{H})$ is an m-isometry and $B \in B(\mathcal{H})$ is an n-isometry for some natural numbers m and n, then the tensor product $A \otimes B$ of A and B is an $(m + n - 1)$-isometry. In this note we generalize the results of [5, Section 4] to prove that: If $A \in B(\mathcal{H})$ is a strict m-isometry, then (i) $A \otimes B$ is an $(m + n - 1)$-isometry if and only if $B \in B(\mathcal{H})$ is an n-isometry, and (ii) $A \otimes B$ is a strict $(m + n - 1)$-isometry if and only if $B \in B(\mathcal{H})$ is a strict n-isometry.

2. Results

Given two complex infinite dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, let $\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$; let, for $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$, $A \otimes B \in B(\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2)$ denote the tensor product operator defined by A and B. Evidently, an operator $A \in B(\mathcal{H})$ is an m-isometry if and only if $(A \otimes I)$ and $(I \otimes A) \in B(\mathcal{H} \overline{\otimes} \mathcal{H})$ are m-isometries. Furthermore, A is a strict m-isometry if and only if $A \otimes I$ and $I \otimes A$ are strict m-isometries. Observe also that $A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$. The following lemma is [7, Theorem 2.10], and provides half of the proof of our main result.

Lemma 2.1. *If $A \in B(\mathcal{H})$ is m-isometric and $B \in B(\mathcal{H})$ is n-isometric, then $A \otimes B$ is $(m + n - 1)$-isometric.*

Let m, n be natural numbers. If an $A \in B(\mathcal{H})$ is an m-isometry, then

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} A^{*m-i} A^{m-i} = 0 \Longleftrightarrow A^{*m} A^m = \sum_{i=1}^{m} (-1)^{i+1} \binom{m}{i} A^{*m-i} A^{m-i}.$$

Furthermore, if A is a strict m-isometry, then A is not a k-isometry for all integers $1 \leq k < m$ and it follows that $\{I, A^*A, A^{*2}A^2, ..., A^{*m-1}A^{m-1}\}$ is a linearly independent set [5, Theorem 3.1]. Observe that if $A \in B(\mathcal{H})$ is m-isometric and $A \otimes B$ is $(m + n - 1)$-isometric, then (it follows from a straightforward calculation that)

$$A^{*m+j} A^{m+j} = \sum_{i=0}^{m-1} (-1)^i \binom{i+j}{j} \binom{m+j}{i+j+1} A^{*m-1-i} A^{m-1-i}$$

for all integers $0 \leq j$ and

$$\begin{aligned} 0 &= \sum_{t=0}^{m+n-1} (-1)^t \binom{m+n-1}{t} A^{*m+n-1-t} A^{m+n-1-t} \otimes B^{*m+n-1-t} B^{m+n-1-t} \\ &= \sum_{t=0}^{n-1} (-1)^t \binom{m+n-1}{t} \{ \sum_{i=0}^{m-1} (-1)^i \binom{n-t-1+i}{n-t-1} \binom{m+n-1-t}{n-t+i} \\ &\quad A^{*m-1-i} A^{m-1-i}) \} \otimes B^{*m+n-1-t} B^{m+n-1-t} \\ &+ \sum_{t=n}^{m+n-1} (-1)^t \binom{m+n-1}{t} A^{*m+n-1-t} A^{m+n-1-t} \otimes B^{*m+n-1-t} B^{m+n-1-t} \\ &= \sum_{p=1}^{m} A^{*m-p} A^{m-p} \otimes \{ (-1)^{m-p} \binom{m+n-p}{m-p} \sum_{t=0}^{n-1} (-1)^t \frac{n-t}{m+n-p-t} \binom{n}{t} \\ &\quad B^{*m+n-t-1} B^{m+n-t-1} + (-1)^n B^{*p-1} B^{p-1} \}. \end{aligned}$$

