Functional Analysis,
Approximation and
Computation
4:1 (2012), 45–59

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Approximating common fixed point of three-step iterative algorithm with errors for four asymptotically nonexpansive mappings

Gurucharan Singh Saluja[a]

[a]*Department of Mathematics and Information Technology,*
Govt. Nagarjuna P.G. College of Science,
Raipur - 492010 (C.G.), India

Abstract. In this paper, we study three-step iterative algorithm with errors for four asymptotically nonexpansive mappings in the framework of uniformly convex Banach spaces. Also we have proved strong convergence theorem for above said algorithm and mappings by using *condition* (GA) which is a generalization of *condition* (A) [17] and a weak convergence theorem by using Opial's condition [12]. The results presented in this paper improve and extend the corresponding results of Khan and Fukhar-ud-din [6], Takahashi and Tamura [20], Boonchari and Saejung [1] and many others from the existing literature.

1. Introduction

Let E be a real Banach space, K be a nonempty subset of E. Throughout the paper, $\mathbb{N}$ denotes the set of positive integers and $F(T) = \{x \in K : Tx = x\}$ the set of fixed points of a mapping T. A mapping $T\colon K \to K$ is said to be nonexpansive if

$$\left\|Tx - Ty\right\| \leq \left\|x - y\right\|,$$

for all $x, y \in K$. T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n\| \leq k_n \left\|x - y\right\|,$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

The class of asymptotically nonexpansive mappings which is an important generalization of that nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. They proved that, if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping of K has a fixed point. Moreover, the set $F(T)$ of fixed points of T is closed and

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 47H09, 47J25.
Keywords. Asymptotically nonexpansive mapping; common fixed point; condition (GA); Opial's condition; three-step iterative algorithm with errors; strong convergence; uniformly convex Banach space; weak convergence.
Received: February 25, 2012; Accepted: March 28, 2012
Communicated by Dragan S. Djordjević
Email address: saluja_1963@rediffmail.com, saluja1963@gmail.com (Gurucharan Singh Saluja)

convex. Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see [4, 6, 7, 13–16, 21] and references therein).

Asymptotically nonexpansive mappings have been widely and extensively studied by many authors in many aspects. One is to approximate a fixed point or a common fixed point of asymptotically nonexpansive mappings by means of an iteratively constructed sequence.

In recent years, Mann iterative scheme [11], Ishikawa iterative scheme [5] and Noor iterative scheme [21] have been studied extensively by many authors. In 1995, Liu [8] introduced iterative schemes with errors as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{u_n\}$ a sequence in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$ is known as Mann iterative scheme with errors.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n + v_n, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ are sequences in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$ is known as Ishikawa iterative scheme with errors.

While it is clear that consideration of errors terms in iterative scheme is an important part of the theory, it is also clear that the iterative scheme with errors introduced by Liu [8], as in (1.1), (1.2) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (1.1), (1.2) which say that they tend to zero as n tends to infinity are, therefore, unreasonable. Xu [22] introduced a more satisfactory error term in the following iterative schemes.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \alpha_n T x_n + \beta_n x_n + \gamma_n u_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in K, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if $\gamma_n = 0$.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \\ y_n &= \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme if $\gamma_n = \gamma'_n = 0$. Chidume and Moore [2] and Takahashi and Tamura [20] studied the above schemes, respectively.

Many authors starting from Das and Debata [3] and including Khan and Takahashi [7], Shahzad and Udomene [18] and Takahashi and Tamura [20] have studied the two mappings case of iterative schemes for different types of mappings.

In 2005, Khan and Fukhar-ud-din [6] generalized iterative scheme (1.4) to the one with errors as follows

$$\begin{aligned} x_1 &= x \in K, \\ y_n &= \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n S y_n + \beta_n x_n + \gamma_n u_n, \quad n \geq 1 \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequence in K with $0 < \delta \leq \alpha_n, \alpha'_n \leq 1 - \delta < 1$.

Recently, Boonchari and Saejung [1] generalize the scheme (1.5) to three nonexpansive mappings with errors as follows:

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x_0 \in K, \\ y_n &= \alpha'_n R x_n + \beta'_n T x_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n R x_n + \beta_n S y_n + \gamma_n u_n, \quad n \geq 1 \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequence in K with $0 < \delta \leq \beta_n, \beta'_n \leq 1 - \delta < 1$.

Inspired by [6], [9] and [1], we extend the scheme (1.6) to the three-step iteration scheme with errors for **four asymptotically nonexpansive mappings** R, S, T and U. The scheme is as follows:

$$\begin{aligned} x_1 &= x_0 \in K, \\ z_n &= \alpha''_n R^n x_n + \beta''_n U^n x_n + \gamma''_n w_n, \\ y_n &= \alpha'_n R^n x_n + \beta'_n T^n z_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n R^n x_n + \beta_n S^n y_n + \gamma_n u_n, \quad n \geq 1 \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$, $\{\gamma''_n\}$ are sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = \alpha'_n+\beta'_n+\gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequence in K with $0 < \delta \leq \beta_n, \beta'_n, \beta''_n \leq 1 - \delta < 1$.

