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On a characterization of L^p-spaces

I. Akbarbaglu[a], S. Maghsoudi[b]

[a]*Department of Mathematics, University of Zanjan, Zanjan, 45195-313, Iran*
[b]*Department of Mathematics, University of Zanjan, Zanjan, 45195-313, Iran*

Abstract. Let $(X, \mathcal{A}, \mu)$ be a positive measure space. Let p and r be real numbers such that $1 \leq r \leq p < \infty$ and let q be given by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. We consider the problem of characterizing those measure spaces for which every μ-measurable function f belongs to $L^q(\mu)$ whenever $fg \in L^r(\mu)$ for every $g \in L^p(\mu)$. For $1 \leq r < p < \infty$, we show that these measure spaces are precisely the semi-finite measure spaces.

1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a positive measure space. For $p \in (0, +\infty)$, let $L^p(\mu)$ denote, as usual, the vector space of all μ-almost everywhere equivalence classes of complex-valued μ-measurable functions f on X for which the integral $\int_X |f|^p \, d\mu$ is finite. For $p = \infty$, define $L^\infty(\mu)$ as the space of all μ-almost everywhere equivalence classes of μ-measurable functions f from X into $\mathbb{C}$ for which there exists a nonnegative number c such that $|f(x)| \leq c$ μ-almost everywhere. The following is a well-known problem:

Let $(X, \mathcal{A}, \mu)$ be a given measure space. Let f be a complex-valued measurable function defined on X such that $fg \in L^1(\mu)$ for every $g \in L^p(\mu)$, where $1 < p \leq \infty$. Show that $f \in L^q(\mu)$, where q is the exponential conjugate to p.

The problem has been formulated in this form in some textbooks; see, for instance, Exercises IV.13.71 in Dunford-Schwartz [2] and Lemma 2 in [7]. It can be easily seen the problem is not true as it is stated: take X to be an arbitrary set and define the measure μ on the power set of X by $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ otherwise, then we obtain a counter-example. The more standard form of this exercise is in the presence of the σ-finitness condition. A correct formulation of the above problem can be found in, for example, Fremlin [4], the most comprehensive text in measure theory up until now, Nielsen [8], and under certain conditions in [3].

Our aim in this note is to prove a more general form of this problem and also establish its converse.

2. Main results

Throughout this note, let $(X, \mathcal{A}, \mu)$ denote a positive measure space. Before stating our main theorem let us recall some more definition. We say that a measure space $(X, \mathcal{A}, \mu)$ is *semi-finite* (or *to have the finite*

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Email addresses: Ibrahim.akbarbaglu@znu.ac.ir (I. Akbarbaglu), s_maghsodi@znu.ac.ir (S. Maghsoudi)

subset property) whenever every measurable set of infinite measure has a measurable subset of finite positive measure. Note, this implies that in a semi-finite measure space $(X, \mathcal{A}, \mu)$ for any given number $c > 0$ and every measurable subset A with $\mu(A) = \infty$, there exists a measurable set $B \subseteq A$ with $c < \mu(B) < \infty$. The proof is simple: let

$$\alpha = \sup\Big\{\mu(F) : F \in \mathcal{A},\ F \subseteq A \text{ and } \mu(F) < \infty\Big\}.$$

Choose a nondecreasing sequence of measurable sets $(F_n)_n$ such that $\mu(F_n) < \infty$, $F_n \subseteq A$, and $\lim_{n\to\infty} \mu(F_n) = \alpha$. Now, let $C = \bigcup_{n=1}^{\infty} F_n$. The assumption that $\mu(C) < \infty$ leads to a contradiction. Hence, we could construct the desired set by a suitable selection of some F_n.

Now we are ready to state our main theorem.

Theorem 2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The following are equivalent.*

(a) *$(X, \mathcal{A}, \mu)$ is a semi-finite measure space.*

(b) *Let p and r be real numbers such that $1 \le r < p < \infty$. Suppose that q is given by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$L^q(\mu) = \Big\{f : X \to \mathbb{C} : f \text{ is measurable and } fg \in L^r(\mu) \text{ for all } g \in L^p(\mu)\Big\}.$$

Proof. (a)⇒(b). Using Hölder's inequality and taking the pair conjugate exponents p/r and q/r, we see that the left hand side of the claimed equality in (b) is contained in the right hand side.

For the converse, take f be a μ-measurable function with $f \notin L^q(\mu)$. So $|f|^r$ is μ-measurable and $|f|^r \notin L^{q/r}(\mu)$. Let $(\varphi_n)_n$ be a nondecreasing sequence of simple measurable functions such that $\varphi_n \nearrow |f|^r$ pointwise μ-almost everywhere. Since $\||f|^r\|_{q/r} = \infty$, by Levi's theorem, we find that $\lim_{n\to\infty} \|\varphi_n\|_{q/r} = \infty$. Now by the semi-finiteness of μ and passing to a subsequence, if necessary, we can choose, for each natural number n, a step function ψ_n with $0 \le \psi_n \le \varphi_n$ and $\|\psi_n\|_{q/r} > n^3$. By Theorem 31.16 in [1], for each n, there is a $g_n \in L^{p/r}(\mu)$ such that $\|g_n\|_{p/r} = 1$ and

$$\int_X \varphi_n g_n \, d\mu > n^3.$$

Put

$$g = \sum_{n=1}^{\infty} \frac{1}{n^2} |g_n|.$$

Since $L^{p/r}(\mu)$ is complete we have $g \in L^{p/r}(\mu)$. But now for each n we have

$$\int_X |f|^r g \, d\mu \ge \int_X \psi_n g \, d\mu \ge n,$$

from which it follows that $f g^{\frac{1}{r}} \notin L^r(\mu)$, while $g^{\frac{1}{r}} \in L^p(\mu)$.