Thus, if A is a strict m-isometry, then

$$(1) \qquad (-1)^{m-p} \binom{m+n-p}{m-p} \sum_{t=0}^{n-1} (-1)^t \frac{n-t}{m+n-p-t} \binom{n}{t} B^{*m+n-t-1} B^{m+n-t-1} + (-1)^n B^{*p-1} B^{p-1} = 0$$

for all $1 \le p \le m$. Choose $p = m$ to conclude

$$B^{*m+n-1}B^{m+n-1} = \sum_{t=1}^{n-1} (-1)^{t+1}\binom{n}{t} B^{*m+n-t-1}B^{m+n-t-1} + (-1)^{n+1}B^{*m-1}B^{m-1}$$

and substitute for $B^{*m+n-1}B^{m+n-1}$ in equation (1) to obtain

$$\begin{aligned} & \binom{m+n-p}{m-p}\{(-1)^{m-p}\frac{n}{m+n-p}\sum_{t=1}^{n-1}(-1)^{t+1}\binom{n}{t}B^{*m+n-t-1}B^{m+n-t-1} \\ + & \sum_{t=1}^{n-1}(-1)^{m-p+t}\frac{n-t}{m+n-p-t}\binom{n}{t}B^{*m+n-t-1}B^{m+n-t-1} \\ + & (-1)^{m+n-p+1}\frac{n}{m+n-p}B^{*m-1}B^{m-1}\} + (-1)^{m+n-p+2}B^{*p-1}B^{p-1} \\ = & \ 0 \end{aligned}$$

for all $1 \le p \le m-1$. Simplifying one obtains

$$\begin{aligned} & \binom{m+n-p-1}{m-p-1}\sum_{t=1}^{n-1}(-1)^{m+t-p+1}\frac{t}{m+n-p-t}\binom{n}{t}B^{*m+n-t-1}B^{m+n-t-1} \\ + & (-1)^{m+n-p+1}\binom{m+n-p-1}{m-p}B^{*m-1}B^{m-1} + (-1)^{m+n-p+2}B^{*p-1}B^{p-1} \\ = & \ 0 \end{aligned}$$

for all $1 \le p \le m-1$.

The following lemma provides the other half of the proof of our main result; it says that a suitably modified version of the preceding equality extends to all $1 \le p \le m-k$ for all integers $0 \le k \le m-1$.

Lemma 2.2. *If $A, B \in B(\mathcal{H})$, A is a strict m-isometry and $A \otimes B$ is an $(m+n-1)$-isometry, then*

$$\begin{aligned} & \binom{m+n-p-k}{m-p-k}\sum_{t=k}^{n-1}(-1)^{m+t-p+k}\frac{t-k+1}{m+n-p-t}\binom{n}{t-k+1}B^{*m+n-t-1}B^{m+n-t-1} \\ + & (-1)^{m+n-p+k}\{\binom{m+n-p-k}{m-p}\binom{m-p-1}{k-1}B^{*m-1}B^{m-1} + (-1)\binom{m+n-p-k}{m-p-1}\binom{m-p-2}{k-2}B^{*m-2}B^{m-2} \\ + & \cdots + (-1)^{k-1}\binom{m+n-p-k}{m-p-k+1}B^{*m-k}B^{m-k} + (-1)^k B^{*p-1}B^{p-1}\} \\ = & \ 0 \end{aligned}$$

for all $1 \le p \le m-k$.

