Functional Analysis,
Approximation and
Computation
4:1 (2012), 33–37

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

On the solvability of a Volterra integral equation of quadratic form in the class of continuous function

E. M. El-Abd[a], S. A. El-Shehawy[b]

[a]Qassim University, College of Science, Department of Mathematics, P.O. Box 6644 Buriedah 81999, Kingdom of Saudi Arabia
[b]Qassim University, College of Science, Department of Mathematics, P.O. Box 6644 Buriedah 81999, Kingdom of Saudi Arabia

Abstract. The goal of this paper is to prove an axistence theorem for the solutions of a Volterra integral equation of quadratic form in the class of continuous function defined on the interval $[o, a]$, $a < \infty$ with the aid of the Hausdorff measure of noncompactness.

0. Introduction

The theory of integral equations plays an important part in the theory of nonlinear analysis; this due to the various applications of integral equations in many branches of mathematical physics such as neutron transportation, radiation and gases kintic theory $(c.f.\ [5,8])$. There are some treatments for solvability such kind of integral equation in $[3,4,7]$. This paper discusses the solvability of the integral equation

$$x(t) = 1 + T x(t) \int_0^t k(t,s)\,\phi(s)\,x(s)\,ds, \qquad t \in [o, a] \tag{1}$$

in the class of bounded continuous functions, where $\phi(t)$ is a given bounded continuous function on the bounded closed interval $[o, a]$, $k(t,s)$ is the kernal of our integral equation and T is a bounded continuous operator of the of the space $C = C[0,a]$ of all continuous real functions on $[0,a]$ into C.

The integral equation (1) is the general form of that in [3], where the authors investigated the existence theorem of the integral equation

$$x(t) = 1 + T x(t) \int_0^1 k(t,s)\,\phi(s)\,x(s)\,ds, \tag{2}$$

in the class $C[0,1]$ of continuous function on $[0,1]$. The space C is a Banach space, with the norm

$$\| x \| = \sup\{ |x(t)| : t \in [o, a],\ x \in C \}. \tag{3}$$

2010 *Mathematics Subject Classification*. 45D05.
Keywords. Volterra integral equation; quadratic form.
Received: October 21, 2010; Accepted: February 5, 2012
Communicated by Dragan S. Djordjević
Email addresses: emelabd@hotmail.com (E. M. El-Abd), shshehawy64@yahoo.com (S. A. El-Shehawy)

1. Preliminaries

Let m_E be the class of all nonempty, bounded subsets of a Banach space E and $B(x, r)$ a closed ball centered at x and with radius r. If θ is the zero vector of E, then we denote $B_r = B(\theta, r)$. For any $X \in m_E$ we define the Hausdorff measure $\chi(X)$ of noncompactness as [2].

$$\chi(X) = \inf\ \{r > 0 : \textit{there exists a finite subset}\ \ Y \subset E \textit{ such that}\ \ X \subset Y + B_r\}.$$

This measure has many properties and applications [1,2]. Also, the modulus of continuity $\omega(x, \varepsilon)$ of a function $x \in X, X \in m_E,\ \varepsilon > 0$ is defined as [4],

$$\omega(x, \varepsilon) = \sup\{\ |x(t) - x(s)| : t, s \in [0, a],\ \ |t - s| \leq \varepsilon\}. \tag{4}$$

If

$$\omega_0(X) = \lim_{\varepsilon \longrightarrow 0} \{\sup\{\omega(x, \varepsilon) : x \epsilon X\}\} \tag{5}$$

then, we have as in [4]

$$\chi(X) = \tfrac{1}{2}\,\omega_0(X). \tag{6}$$

In the sequel of this section, we will recall Darbo condition and Darbo fixed point theorem,

Definition 1. (Darbo condition) [2] Let M be a nonempty subset of a Banach space E and $T : M \longrightarrow E$ a continuous operator that transforms bounded sets onto bounded ones ;

T is said to satisfy Darbo condition with a constant $\alpha \geq 0$ (with respect to χ) if for any subset $X \subset M$ we have

$$\chi(TX) \leq \ \alpha\,\chi(X). \tag{7}$$

Note that if $0 \leq \alpha < 1$, then the operator T is said to be a contraction with respect to χ.

Theorem 1. [6] Let Q be a nonempty bounded closed convex subset of a Banach space E and let $T : Q \longrightarrow Q$ be a contraction with respect to χ. Then T has at least one fixed point in Q.