(b)⇒(a). Let E be any measurable set with $\mu(E) = \infty$, and let p, q and r be as in (b). Then $\chi_E \notin L^q(\mu)$ (here, χ_E denotes the indicator of the set E). Hence, by the assumption, there exists a positive function $g \in L^p(\mu)$ such that $\int_X |g\chi_E|^r \, d\mu = \infty$. But $g\chi_E \in L^p(\mu)$ and $\|g\chi_E\|_p > 0$; this is because of our assumption about E. So, there is a simple measurable function φ with $0 \le \varphi \le g\chi_E$ such that $\|\varphi\|_p > 0$. This, in turn, guarantees the existence of a measurable subset $F \subseteq E$ with $0 < \mu(F) < \infty$; which completes the proof. □

Remark 2.2. A small modification in the above proof shows that (a)⇒(b) is true for the case when $r = p$ and $q = \infty$. But we should note that (b)⇒(a) dose not hold in this case. We give a counter-example.

Consider $X = \mathbb{N} \cup \{0\}$ and let $\mathcal{A}$ be the power set of X. Define the measure μ on $\mathcal{A}$ by setting $\mu(\{0\}) = \infty$ and $\mu(\{n\}) = 1$ for all $n \in X \setminus \{0\}$. Obviously, μ is not semi-finite. We show that

$$L^\infty(\mu) = \Big\{f : X \to \mathbb{C} : f \text{ is measurable and } fg \in L^p(\mu) \text{ for all } g \in L^p(\mu)\Big\}.$$

To see this, suppose to the contrary that there is a measurable function f with $f \notin L^\infty(\mu)$ such that $fg \in L^p(\mu)$ for every $g \in L^p(\mu)$. For each $m \in \mathbb{N}$, put

$$E_m = \Big\{x \in X : \big|f(x)\big| > \max\Big(m^3, \big|f(0)\big|\Big)\Big\}.$$

Then E_m is measurable and $\mu(E_m) > 0$ for each $m \in \mathbb{N}$. Hence, for any $m \in \mathbb{N}$, we can find an element $n_m \in \mathbb{N}$ with $n_m \in E_m$. Put $g = \sum_{m=1}^{\infty} m^{-2}\chi_{\{n_m\}}$. Then $0 \le g \in L^p(\mu)$, but $m^p \le \sum_{n=1}^{\infty} f^p(n)g^p(n) < \infty$; which is a contradiction.

3. The case of locally compact groups

Let G be a locally compact group, that is, a group with a Hausdorff topology such that the multiplication and inverse operations are continuous; for more details see [5]. It is a well-known and fundamental fact in Harmonic analysis that on every locally compact group there is a non-zero left translation invariant regular Borel measure which is also unique up to a positive constant. Let us remark that the Haar measure on a locally compact group is semi-finite if and only if it is discreet or σ-compact; see Theorem 16.14 in [5].

Proposition 3.1. *Let G be a locally compact group with a left Haar measure λ and let $1 \le p < \infty$. Then λ is semi-finite if and only if*

$$L^\infty(G) = \Big\{f : G \to \mathbb{C} : f \text{ is measurable and } fg \in L^p(G) \text{ for all } g \in L^p(G)\Big\}. \quad (*)$$

Proof. The "only if" part follows from Remark 2.2. It only remains to prove the "if" part. For this, suppose that G is not semi-finite. So G is neither discrete nor σ-compact. It is a well known fact that every locally compact group G contains an open and closed σ-compact subgroup H; see Proposition 2.4 in [**?**]. Since the quotient space G/H is uncountable, we can choose an infinite sequence $(E_n)_n$ of mutually disjoint subsets of $G \setminus H$ with $\lambda(E_n) = \infty$ for each $n \in \mathbb{N}$. In particular, for each n, the intersection of E_n with any measurable subset with finite measure has measure zero. Now set $f = \sum_{n=1}^{\infty} 2^n \chi_{E_n}$. Plainly, f is measurable and $f \notin L^\infty(G)$. Moreover, if $g \in L^p(G)$ then since $\{x \in G : g(x) \neq 0\}$ is σ-compact, we find that $\int_G fg\, d\lambda = 0$ and, by the assumption, $f \in L^\infty(G)$. But this is a contradiction. □

At the end we must warn the reader of the difference in defining $L^\infty(X, \mu)$, X a locally compact space and μ a Radon measure, in measure theory textbooks. Some authors define $L^\infty(X, \mu)$ as the vector space of all locally almost everywhere equivalence classes of all μ-measurable complex-valued functions, that are bounded except for possibly on a locally null set. (A subset A is a *locally null set* if $A \cap K$ is null for every compact subset K.) This makes a big difference; with this new definition $(*)$ is true for all locally compact groups; for a proof see Theorem 20.15 in [6].

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