Proof. We use induction. As seen above, the lemma holds for $k = 1$; assume that it is true for $k = 1, 2, ..., r$, i.e., the equality of the statement of the lemma holds for $k = r$. Choose $p = m - r$ to obtain

$$\begin{aligned} B^{*m+n-r-1}B^{m+n-r-1} = & \sum_{t=r+1}^{n-1}(-1)^{t+r+1}\frac{t-r+1}{n+r-t}\binom{n}{t-r+1}B^{*m+n-t-1}B^{m+n-t-1} \\ + & (-1)^{n+r+1}\{\binom{n}{r}B^{*m-1}B^{m-1} + (-1)\binom{n}{r-1}B^{*m-2}B^{m-2} \\ + & \cdots + (-1)^{r-1}\binom{n}{1}B^{*m-r}B^{m-r} + (-1)^r B^{*m-r-1}B^{m-r-1}\}, \end{aligned}$$

and then substituting back for $B^{*m+n-r-1}B^{m+n-r-1}$ we have

$$\begin{aligned}
0 &= \binom{m+n-p-r}{m-p-r}\{(-1)^{m-p+2r}\frac{1}{m+n-p-r}\binom{n}{1}\sum_{t=r+1}^{n-1}(-1)^{t+r+1}\frac{t-r+1}{n+r-t}\binom{n}{t-r+1}\\
&+ \sum_{t=r+1}^{n-1}(-1)^{m+t-p+r}\frac{t-r+1}{m+n-p-t}\binom{n}{t-r+1}\}B^{*m+n-t-1}B^{m+n-t-1}\\
&+ \binom{m+n-p-r}{m-p-r}(-1)^{m-p+2r}\frac{1}{m+n-p-r}\binom{n}{1}(-1)^{n+r+1}\{\binom{n}{r}B^{*m-1}B^{m-1}\\
&+ (-1)\binom{n}{r-1}B^{*m-2}B^{m-2}+\cdots+(-1)^{r-1}\binom{n}{1}B^{*m-r}B^{m-r}+(-1)^{r}B^{*m-r-1}B^{m-r-1}\}\\
&+ (-1)^{m+n-p+r}\{\binom{m+n-p-r}{m-p}\binom{m-p-1}{r-1}B^{*m-1}B^{m-1}+(-1)\binom{m+n-p-r}{m-p-1}\binom{m-p-2}{r-2}B^{*m-2}B^{m-2}\\
&+ \cdots+(-1)^{r-1}\binom{m+n-p-r}{m-p-r+1}B^{*m-r}B^{m-r}+(-1)^{r}B^{*p-1}B^{p-1}\}\\
&= \sum_{t=r+1}^{n-1}(-1)^{m+t-p+r+1}\{\frac{t-r+1}{n+r-t}\binom{m+n-p-r-1}{m-p-r}\binom{n}{t-r+1}-\frac{t-r+1}{m+n-p-t}\binom{m+n-p-r}{m-p-r}\\
&\quad \binom{n}{t-r+1}\}B^{*m+n-t-1}B^{m+n-t-1}\\
&+ (-1)^{m+n-p+r+1}\{\binom{m+n-p-r-1}{m-p-r}\binom{n}{r}-\binom{m+n-p-r}{m-p}\binom{m-p-2}{r-1}\}B^{*m-1}B^{m-1}\\
&+ (-1)^{m+n-p+r+2}\{\binom{m+n-p-r-1}{m-p-r}\binom{n}{r-1}-\binom{m+n-p-r}{m-p-1}\binom{m-p-2}{r-2}\}B^{*m-2}B^{m-2}\\
&+ \cdots+(-1)^{m+n-p+2r}\{\binom{m+n-p-r-1}{m-p-r}\binom{n}{1}-\binom{m+n-p-r}{m-p-r+1}\}B^{*m-r}B^{m-r}\\
&+ (-1)^{m+n-p+2r+1}\binom{m+n-p-r-1}{m-p-r}B^{*m-r-1}B^{m-r-1}+(-1)^{m+n-p+2r+2}B^{*p-1}B^{p-1}.
\end{aligned}$$