2. Main result

In this section, we will discuss the solvability of our integral equation (1) in the space $C[0, a]$ under the following assumptions;

(i) The kernal

$$k : [0, a] \times [0, a] \longrightarrow R$$

is continuous with respect to t and s and for each $t \in [0, a]$, the integral

$$\int_o^t |k(t, s)|\, ds \quad \text{exists;}$$

(ii) ϕ is continuous function defined on the closed interval $[0, a]$;

(iii) $T : C \longrightarrow C$ is a bounded continuous operator satisfying Darbo condition with a constant α and there a nonnegative constant b such that

$$\|Tx\| \leq b\,\|x\|, \text{ for each } \ x \in C.$$

(iv) There exists a bounded function

$$u\ ; [0, a] \longrightarrow R_+$$

such that

$$u(0) = \lim_{t \longrightarrow 0^+} u(t) = 0$$

and

$$\int_o^t \left| k(t_2,s) - k(t_1,s) \right| ds \le u(|t_2 - t_1|),$$

for all $t, t_1, t_2 \in [0,a]$.

Notice that, by using (i) and (iv), we deduce for $t \in [0,a]$ that

$$\int_o^t |k(t,s)|\,ds \le \int_o^t |k(t,s) - k(0,s)|\,ds + \int_o^t |k(0,s)|\,ds$$
$$\le u(t) + \int_o^t |k(0,s)|\,ds.$$

If $q = \sup\left\{\int_o^t |k(t,s)|\,ds : t \in [0,a]\right\}$, then $q < \infty$.

Furthermore, let $Q = \sup\left\{\int_o^t |k(t,s)|\left|\phi(s)\right| ds\right\}$, $t \in [0,a]$, then

$$Q \le q\left\|\phi\right\|,$$

which means that Q is bounded. Also, we assume that

(v) $bQ < \frac{1}{4}$,

(vi) $\alpha < 4b$.

Then, we can formulate the following existence theorem.

Theorem 2. If the above assumptions $(i)-(vi)$ are satisfied, then there exists at least a continuous function x satisfying (1).

Proof. Let A be an operator defined as

$$(Ax)(t) = 1 + Tx(t)\int_0^t k(t,s)\,\phi(s)\,x(s)\,ds, \qquad t \in [o,a]. \tag{8}$$

First, observe that due to the assumptions (i), (ii), (iii), the operator A transforms the space C into itself. Moreover, we have

$$(Ax)(t) \le 1 + |(Tx)(t)|\int_0^t |k(t,s)|\left|\phi(s)\right| |x(s)|\,ds,$$

$$\le 1 + bQ\,\|x\|^2.$$

Using (v), we see that the operator A transforms the ball B_r into itself for $r_0 \le r \le r_1$, where

$$r_0 = \frac{1-\sqrt{1-4bQ}}{2bQ}, \qquad r_1 = \frac{1+\sqrt{1-4bQ}}{2bQ}.$$

Since $0 < bQ < \frac{1}{4}$ then $r_0 < 1$, $r_1 > 1$, thus we will consider the case $r_0 = r$. Note that it is easy to see that B_r is nonempty, bounded closed and convex. In the sequel, we will show that the operator A is contraction with respect to the Hausdorff measure of noncompactness. For this, let X be a nonempty subset of B_r. Fix $x \in X$, then for arbitrary chosen t_1, $t_2 \in [0,a]$, we have

$$|(Ax)(t_2) - (Ax)(t_1)| =$$
$$\left|(Tx)(t_2)\int_0^{t_2} k(t_2,s)\phi(s)\,x(s)\,ds - (Tx)(t_1)\int_0^{t_1} k(t_1,s)\,\phi(s)\,x(s)\,ds\right|$$

$$\leq |(Tx)(t_2)| \int_0^{t_2} |k(t_2,s) - k(t_1,s)| \left|\phi(s)\right| |x(s)|\, ds$$
$$+ |(Tx)(t_2)| \int_{t_1}^{t_2} |k(t_2,s)| \left|\phi(s)\right| |x(s)|\, ds$$
$$+ |(Tx)(t_2) - (Tx)(t_1)| \int_0^{t_1} |k(t_1,s)| \left|\phi(s)\right| |x(s)|\, ds$$
$$\leq b\, \|x\|^2 \left\|\phi\right\| u(|t_2 - t_1|) + |(Tx)(t_2)|\, J + \omega(Tx, |t_2 - t_1|)\, Q\, \|x\|,$$

where
$$J = \int_{t_1}^{t_2} |k(t_2, s)| \left|\phi(s)\right| |x(s)|\, ds.$$

Since , $x \in B_r$, then we have

$$|(Ax)(t_2) - (Ax)(t_1)|\, b\, r^2 \left\|\phi\right\| u(|t_2 - t_1|) + |(Tx)(t_2)|\, J + \omega(Tx, |t_2 - t_1|)\, Q\, r.$$