The purpose of this paper is to study newly defined iteration scheme (1.7) and prove weak and strong convergence theorems for said scheme in the framework of uniformly convex Banach spaces.

2. Preliminaries

Let E be a Banach space and let K be a nonempty closed convex subset of E. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A Banach space E is said to satisfy Opial's condition [12] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta_E(\varepsilon)$ of convexity of E by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

A mapping $T\colon K \to E$ is said to be demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in K and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ it follows that $x \in K$ and $Tx = y$.

Next we state the following useful lemmas to prove our main results:

Lemma 2.1. *([19]) Let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \le (1+\beta_n)\alpha_n + r_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^\infty \beta_n < \infty$ and $\sum_{n=1}^\infty r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists. In particular, $\{\alpha_n\}_{n=1}^\infty$ has a subsequence which converges to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2. *([15]) Let E be a uniformly convex Banach space and $0 < \alpha \le t_n \le \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \le a$, $\limsup_{n\to\infty} \|y_n\| \le a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \ge 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3. *([15], Lemma 1.6) Let E be a uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of E and $T\colon K \to K$ be asymptotically nonexpansive. Then $I - T$ is demiclosed with respect to zero.*

3. Main Results

In this section, we shall prove the weak and strong convergence theorems of the iteration scheme (1.7) to a common fixed point of the asymptotically nonexpansive mappings R, S, T and U.

Lemma 3.1. *Let E be a uniformly convex Banach space and K be its nonempty closed convex subset. Let $R, S, T, U\colon K \to K$ be four asymptotically nonexpansive mappings with sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $\sum_{n=1}^\infty (k_n^3 - 1) < \infty$. Let $\{x_n\}$ be the sequence as defined in (1.7) with the restrictions $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \gamma'_n < \infty$ and $\sum_{n=1}^\infty \gamma''_n < \infty$. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \ne \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$.*

Proof. Let $p \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$. Since R, S, T, U are asymptotically nonexpansive mappings, from (1.7) we have

$$\begin{aligned}
\|z_n - p\| &= \|\alpha''_n R^n x_n + \beta''_n U^n x_n + \gamma''_n w_n - p\| \\
&\le \alpha''_n \|R^n x_n - p\| + \beta''_n \|U^n x_n - p\| + \gamma''_n \|w_n - p\| \\
&\le \alpha''_n k_n \|x_n - p\| + \beta''_n k_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&\le (\alpha''_n + \beta''_n) k_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&= (1 - \gamma''_n) k_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&\le k_n \|x_n - p\| + A_n \qquad (3.1)
\end{aligned}$$

where $A_n = \gamma''_n \left\|w_n - p\right\|$. Since $\sum_{n=1}^{\infty} \gamma''_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$. Again from (1.7) and (3.1), we have

$$\begin{aligned}
\left\|y_n - p\right\| &= \left\|\alpha'_n R^n x_n + \beta'_n T^n z_n + \gamma'_n v_n - p\right\| \\
&\le \alpha'_n \left\|R^n x_n - p\right\| + \beta'_n \left\|T^n z_n - p\right\| + \gamma'_n \left\|v_n - p\right\| \\
&\le \alpha'_n k_n \left\|x_n - p\right\| + \beta'_n k_n \left\|z_n - p\right\| + \gamma'_n \left\|v_n - p\right\| \\
&\le \alpha'_n k_n \left\|x_n - p\right\| + \beta'_n k_n [k_n \left\|x_n - p\right\| + A_n] \\
&\quad + \gamma'_n \left\|v_n - p\right\| \\
&\le (\alpha'_n + \beta'_n) k_n^2 \left\|x_n - p\right\| + A_n k_n \beta'_n \\
&\quad + \gamma'_n \left\|v_n - p\right\| \\
&= (1 - \gamma'_n) k_n^2 \left\|x_n - p\right\| + A_n k_n \beta'_n \\
&\quad + \gamma'_n \left\|v_n - p\right\| \\
&\le k_n^2 \left\|x_n - p\right\| + B_n \qquad (3.2)
\end{aligned}$$

where $B_n = A_n k_n \beta'_n + \gamma'_n \left\|v_n - p\right\|$. Since $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} A_n < \infty$, it follows that $\sum_{n=1}^{\infty} B_n < \infty$. Again from (1.7) and (3.2), we have

$$\begin{aligned}
\left\|x_{n+1} - p\right\| &= \left\|\alpha_n R^n x_n + \beta_n S^n y_n + \gamma_n u_n - p\right\| \\
&\le \alpha_n \left\|R^n x_n - p\right\| + \beta_n \left\|S^n y_n - p\right\| + \gamma_n \left\|u_n - p\right\| \\
&\le \alpha_n k_n \left\|x_n - p\right\| + \beta_n k_n \left\|y_n - p\right\| + \gamma_n \left\|u_n - p\right\| \\
&\le \alpha_n k_n \left\|x_n - p\right\| + \beta_n k_n [k_n^2 \left\|x_n - p\right\| + B_n] + \gamma_n \left\|u_n - p\right\| \\
&\le (a_n + b_n) k_n^3 \left\|x_n - p\right\| + \beta_n k_n B_n + \gamma_n \left\|u_n - p\right\| \\
&= (1 - \gamma_n) k_n^3 \left\|x_n - p\right\| + \beta_n k_n B_n + \gamma_n \left\|u_n - p\right\| \\
&\le k_n^3 \left\|x_n - p\right\| + H_n \\
&= [1 + (k_n^3 - 1)] \left\|x_n - p\right\| + H_n \qquad (3.3)
\end{aligned}$$