Simplifying one obtains

$$\begin{aligned}
&\binom{m+n-p-r-1}{m-p-r-1}\sum_{t=r+1}^{n-1}(-1)^{m-p+t+r+1}\frac{t-r}{m+n-p-t}\binom{n}{t-r}B^{*m+n-t-1}B^{m+n-t-1}\\
+\ &(-1)^{m+n-p+r+1}\{\binom{m+n-p-r-1}{m-p}\binom{m-p-1}{r}B^{*m-1}B^{m-1}\\
+\ &(-1)\binom{m+n-p-r-1}{m-p-1}\binom{m-p-2}{r-1}B^{*m-2}B^{m-2}+\cdots+(-1)^{r-1}\binom{m+n-p-r-1}{m-p-r+1}\binom{m-p-r}{1}B^{*m-r}B^{m-r}\\
+\ &(-1)^{r}\binom{m+n-p-r-1}{m-p-r}B^{*m-r-1}B^{m-r-1}+(-1)^{r+1}B^{*p-1}B^{p-1}\}\\
=\ &0.
\end{aligned}$$

This proves the lemma. □

We prove now our main result.

Theorem 2.3. *Let $A \in B(\mathcal{H})$ be a strict m-isometry. Then:*
(i) $A \otimes B$ is an $(m+n-1)$-isometry if and only if $B \in B(\mathcal{H})$ is an n-isometry.
(ii) $A \otimes B$ is a strict $(m+n-1)$-isometry if and only if $B \in B(\mathcal{H})$ is a strict n-isometry.

Proof. (i) The implication "A is an m-isometry and B is an n-isometry implies $A \otimes B$ is an $(m+n-1)$-isometry is Lemma 2.1". Conversely, if A is a strict m-isometry and $A \otimes B$ is an $(m+n-1)$-isometry, then choosing

$k = m-1$ in Lemma 2.2 we have $p = 1$ and hence:

$$\begin{aligned} 0 &= \sum_{t=m-1}^{n-1}(-1)^t\frac{t-m+2}{m+n-t-1}\binom{n}{t-m+2}B^{*m+n-t-1}B^{m+n-t-1} + (-1)^n\{\binom{n}{m-1}B^{*m-1}B^{m-1} \\ &+ (-1)\binom{n}{m-2}B^{*m-2}B^{m-2} + \cdots + (-1)^{m-2}\binom{n}{1}B^*B + (-1)^{m-1}I\} \\ &= \sum_{t=0}^{n-m}(-1)^{t+m-1}\binom{n}{t}B^{*n-t}B^{n-t} + (-1)^n\{\binom{n}{m-1}B^{*m-1}B^{m-1} \\ &+ (-1)\binom{n}{m-2}B^{*m-2}B^{m-2} + \cdots + (-1)^{m-2}\binom{n}{1}B^*B + (-1)^{m-1}I\} \\ &= \sum_{t=0}^{n}(-1)^t\binom{n}{t}B^{*n-t}B^{n-t}. \end{aligned}$$

This proves (i).

(ii) It is clear from the above that if $A \otimes B$ is an $(m+n-1)$-isometry, then B is an n-isometry: we claim that B is strict if and only if $A \otimes B$ is strict. Suppose, to start with, that $A \otimes B$ is strict but B is not a strict n-isometry. Then there exists an integer k, $1 \le k < n$, such that B is a k-isometry, and hence $A \otimes B$ is an $(m+k-1)$-isometry (see Lemma 2.1). Since $m+k-1 < m+n-1$, we have a contradiction. If, instead, B is strict and $A \otimes B$ is not a strict $(m+n-1)$-isometry, then $A \otimes B$ is a k-isometry for some $m \le k < m+n-1$; hence, see part (i), B is a $(k-m+1)$-isometry. Since $k-m+1 < n$, this is a contradiction. □

Since an $A \in B(\mathcal{H})$ is a strict m-isometry if and only if $I \otimes A$ and $A \otimes I$ are strict m-isometries, and since $A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$, Theorem 2.3 implies:

Corollary 2.4. *If $A \in B(\mathcal{H})$ (resp., $B \in B(\mathcal{H})$) is a strict m-isometry, then: (i) $A \otimes B$ is an $(m+n-1)$-isometry if and only if B (resp., A) is an n-isometry; (ii) $A \otimes B$ is a strict $(m+n-1)$-isometry if and only if B (resp., A) is a strict n-isometry.*