Taking the limit as $t_2 \longrightarrow t_1$ and using (iv) we deduce that
$$\omega_0(AX) \leq r\, Q\, \omega_0(TX).$$

Consequently, we get
$$\chi(AX) \leq r\, Q\, \chi(Tx) \leq \alpha\, r\, Q\, \chi(X).$$

Due to (v) , (iv) and our case where we take $r = r_0$, $r_0 < 1$, we have

$$\alpha r\, Q = \alpha r_0 Q < 4b\, r_0 Q < r_0 < 1.$$

Hence, A is a contraction with respect χ on B_r . Apply *Theorem* 1, we deduce that the operator A has at least a fixed point x in C which is the solution of our integral equation (1). This completes the proof.

Example. Consider the integral equation

$$x(t) = 1 + \frac{t\, x(t)}{1+t} \int_0^t \frac{s\, e^{-(t^2+s^2)}}{s+4}\, x(s)\, ds, \quad t \in [0, a] \tag{9}$$

where
$$(Tx)(t) = \frac{t\, x(t)}{1+t} \;\; and \;\; \phi(s) = \frac{s}{s+4}, \; 0 \leq s < t, \quad t \in [0, a]$$

are continuous on $[0, a]$, and $k(t,s) = e^{-(t^2 + s^2)}$ is continuous on $[o, a] \times [0, a]$.

Note that the operator T transforms the continuous functions into continuous functions and

$$\int_o^t |k(t,s)|\, ds \leq \int_\infty^t |k(t,s)|\, ds = \frac{\sqrt{\pi}}{2} e^{-t^2} < \frac{\sqrt{\pi}}{2}, \quad for\, t \in [0, a] .$$

Also, T satisfies Darbo condition with constant $\alpha = 1$ and
$\|Tx\| \leq \|x\|$, *i.e* $b = 1$.

The function u defined in (iv) can be taken as
$$u(|t_2 - t_1|) = \left| e^{-t_2^2} - e^{-t_1^2} \right| \longrightarrow 0, \qquad as\;\; t_2 \longrightarrow t_1.$$

Finally, we can see that
$$\left\|\phi\right\| \leq 1, \quad q = \frac{\sqrt{\pi}}{2} \text{ and } Q = \frac{1}{8}.$$

Then, y using *Theorem* 2, the integral equation (9) is solvable in the space $C[0, a]$.

References

[1] J. Appell, On the solvability of nonlinear noncompact problems in function spaces with applications to integral and differential equations, Boll. Un. Mat. Ital. (6), 1-B (1982), 1161-1177.

[2] J. Banaś and K. Goebel, Measures of noncompactness in Banach spaces, Lect. Notes in pure and Appl. Math. vol. 60, Deker, New York, Basel (1980).

[3] J. Banaś, M. Lecko and W. G. El-Sayed, Existence theorem for some quadratic integral equations, J. Math. Anal. Appl. 222 (1988), 276-285.

[4] M. M. Bourai, W. G. El-Sayed and M. I. Abbas, Monotonic solutions of a class of quadratic singular integral equations of Volterra type, Int. J. Contemp. Math. Sci., 2 (2) (2007), 89-102.

[5] L. W. Busbrige, The Mathematics of Radiative Transfer, Cambridge Univ. Press. Cambridge, England, 1960.

[6] G. Darbo, Punti Uniti in transformazoioni a codominio noncompactta, Rend, Sem. Mat. Univ. Padova, 24 (1955), 84-92.

[7] W. G. El-Sayed and B. Rzepka, Nondecreasing solutions of a quadratic integral equations of Uryshon type, Comput. Math. Appl. 51 (2006), 1065-1074.

[8] C. T. Kelly, Approximation of solutions of some quadratic integral equations in transport theory, J. Integral Eq. 4 (1982), 221-237.