where $H_n = \beta_n k_n B_n + \gamma_n \left\|u_n - p\right\|$. Since $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, it follows that $\sum_{n=1}^{\infty} H_n < \infty$ and by assumption of the theorem $\sum_{n=1}^{\infty} (k_n^3 - 1) < \infty$. Hence by Lemma 2.1, we have $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists. This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and K be its nonempty closed convex subset. Let $R, S, T, U\colon K \to K$ be four asymptotically nonexpansive mappings with sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$. Let $\{x_n\}$ be the sequence as defined in (1.7) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$ and $0 < t_1 \le \beta_n, \beta'_n, \beta''_n \le t_2 < 1$ for some $t_1, t_2 \in (0, 1)$. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$,*

$$\left\|x - Sy\right\| \le \left\|Rx - Sy\right\|, \ \forall x, y \in K. \qquad (3.4)$$

and

$$\|x - Rx\| \le \|Ux - Rx\|, \ \forall x \in K. \qquad (3.5)$$

Then

$$\lim_{n\to\infty} \|Rx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ux_n - x_n\| = 0,$$

for all $p \in \mathcal{F}$.

Proof. From Lemma 3.1 we get $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = r$. Then if $r = 0$, we are done. Assume that $r > 0$. Next, we show that $\lim_{n\to\infty}\|R^n x_n - S^n y_n\| = 0$. We note that $\{u_n - R^n x_n - p\}$ is a bounded sequence, so $\lim_{n\to\infty}\gamma_n\|u_n - R^n x_n - p\| = 0$. From (3.2) we have

$$\|y_n - p\| \le k_n^2\|x_n - p\| + B_n,\ \ n \ge 1,$$

where $B_n = A_n k_n \beta_n' + \gamma_n'\|v_n - p\|$ such that $\sum_{n=1}^{\infty} B_n < \infty$.

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\|y_n - p\| &\le \limsup_{n\to\infty}[k_n^2\|x_n - p\| + B_n]\\ &= \limsup_{n\to\infty}\|x_n - p\| = r. \end{aligned}\tag{3.6}$$

Note that

$$\limsup_{n\to\infty}\|S^n y_n - p\| \le \limsup_{n\to\infty} k_n\|y_n - p\| = r. \tag{3.7}$$

Also,

$$\limsup_{n\to\infty}\|R^n x_n - p\| \le \limsup_{n\to\infty} k_n\|x_n - p\| = r. \tag{3.8}$$

Observe that

$$\begin{aligned} r &= \lim_{n\to\infty}\|x_{n+1} - p\|\\ &= \lim_{n\to\infty}\|\alpha_n R^n x_n + \beta_n S^n y_n + \gamma_n u_n - p\|\\ &= \lim_{n\to\infty}\|(1-\beta_n)R^n x_n + \beta_n S^n y_n + \gamma_n u_n - \gamma_n R^n x_n - p\|\\ &= \lim_{n\to\infty}\|(1-\beta_n)(R^n x_n - p) + \beta_n(S^n y_n - p) + \gamma_n(u_n - R^n x_n - p)\|\\ &= \lim_{n\to\infty}\|(1-\beta_n)(R^n x_n - p) + \beta_n(S^n y_n - p)\|. \end{aligned}\tag{3.9}$$

From (3.7) - (3.9) and using Lemma 2.2 we have

$$\lim_{n\to\infty}\|R^n x_n - S^n y_n\| = 0. \tag{3.10}$$

Using (3.4), it follows then that

$$\begin{aligned}\|R^n x_n - x_n\| &\le \|R^n x_n - S^n y_n\| + \|S^n y_n - x_n\|\\ &\le 2\|R^n x_n - S^n y_n\| \to 0 \ \ as\ \ n\to\infty, \end{aligned}\tag{3.11}$$

and hence

$$\begin{aligned}\|S^n y_n - x_n\| &\le \|S^n y_n - R^n x_n\| + \|R^n x_n - x_n\|\\ &\to 0 \ \ as\ \ n\to\infty. \end{aligned}\tag{3.12}$$

Again, we observe that for each $n \geq 1$,

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\leq \|x_n - S^n y_n\| + k_n \|y_n - p\|, \end{aligned} \tag{3.13}$$

using (3.12), we obtain

$$r = \lim_{n\to\infty} \|x_n - p\| \leq \liminf_{n\to\infty} \|y_n - p\|.$$

This together with (3.6) gives

$$\lim_{n\to\infty} \|y_n - p\| = r. \tag{3.14}$$

Now from (3.1)we have

$$\|z_n - p\| \leq k_n \|x_n - p\| + A_n \ \ n \geq 1, \tag{3.15}$$

where $A_n = \gamma''_n \|w_n - p\|$ such that $\sum_{n=1}^{\infty} A_n < \infty$.