If $C_2(\mathcal{H})$ denotes the (separable) Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$ and $L_A \in B(C_2(\mathcal{H}))$ (resp., $R_A \in B(C_2(\mathcal{H}))$) is the operator $L_A(T) = AT$ (resp., $R_A(T) = TA$) of left (resp., right) multiplication by A, then the tensor product $A \otimes B^*$ may be identified with the (length one elementary) operator $L_AR_B|_{C_2(\mathcal{H})}$, $L_AR_B(T) = ATB$ for all $T \in C_2(\mathcal{H})$ [6]. Consequently, Corollary 2.4 implies the following generalization of the results of [5, Section 4].

Corollary 2.5. *If $A \in B(\mathcal{H})$ (resp., $B^* \in B(\mathcal{H})$) is a strict m-isometry, then: (i) $L_AR_B|_{B(C_2(\mathcal{H}))}$ is an $(m+n-1)$-isometry if and only if B^* (resp., A) is an n-isometry; (ii) $L_AR_B|_{B(C_2(\mathcal{H}))}$ is a strict $(m+n-1)$-isometry if and only if B^* (resp., A) is a strict n-isometry.*

We conclude this paper with the following conjecture:

Let $A, B \in B(\mathcal{H})$ be such that $A \otimes B$ is a strict n-isometry for some integer $n \ge 1$. Then there exists an integer p, $0 \le p \le n-1$, and a non-zero real number c such that cA (or, cB) is a strict $(n-p)$-isometry and $\frac{1}{c}B$ (resp., $\frac{1}{c}A$) is a strict $(p+1)$-isometry.

The following observations lend support to this conjecture. If $A \otimes B$ is isometric (hence, strictly 1-isometric), then $(A \otimes B)^*(A \otimes B) - I \otimes I = 0$ if and only if there exists a scalar $c > 0$ such that $A^*A = cI$ and $B^*B = \frac{1}{c}I$, equivalently if and only if $\frac{1}{\sqrt{c}}A$ and $\sqrt{c}B$ are isometric. If $A \otimes B$ is strictly 2-isometric, then there exists a scalar $c > 0$ such that cA (resp., cB) is strictly 2-isometric and $\frac{1}{c}B$ (resp., $\frac{1}{c}A$) is isometric (see [4, Theorem 3.2] and [7]). Finally, if $A \otimes B$ is strictly 3-isometric, then there exists a non-zero real number c such that either cA and $\frac{1}{c}B$ are strictly 2-isometric, or, cA (resp., cB) is strictly 3-isometric and $\frac{1}{c}B$ (resp., $\frac{1}{c}A$) is isometric (see [4, Theorem 4.1] and [7]).

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References

[1] J. Agler, M. Stankus, *n-isometric transformations of Hilbert space I*, Integral Equat. Oper. Th. **21**(4)(1995) 383-429.
[2] J. Agler, M. Stankus, *n-isometric transformations of Hilbert space II*, Integral Equat. Oper. Th. **23**(1)(1995) 1-48.
[3] J. Agler, M. Stankus, *n-isometric transformations of Hilbert space III*, Integral Equat. Oper. Th. **24**(4)(1996), 379-421.
[4] F. Botelho, J. Jamison, *Isometric properties of elementary operators*, Linear Alg. Appl. **432**(1)(2010) 357-365.
[5] F. Botelho, J. Jamison, B. Zheng, *Strict isometries of arbitrary order*, Linear Alg. Appl. 436 (2012) 3303-3314. (doi:10.1016/j.laa.2011.11.022)
[6] A. Brown, C. Pearcy, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc. **17**(1966), 162-166.
[7] B. P. Duggal, *Tensor product of n-isometries*, Linear Alg. Appl. 437(2012) 307-318. (doi:10.1016/j.laa.2012.02.017)