Tensor product of \( n \)-isometries II

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Abstract. A Hilbert space operator \( A \in B(\mathcal{H}) \) is an \( m \)-isometry for some natural number \( m \) if
\[
\sum_{i=0}^{m} (-1)^i \binom{m}{i} A^{m-i} A^{n-i} = 0;
\]
A is a strict \( m \)-isometry if \( A \) is not a \( k \)-isometry for every integer \( 1 \leq k < m \). Let \( A, B \in B(\mathcal{H}) \), and let \( m, n \) be some natural numbers. If \( A \) (resp., \( B \)) is a strict \( m \)-isometry, then: (i) the tensor product \( A \otimes B \) is an \((m+n-1)\)-isometry if and only if \( B \) (resp., \( A \)) is an \( n \)-isometry; (ii) \( A \otimes B \) is a strict \((m+n-1)\)-isometry if and only if \( B \) (resp., \( A \)) is a strict \( n \)-isometry. This generalizes some results of Botelho, Jamison and Zheng [5, Section 4].

To Harrison Henry Duggal on his birthday

1. Introduction

An operator \( A \in B(\mathcal{H}) \), the algebra of operators (equivalently, bounded linear transformations) on a complex infinite dimensional Hilbert space \( \mathcal{H} \) into itself, is an \( m \)-isometry for some integer \( m \geq 1 \) if
\[
\sum_{i=0}^{m} (-1)^i \binom{m}{i} A^{m-i} A^{n-i} = 0;
\]
A is a strict \( m \)-isometry if \( A \) is not an \( (m-1) \)-isometry. Evidently, an \( m \)-isometric operator is \( k \)-isometric for all integers \( k \geq m \); hence if an \( A \in B(\mathcal{H}) \) is a strict \( m \)-isometry, then it is not a \( k \)-isometry for all integers \( 1 \leq k < m \). The class of \( m \)-isometric operators is a generalization of the class of isometric operators, and a detailed study of \( m \)-isometric operators has been carried out by Agler and Stankus in a series of papers [1–3]. For \( A, B \in B(\mathcal{H}) \), let \( L_A R_B \) denote the (length one) elementary operator of left multiplication by \( A \) and right multiplication by \( B \). A characterization of \( L_A R_B |_{C_2(\mathcal{H})} \), where \( C_2(\mathcal{H}) \) is the Hilbert-Schmidt class, which are either 2-isometries or 3-isometries, and a sufficient condition for \( L_A R_B \) to be an \( m \)-isometry, has been carried out by Botelho and Jamison [4]. More recently, Botelho et al. [5] have proved that if \( A \) (resp.,...
B) is a strict $m$-isometry for $m = 2$ or 3, then: (i) $L_A R_B$ is an $(n + m - 1)$-isometry if and only if $B^*$ (resp., $A$) is an $n$-isometry; (ii) $L_A R_B$ is a strict $(n + m - 1)$-isometry if and only if $B^*$ (resp., $A$) is a strict $n$-isometry. Generalizing [4, Proposition 4.1], see also [4, Remark 4.1], the author proved in [7, Theorem 2.10] that if $A \in B(\mathcal{H})$ is an $m$-isometry and $B \in B(\mathcal{H})$ is an $n$-isometry for some natural numbers $m$ and $n$, then the tensor product $A \otimes B$ of $A$ and $B$ is an $(m + n - 1)$-isometry. In this note we generalize the results of [5, Section 4] to prove that: If $A \in B(\mathcal{H})$ is a strict $m$-isometry, then (i) $A \otimes B$ is an $(m + n - 1)$-isometry if and only if $B \in B(\mathcal{H})$ is an $n$-isometry, and (ii) $A \otimes B$ is a strict $(m + n - 1)$-isometry if and only if $B \in B(\mathcal{H})$ is a strict $n$-isometry.

2. Results

Given two complex infinite dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, let $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$; let, for $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$, $A \otimes B \in B(\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2)$ denote the tensor product operator defined by $A$ and $B$. Evidently, an operator $A \in B(\mathcal{H})$ is an $m$-isometry if and only if $(A \otimes I)$ and $(I \otimes A) \in B(\mathcal{H} \bar{\otimes} \mathcal{H})$ are $m$-isometries. Furthermore, $A$ is a strict $m$-isometry if and only if $A \otimes I$ and $I \otimes A$ are strict $m$-isometries. Observe also that $A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$. The following lemma is [7, Theorem 2.10], and provides half of the proof of our main result.