Taking $\limsup_{n\to\infty}$ on both sides of (3.15), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \|z_n - p\| &\leq \limsup_{n\to\infty}[k_n \|x_n - p\| + A_n] \\ &\leq \lim_{n\to\infty} \|x_n - p\| = r. \end{aligned} \tag{3.16}$$

Also,

$$\limsup_{n\to\infty} \|T^n z_n - p\| \leq \limsup_{n\to\infty}[k_n \|z_n - p\|] = r, \tag{3.17}$$

and

$$\limsup_{n\to\infty} \|R^n x_n - p\| \leq \limsup_{n\to\infty}[k_n \|x_n - p\|] = r. \tag{3.18}$$

Now from (3.14) and the boundedness of the sequence $\{v_n - R^n x_n - p\}$, we have

$$\begin{aligned} r &= \lim_{n\to\infty} \|y_n - p\| \\ &= \lim_{n\to\infty} \|\alpha'_n R^n x_n + \beta'_n T^n z_n + \gamma'_n v_n - p\| \\ &= \lim_{n\to\infty} \|(1 - \beta'_n) R^n x_n + \beta'_n T^n z_n + \gamma'_n v_n - \gamma'_n R^n x_n - p\| \\ &= \lim_{n\to\infty} \|(1 - \beta'_n)(R^n x_n - p) + \beta'_n (T^n z_n - p) + \gamma'_n (v_n - R^n x_n - p)\| \\ &= \lim_{n\to\infty} \|(1 - \beta'_n)(R^n x_n - p) + \beta'_n (T^n z_n - p)\|. \end{aligned} \tag{3.19}$$

From (3.17), (3.18) and (3.19), using Lemma 2.2 we have

$$\lim_{n\to\infty} \|R^n x_n - T^n z_n\| = 0, \tag{3.20}$$

and hence

$$\begin{aligned} \|T^n z_n - x_n\| &\leq \|T^n z_n - R^n x_n\| + \|R^n x_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.21}$$

Again note that

$$\limsup_{n\to\infty} \left\|U^n x_n - p\right\| \leq \limsup_{n\to\infty} [k_n \left\|x_n - p\right\|] = r, \tag{3.22}$$

and

$$\limsup_{n\to\infty} \left\|R^n x_n - p\right\| \leq \limsup_{n\to\infty} [k_n \left\|x_n - p\right\|] = r. \tag{3.23}$$

Also,

$$\begin{aligned} \left\|x_n - p\right\| &\leq \|x_n - T^n z_n\| + \left\|T^n z_n - p\right\| \\ &\leq \|x_n - T^n z_n\| + k_n \left\|z_n - p\right\|. \end{aligned} \tag{3.24}$$

Using (3.21) in (3.24), we obtain

$$r = \left\|x_n - p\right\| \leq \liminf_{n\to\infty} \left\|z_n - p\right\|. \tag{3.25}$$

This together with (3.16) gives

$$\lim_{n\to\infty} \left\|z_n - p\right\| = r. \tag{3.26}$$

Now from (3.26) and the boundedness of the sequence $\{w_n - R^n x_n - p\}$, we have

$$\begin{aligned} r &= \lim_{n\to\infty} \left\|z_n - p\right\| \\ &= \lim_{n\to\infty} \left\|\alpha''_n R^n x_n + \beta''_n U^n x_n + \gamma''_n w_n - p\right\| \\ &= \lim_{n\to\infty} \left\|(1 - \beta''_n) R^n x_n + \beta''_n U^n x_n + \gamma''_n w_n - \gamma''_n R^n x_n - p\right\| \\ &= \lim_{n\to\infty} \left\|(1 - \beta''_n)(R^n x_n - p) + \beta''_n (U^n x_n - p) + \gamma''_n (w_n - R^n x_n - p)\right\| \\ &= \lim_{n\to\infty} \left\|(1 - \beta''_n)(R^n x_n - p) + \beta''_n (U^n x_n - p)\right\|. \end{aligned} \tag{3.27}$$

From (3.22), (3.23) and (3.27), using Lemma 2.2 we have

$$\lim_{n\to\infty} \|R^n x_n - U^n x_n\| = 0. \tag{3.28}$$

Using (3.5), it follows that

$$\begin{aligned} \|U^n x_n - x_n\| &\leq \|U^n x_n - R^n x_n\| + \|R^n x_n - x_n\| \\ &\leq 2\|U^n x_n - R^n x_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.29}$$

Consequently, we have

$$\begin{aligned} \|x_n - T^n x_n\| &\leq \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \\ &\leq \|x_n - T^n z_n\| + k_n \|z_n - x_n\| \\ &\leq \|x_n - T^n z_n\| \\ &\quad + k_n \left\|\alpha''_n R^n x_n + \beta''_n U^n x_n + \gamma''_n w_n - x_n\right\| \\ &\leq \|x_n - T^n z_n\| + k_n \alpha''_n \|R^n x_n - x_n\| \\ &\quad + k_n \beta''_n \|U^n x_n - x_n\| + k_n \gamma''_n \|w_n - x_n\|. \end{aligned} \tag{3.30}$$