**Lemma 2.1.** If $A \in B(\mathcal{H})$ is an $m$-isometric and $B \in B(\mathcal{H})$ is an $n$-isometric, then $A \otimes B$ is $(m + n - 1)$-isometric.

Let $m, n$ be natural numbers. If $A \in B(\mathcal{H})$ is an $m$-isometry, then

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} A^{m-i} A^{m-i} = 0 \iff A^{m} A^{m} = \sum_{i=1}^{m} (-1)^{i+1} \binom{m}{i} A^{m-i} A^{m-i}.$$ 

Furthermore, if $A$ is a strict $m$-isometry, then $A$ is not a $k$-isometry for all integers $1 \leq k < m$ and it follows that $\{I, A^{*} A, A^{*} A^{2}, ..., A^{*} A^{m-1}\}$ is a linearly independent set [5, Theorem 3.1]. Observe that if $A \in B(\mathcal{H})$ is $m$-isometric and $A \otimes B$ is $(m + n - 1)$-isometric, then (it follows from a straightforward calculation that)

$$A^{m+i} A^{m+j} = \sum_{i=0}^{m-1} (-1)^{i+j} \binom{m+j}{i+j} A^{m-i} A^{m-1-i}$$

for all integers $0 \leq j$ and

$$0 = \sum_{i=0}^{m+n-1} (-1)^i \binom{m+n-1}{i} A^{m+n-1-i} A^{m+n-1-i} \otimes B^{m+n-1-i} B^{m+n-1-i}$$

$$= \sum_{i=0}^{m-1} (-1)^i \binom{m+n-1}{i} \left( \sum_{k=0}^{m-1} (-1)^{i+k} \binom{m+n-1-i}{k} A^{m-i-k} A^{m-i-k} \right)$$

$$\otimes B^{m+n-1-i} B^{m+n-1-i}$$

$$+ \sum_{i=0}^{m+n-1} (-1)^i \binom{m+n-1}{i} A^{m+n-1-i} A^{m+n-1-i} \otimes B^{m+n-1-i} B^{m+n-1-i}$$

$$= \sum_{i=1}^{m} A^{m-i} A^{m-i} \otimes \left( (-1)^{m-p} \binom{m+n-1}{m-p} \sum_{i=0}^{n-1} (-1)^i \frac{n-t}{m+n-p-t} \binom{t}{i} \right)$$

$$B^{m+n-1-i} B^{m+n-1-i} + (-1)^{m-n} B^{n-1} B^{n-1}.$$ 

Thus, if $A$ is a strict $m$-isometry, then

$$(1) \ - (-1)^{m-n} \binom{m+n-1}{m-p} \sum_{i=0}^{n-1} (-1)^i \frac{n-t}{m+n-p-t} \binom{t}{i} B^{m+n-1-i} B^{m+n-1-i} + (-1)^{m-n} B^{n-1} B^{n-1} = 0$$
for all $1 \leq p \leq m$. Choose $p = m$ to conclude

$$B^{m+n-1}B^{m+n-1} = \sum_{i=1}^{n-1} (-1)^{i+1}(\binom{n}{i})B^{m+n-i-1}B^{m+n-i-1} + (-1)^{n+1}B^{m-1}B^{m-1}$$

and substitute for $B^{m+n-1}B^{m+n-1}$ in equation (1) to obtain

$$\sum_{i=1}^{n-1} (-1)^{m-p+1} \frac{n}{m+n-p} \binom{n}{i} B^{m+n-i-1}B^{m+n-i-1} + \sum_{i=1}^{n-1} (-1)^{m-p+1} \frac{n-1}{m+n-p-1} \binom{n}{i} B^{m+n-i-1}B^{m+n-i-1}$$

$$\sum_{i=1}^{n-1} (-1)^{m-p+1} \frac{n-1}{m+n-p-1} \binom{n}{i} B^{m+n-i-1}B^{m+n-i-1} + (-1)^{n+1}B^{m-1}B^{m-1} = 0$$

for all $1 \leq p \leq m-1$. Simplifying one obtains

$$\sum_{i=1}^{n-1} (-1)^{m-p+1} \frac{n}{m+n-p} \binom{n}{i} B^{m+n-i-1}B^{m+n-i-1} + (-1)^{n+1}B^{m-1}B^{m-1} = 0$$

for all $1 \leq p \leq m-1$.

The following lemma provides the other half of the proof of our main result; it says that a suitably modified version of the preceding equality extends to all $1 \leq p \leq m - k$ for all integers $0 \leq k \leq m - 1$.

**Lemma 2.2.** If $A, B \in B(H)$, $A$ is a strict $m$-isometry and $A \otimes B$ is an $(m + n - 1)$-isometry, then

$$\sum_{i=1}^{n-1} (-1)^{m-p+1} \frac{n}{m+n-p} \binom{n}{i} B^{m+n-i-1}B^{m+n-i-1} + (-1)^{n+1}B^{m-1}B^{m-1} = 0$$

for all $1 \leq p \leq m - k$.

**Proof.** We use induction. As seen above, the lemma holds for $k = 1$; assume that it is true for $k = 1, 2, ..., r$, i.e., the equality of the statement of the lemma holds for $k = r$. Choose $p = m - r$ to obtain

$$B^{m+n-r-1}B^{m+n-r-1} = \sum_{i=r+1}^{n-1} (-1)^{i+r+1} \frac{n-r+1}{n+r-l} \binom{n}{i} B^{m+n-i-1}B^{m+n-i-1} + (-1)^{n+r+1}B^{m-1}B^{m-1} + (-1)^{r+1}B^{m-r}B^{m-r}$$

for all $1 \leq p \leq m - k$. 

and then substituting back for $B^{m+n-r-1}B^{m+n-r-1}$ we have

\[
0 = \left( \binom{m+n-p-r}{m-p} \right) (-1)^{m+2p} \frac{n}{m+n-p-r} \sum_{r=1}^{n-1} (-1)^{r+1} \frac{t-r+1}{n+r-t} \binom{n}{r+1} \cdot B^{m+n-1}B^{m+n-1}
\]

Simplifying one obtains

\[
\left( \binom{n}{r+1} \right) B^{m+n-l-1}B^{m+n-l-1}
\]

We prove now our main result.

**Theorem 2.3.** Let $A \in B(\mathcal{H})$ be a strict $m$-isometry. Then:

(i) $A \otimes B$ is an $(m+n-1)$-isometry if and only if $B \in B(\mathcal{H})$ is an $n$-isometry.

(ii) $A \otimes B$ is a strict $(m+n-1)$-isometry if and only if $B \in B(\mathcal{H})$ is a strict $n$-isometry.

**Proof.** (i) The implication “$A$ is an $m$-isometry and $B$ is an $n$-isometry implies $A \otimes B$ is an $(m+n-1)$-isometry is Lemma 2.1”. Conversely, if $A$ is a strict $m$-isometry and $A \otimes B$ is an $(m+n-1)$-isometry, then choosing
\( k = m - 1 \) in Lemma 2.2 we have \( p = 1 \) and hence:

\[
0 = \sum_{t=0}^{n-m} (-1)^t \frac{t - m + 2}{m + n - t - 1} (m_{n-t}^{n}) B^{m+n-t-1} B^{m-t-1} + (-1)^n (m_{n-1}) B^{m-1} B^{m-1}
\]

This proves (i).