Using (3.11), (3.21) and (3.29) in (3.30), we have

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.31}$$

And

$$\begin{aligned} \|x_n - S^n x_n\| &\leq \left\|x_n - S^n y_n\right\| + \left\|S^n y_n - S^n x_n\right\| \\ &\leq \left\|x_n - S^n y_n\right\| + k_n \left\|y_n - x_n\right\| \\ &\leq \left\|x_n - S^n y_n\right\| + k_n \left\|\alpha'_n R^n x_n + \beta'_n T^n z_n + \gamma'_n v_n - x_n\right\| \\ &\leq \left\|x_n - S^n y_n\right\| + k_n[\alpha'_n \|R^n x_n - x_n\| + \beta'_n \|T^n z_n - x_n\| \\ &\quad + \gamma'_n \|v_n - x_n\|], \end{aligned} \tag{3.32}$$

using (3.11), (3.12) and (3.21) in (3.32), we have

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0. \tag{3.33}$$

Again note that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n \|R^n x_n - x_n\| + \beta_n \left\|S^n y_n - x_n\right\| \\ &\quad + \gamma_n \|u_n - x_n\|, \end{aligned} \tag{3.34}$$

using (3.11) and (3.12) in (3.34), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.35}$$

And

$$\begin{aligned} \left\|y_n - x_n\right\| &\leq \alpha'_n \|R^n x_n - x_n\| + \beta'_n \|T^n z_n - x_n\| \\ &\quad + \gamma'_n \|v_n - x_n\|, \end{aligned} \tag{3.36}$$

using (3.11) and (3.21) in (3.36), we obtain

$$\lim_{n\to\infty} \left\|y_n - x_n\right\| = 0. \tag{3.37}$$

Now, we have

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| \\ &+ \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| \\ &+ \left\|T^{n+1}x_n - Tx_n\right\|. \end{aligned} \tag{3.38}$$

Since T is uniformly L-Lipschitzian, we obtain that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| \\ &+ L\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\|. \end{aligned} \tag{3.39}$$

Using (3.30) and (3.34) in (3.39), we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.40}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - Rx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Ux_n\| = 0. \tag{3.41}$$

This completes the proof. □

We first establish the weak convergence theorem for the iteration scheme (1.7).

Theorem 3.3. *Let E be a uniformly convex Banach space satisfies the Opial's condition and K, R, S, T, U and $\{x_n\}$ be as in Lemma 3.2. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, $0 < t_1 \leq \beta_n, \beta'_n, \beta''_n \leq t_2 < 1$ for some $t_1, t_2 \in (0,1)$ and R, S, U satisfy the condition (3.4) and (3.5), then $\{x_n\}$ converges weakly to a common fixed point of the mappings R, S, T and U.*

Proof. Let $p \in \mathcal{F}$. By Lemma 3.1, we get $\lim_{n\to\infty} \left\|x_n - p\right\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F}$. To prove this, let q_1 and q_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 3.2, $\lim_{n\to\infty} \|x_n - Rx_n\| = 0$ and $I - R$ is demiclosed with respect to zero by Lemma 2.3, therefore we obtain $Rq_1 = q_1$. Similarly, $Sq_1 = q_1$, $Tq_1 = q_1$ and $Uq_1 = q_1$. Again in the same way as above, we can prove that $q_2 \in F(R) \cap F(S) \cap F(T) \cap F(U)$. Next, we prove the uniqueness. For this we suppose that $q_1 \neq q_2$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty} \left\|x_n - q_1\right\| &= \lim_{i\to\infty} \left\|x_{n_i} - q_1\right\| < \lim_{i\to\infty} \left\|x_{n_i} - q_2\right\| = \lim_{n\to\infty} \left\|x_n - q_2\right\| = \lim_{j\to\infty} \left\|x_{n_j} - q_2\right\| \\ &< \lim_{j\to\infty} \left\|x_{n_j} - q_1\right\| = \lim_{n\to\infty} \left\|x_n - q_1\right\|. \end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a common fixed point in $\mathcal{F}$. This completes the proof. □

Our next aim is to prove a strong convergence theorem.

Recall that the following:

A mapping $T\colon K \to K$ where K is a subset of E, is said to satisfy condition (A) [17] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$

for all $x \in K$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

Senter and Dotson [17] approximated fixed points of nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [10] and Tan and Xu [19] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same *condition* (A) which is weaker than the requirement that T is demicompact. We modify this condition for four mappings R, S, T and $U\colon K \to K$ as follows.

Four mappings R, S, T and $U\colon K \to K$ where K is a subset of E, are said to satisfy *condition* (GA) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $a_1\|x - Rx\| + a_2\|x - Sx\| + a_3\|x - Tx\| + a_4\|x - Ux\| \geq f(d(x,\mathcal{F}))$ for all $x \in K$, where $d(x,\mathcal{F}) = \inf\Big\{\big\|x - p\big\| : p \in \mathcal{F}\Big\}$ and a_1, a_2, a_3 and a_4 are four nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 = 1$.

Remark 3.4. *Condition* (GA) *reduces to condition* (A) *[17] when* $R = S = T = U$.

Theorem 3.5. *Let E be a uniformly convex Banach space and K, $\{x_n\}$ be taken as in Lemma 3.2. Let R, S, T, $U\colon K \to K$ be four asymptotically nonexpansive mappings satisfying condition (GA). If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, $0 < t_1 \leq \beta_n, \beta'_n, \beta''_n \leq t_2 < 1$ for some $t_1, t_2 \in (0,1)$ and R, S, U satisfy the condition (3.4) and (3.5), then $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U.*

Proof. By Lemma 3.1, we know that $\lim_{n\to\infty}\big\|x_n - p\big\|$ exists for all $p \in \mathcal{F}$. Let $\lim_{n\to\infty}\big\|x_n - p\big\| = r$ for some $r \geq 0$. If $r = 0$, we are done. Suppose that $r > 0$. By Lemma 3.2 we know that

$$\lim_{n\to\infty}\|Rx_n - x_n\| = \lim_{n\to\infty}\|Sx_n - x_n\| = \lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|Ux_n - x_n\| = 0.$$

From (3.3), we have

$$\big\|x_{n+1} - p\big\| \quad\leq\quad (1 + \theta_n)\big\|x_n - p\big\| + H_n, \qquad \forall n \geq 1, \tag{3.42}$$

where $\theta_n = (k_n^3 - 1)$ and $H_n = \beta_n k_n B_n + \gamma_n\big\|u_n - p\big\|$ with $\sum_{n=1}^{\infty}\theta_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$.

This implies that $d(x_{n+1},\mathcal{F}) \leq (1 + \theta_n)d(x_n,\mathcal{F}) + H_n$ and hence $\lim_{n\to\infty} d(x_n,\mathcal{F})$ exists by virtue of Lemma 2.1. By condition (GA), we have $\lim_{n\to\infty} f(d(x_n,\mathcal{F})) = 0$.

Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. In fact, since for any $x > 0$, $1 + x \leq exp(x)$, therefore, for any $m, n \geq 1$ and for given $p \in \mathcal{F}$, from (3.42), we have

$$\begin{aligned}
\left\|x_{n+m}-p\right\| &\leq (1+\theta_{n+m-1})\left\|x_{n+m-1}-p\right\| + H_{n+m-1} \\
&\leq exp\{\theta_{n+m-1}\}\left\|x_{n+m-1}-p\right\| + H_{n+m-1} \\
&\leq exp\{\theta_{n+m-1}\}[exp\{\theta_{n+m-2}\}\left\|x_{n+m-2}-p\right\| + H_{n+m-2}] \\
&\quad +H_{n+m-1} \\
&= exp\{\theta_{n+m-1}+\theta_{n+m-2}\}\left\|x_{n+m-2}-p\right\| \\
&\quad +exp\{\theta_{n+m-1}\}H_{n+m-2} + H_{n+m-1} \\
&\leq exp\{\theta_{n+m-1}+\theta_{n+m-2}\}\left\|x_{n+m-2}-p\right\| \\
&\quad +exp\{\theta_{n+m-1}\}[H_{n+m-1} + H_{n+m-2}] \\
&\leq \dots \\
&\leq exp\Big\{\sum_{k=n}^{n+m-1}\theta_k\Big\}\left\|x_n-p\right\| + exp\Big\{\sum_{k=n+1}^{n+m-1}\theta_k\Big\}\sum_{k=n}^{n+m-1}H_k \\
&\leq exp\Big\{\sum_{k=n}^{n+m-1}\theta_k\Big\}\left\|x_n-p\right\| + exp\Big\{\sum_{k=n}^{n+m-1}\theta_k\Big\}\sum_{k=n}^{n+m-1}H_k \\
&\leq K\Big(\left\|x_n-p\right\| + \sum_{k=n}^{n+m-1}H_k\Big) < \infty, \qquad (3.43)
\end{aligned}$$

where $K = exp\Big\{\sum_{k=1}^{\infty}\theta_k\Big\} < \infty$. Since

$$\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0, \qquad \sum_{n=1}^{\infty} H_n < \infty \qquad (3.44)$$

for any given $\varepsilon > 0$, there exists a positive integer n_1 such that

$$d(x_n,\mathcal{F}) < \frac{\varepsilon}{4(K+1)}, \qquad \sum_{k=n}^{\infty} H_k < \frac{\varepsilon}{2K} \qquad \forall n \geq n_1. \qquad (3.45)$$

Hence, there exists $q \in \mathcal{F}$ such that

$$\left\|x_n-q\right\| < \frac{\varepsilon}{2(K+1)} \qquad \forall n \geq n_1. \qquad (3.46)$$