(ii) It is clear from the above that if \( A \otimes B \) is an \((m + n - 1)\)-isometry, then \( B \) is an \( n \)-isometry: we claim that \( B \) is strict if and only if \( A \otimes B \) is strict. Suppose, to start with, that \( A \otimes B \) is strict but \( B \) is not a strict \( n \)-isometry. Then there exists an integer \( k, 1 \leq k < n \), such that \( B \) is a \( k \)-isometry, and hence \( A \otimes B \) is an \((m + k - 1)\)-isometry (see Lemma 2.1). Since \( m + k - 1 < m + n - 1 \), we have a contradiction. If, instead, \( B \) is strict and \( A \otimes B \) is not a strict \((m + n - 1)\)-isometry, then \( A \otimes B \) is a \( k \)-isometry for some \( m \leq k < m + n - 1 \); hence, see part (i), \( B \) is a \((k - m + 1)\)-isometry. Since \( k - m + 1 < n \), this is a contradiction. \( \square \)

Since an \( A \in B(\mathcal{H}) \) is a strict \( m \)-isometry if and only if \( I \otimes A \) and \( A \otimes I \) are strict \( m \)-isometries, and since \( A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I) \), Theorem 2.3 implies:

**Corollary 2.4.** If \( A \in B(\mathcal{H}) \) (resp., \( B \in B(\mathcal{H}) \)) is a strict \( m \)-isometry, then: (i) \( A \otimes B \) is an \((m + n - 1)\)-isometry if and only if \( B \) (resp., \( A \)) is an \( n \)-isometry; (ii) \( A \otimes B \) is a strict \((m + n - 1)\)-isometry if and only if \( B \) (resp., \( A \)) is a strict \( n \)-isometry.

If \( C_2(\mathcal{H}) \) denotes the (separable) Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H} \) and \( L_A \in B(C_2(\mathcal{H})) \) (resp., \( R_A \in B(C_2(\mathcal{H})) \)) is the operator \( L_A(T) = AT \) (resp., \( R_A(T) = TA \)) of left (resp., right) multiplication by \( A \), then the tensor product \( A \otimes B \) may be identified with the (length one elementary) operator \( L_A R_B |_{B(C_2(\mathcal{H}))} \), \( L_A R_B(T) = ATB \) for all \( T \in C_2(\mathcal{H}) \) [6]. Consequently, Corollary 2.4 implies the following generalization of the results of [5, Section 4].

**Corollary 2.5.** If \( A \in B(\mathcal{H}) \) (resp., \( B \in B(\mathcal{H}) \)) is a strict \( m \)-isometry, then: (i) \( L_A R_B |_{B(C_2(\mathcal{H}))} \) is an \((m + n - 1)\)-isometry if and only if \( B \) (resp., \( A \)) is an \( n \)-isometry; (ii) \( L_A R_B |_{B(C_2(\mathcal{H}))} \) is a strict \((m + n - 1)\)-isometry if and only if \( B \) (resp., \( A \)) is a strict \( n \)-isometry.

We conclude this paper with the following conjecture:

Let \( A, B \in B(\mathcal{H}) \) be such that \( A \otimes B \) is a strict \( n \)-isometry for some integer \( n \geq 1 \). Then there exists an integer \( p, 0 \leq p \leq n - 1 \), and a non-zero real number \( c \) such that \( cA \) (or, \( cB \)) is a strict \((n - p)\)-isometry and \( \frac{1}{c} B \) (resp., \( \frac{1}{c} A \)) is a strict \((p + 1)\)-isometry.

The following observations lend support to this conjecture. If \( A \otimes B \) is isometric (hence, strictly 1-isometric), then \((A \otimes B)'(A \otimes B) - I \otimes I = 0 \) if and only if there exists a scalar \( c > 0 \) such that \( A'A = cl \) and \( B'B = \frac{1}{c} I \), equivalently if and only if \( \frac{1}{c} A \) and \( \sqrt{c} B \) are isometric. If \( A \otimes B \) is strictly 2-isometric, then there exists a scalar \( c > 0 \) such that \( cA \) (resp., \( cB \)) is strictly 2-isometric and \( \frac{1}{c} B \) (resp., \( \frac{1}{c} A \)) is isometric (see [4, Theorem 3.2] and [7]). Finally, if \( A \otimes B \) is strictly 3-isometric, then there exists a non-zero real number \( c \) such that either \( cA \) and \( \frac{1}{c} B \) are strictly 2-isometric, or \( cA \) (resp., \( cB \)) is strictly 3-isometric and \( \frac{1}{c} B \) (resp., \( \frac{1}{c} A \)) is isometric (see [4, Theorem 4.1] and [7]).

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References