Consequently, for any $n \geq n_1$ and $m \geq 1$, from (3.43), we have

$$\begin{aligned}
\|x_{n+m}-x_n\| &\leq \left\|x_{n+m}-q\right\| + \left\|x_n-q\right\| \\
&\leq K\Big\{\left\|x_n-q\right\| + \sum_{k=n}^{n+m-1}H_k\Big\} + \left\|x_n-q\right\| \\
&\leq (K+1)\left\|x_n-q\right\| + K\Big(\sum_{k=n}^{n+m-1}H_k\Big) \\
&< (K+1).\frac{\varepsilon}{2(K+1)} + K.\frac{\varepsilon}{2K} = \varepsilon. \qquad (3.47)
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in \mathcal{F}$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_2 such that $\left\|x_n - q^*\right\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_2$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, there exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$ we have $d(x_n, \mathcal{F}) < \frac{\varepsilon_1}{5}$ and in particular we have $d(x_{n_3}, \mathcal{F}) \leq \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in \mathcal{F}$ such that $\left\|x_{n_3} - w^*\right\| < \frac{\varepsilon_1}{4}$. For any $n \geq n_3$, we have

$$\begin{aligned} \left\|Rq^* - q^*\right\| &\leq \left\|Rq^* - w^*\right\| + \left\|w^* - q^*\right\| \\ &\leq 2\left\|q^* - w^*\right\| \\ &\leq 2\Big(\left\|q^* - x_{n_3}\right\| + \left\|x_{n_3} - w^*\right\|\Big) \\ &< 2\Big(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}\Big) \\ &< \varepsilon_1. \end{aligned}$$

This implies that $Rq^* = q^*$. Similarly, we can show that $Sq^* = q^*$, $Tq^* = q^*$ and $Uq^* = q^*$. Hence $q^* \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U. This completes the proof. □

For our next result, we shall need the following definition:

Definition 3.6. *Let K be a nonempty closed subset of a Banach space E. A mapping $T \colon K \to K$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\lim_{n\to\infty} x_{n_j} = x \in K$.*

Theorem 3.7. *Let E be a uniformly convex Banach space and K, $\{x_n\}$ be taken as in Lemma 3.2. Let R, S, T, $U \colon K \to K$ be four asymptotically nonexpansive mappings. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, $0 < t_1 \leq \beta_n, \beta'_n, \beta''_n \leq t_2 < 1$ for some $t_1, t_2 \in (0, 1)$ and R, S, U satisfy the condition (3.4) and (3.5). Suppose one of the mappings in $\{R, S, T, U\}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U.*

Proof. Suppose R is semi-compact. By Lemma 3.2, we have $\lim_{n\to\infty} \|x_n - Rx_n\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x^* \in K$. Now Lemma 3.2 guarantees that $\lim_{n_j\to\infty} \left\|x_{n_j} - Rx_{n_j}\right\| = 0$, $\lim_{n_j\to\infty} \left\|x_{n_j} - Sx_{n_j}\right\| = 0$, $\lim_{n_j\to\infty} \left\|x_{n_j} - Tx_{n_j}\right\| = 0$, $\lim_{n_j\to\infty} \left\|x_{n_j} - Ux_{n_j}\right\| = 0$ and so $\|x^* - Rx^*\| = 0$, $\|x^* - Sx^*\| = 0$, $\|x^* - Tx^*\| = 0$, $\|x^* - Ux^*\| = 0$. This implies that $x^* \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, it follows, as in the proof of Theorem 3.5, that $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U. This completes the proof. □

Remark 3.8. *(i) Our results extend the corresponding results of Boonchari and Saejung [1] to the case of more general class of nonexpansive mappings considered in this paper.*

(ii) Our results also extend and improve the corresponding results of Khan and Fukhar-ud-din [6] in the following ways:

(a) We remove the boundedness of K.

(b) The identity mapping and nonexpansive mapping in [6] is replaced by the more general asymptotically nonexpansive mapping.

(iii) Our results also extend and improve the corresponding results of Takahashi and Tamura [20] to the case of three-step iteration scheme with errors for four asymptotically nonexpansive mappings considered in this paper.

The following example shows that our results extend substantially the corresponding results in [6], [7] and [13] - [16].

Example 3.9. *Let E be the real line with the usual norm* $|.|$ *and let* $K = [-1, 1]$. *Define* $R, S, T, U \colon K \to K$ *by*

$$R(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

$$S(x) = \begin{cases} -\sin x, & \text{if } x \in [0,1], \\ \sin x, & \text{if } x \in [-1,0). \end{cases}$$

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1], \\ -\frac{x}{2}, & \text{if } x \in [-1,0). \end{cases}$$

and

$$U(x) = \begin{cases} \frac{x}{3}, & \text{if } x \in [0,1], \\ -\frac{x}{3}, & \text{if } x \in [-1,0). \end{cases}$$

for $x \in K$. *Obviously,* $F(R) \cap F(S) \cap F(T) \cap F(U) = \{0\}$. *Now we check that S is nonexpansive. In fact, if x and* $y \in [0,1]$ *or if x and* $y \in [-1,0)$, *then* $|Sx - Sy| = |\sin x - \sin y| = 2|\cos\frac{x+y}{2}\sin\frac{x-y}{2}| = 2|\sin\frac{x-y}{2}| \leq 2.|\frac{x-y}{2}| = |x-y|$; *if* $x \in [0,1]$ *and* $y \in [-1,0)$ *or* $x \in [-1,0)$ *and* $y \in [0,1]$, *then*

$$|Sx - Sy| = |\sin x + \sin y| = 2\Big|\sin\frac{x+y}{2}\cos\frac{x-y}{2}\Big| \leq |x+y| \leq |x-y|.$$

That is, S is nonexpansive. Similarly, we can verify that R, T and U are nonexpansive. Thus S is uniformly 1*-Lipschitzian and asymptotically nonexpansive with constant sequence* $\{k_n\} = \{1\}$ *for each* $n \geq 1$. *Similarly, we can show that R, T and U are also uniformly* 1*-Lipschitzian and asymptotically nonexpansive with constant sequence* $\{k_n\} = \{1\}$ *for each* $n \geq 1$. *In order to show that R, S and U satisfy the conditions (3.4) and (3.5), we have to consider the following cases:*

Case 1. Suppose that x and $y \in [0,1]$. *It follows that*

$$|x - Sy| = |x + \sin y| = |Rx - Sy|;$$

Case 2. Suppose that x and $y \in [-1,0)$. *Then we easily see that*

$$|x - Sy| = |x - \sin y| \leq |-x - \sin y| = |Rx - Sy|;$$

Case 3. Suppose that $x \in [-1,0)$ *and* $y \in [0,1]$. *It is easy to verify that*

$$|x - Sy| = |x + \sin y| \leq |-x + \sin y| = |Rx - Sy|;$$

Case 4. Suppose that $x \in [0,1]$ and $y \in [-1,0)$. It follows that

$$|x - Sy| = |x - sin\, y| = |Rx - Sy|.$$

Hence (3.4) is satisfied. Similarly, we can show that the condition (3.5) is also satisfied. Moreover, it is not difficult to see that asymptotically nonexpansive mappings R, S, T and U satisfy condition (GA).

References

[1] D. Boonchari, S. Saejung, Weak and strong convergence of a scheme with errors for three nonexpansive mappings, Rostocker Mathematisches Kolloquium 63 (2008) 25–35.
[2] C.E. Chidume, C. Moore, Fixed point iteration for pseudocontractive maps, Proceeding of the American Mathematical Society 127(4) (1999) 1163–1170.
[3] G. Das, J. P. Debate, Fixed points of quasi-nonexpansive mappings, Indian Journal of Pure and Applied Mathematics 17(11) (1986) 1263–1269.
[4] K. Goebel, W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proceeding of the American Mathematical Society 35 (1972) 171–174.
[5] S. Ishikawa, Fixed points by a new iteration method, Proceeding of the American Mathematical Society 44 (1974) 147–150.
[6] S. H. Khan, H. Fkhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Analysis 61(8) (2005) 1295–1301.
[7] S. H. Khan, W. Takahashi, Approximating common fixed points of two asymptotically nonexpansive mappings, Scientiae Mathematicae Japonicae 53(1) (2001) 143–148.
[8] L. S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, Journal of Mathematical Analysis and Applications 194(1) (1995) 114–125.
[9] Z. Liu, R. P. Agarwal, C. Feng, S. M. Kang, Weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 44 (2005) 83–96.
[10] M. Maiti, M. K. Ghosh, Approximating fixed points by Ishikawa iterates, Bulletin of the Australian Mathematical Society 40(1) (1989) 113–117.
[11] W. R. Mann, Mean value methods in iteration, Proceeding of the American Mathematical Society 4 (1953) 506–510.
[12] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bulletin of the American Mathematical Society 73 (1967) 591–597.
[13] M. O. Osilike, S. C. Aniagbosor, Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings, Mathematics and Computer Modelling 32 (2000) 1181–1191.
[14] B. E. Rhoades, Fixed point iteration for certain nonlinear mappings, Journal of Mathematical Analysis and Applications 183 (1994) 118–120.
[15] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bulletin of the Australian Mathematical Society 43(1) (1991) 153–159.
[16] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, Journal of Mathematical Analysis and Applications 158 (1991) 407–413.
[17] H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, Proceeding of the American Mathematical Society 44 (1974) 375–380.
[18] N. Shahzad, A. Udomene, Approximating common fixed points of two asymptotically quasi nonexpansive mappings in Banach spaces, Fixed point Theory and Applications (2006), Article ID 18909, Pages 1-10.
[19] K. K. Tan, H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, Journal of Mathematical Analysis and Applications 178 (1993) 301–308.
[20] W. Takahashi, T. Tamura, Convergence theorems for a pair of nonexpansive mappings, Journal of Convex Analysis 5(1) (1998) 45–56.
[21] B. L. Xu, M. A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, Journal of Mathematical Analysis and Applications 267(2) (2002) 444–453.
[22] Y. Xu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations, Journal of Mathematical Analysis and Applications 224(1) (1998) 91–101.