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Almost Jordan homomorphisms and Jordan derivations on fuzzy Banach algebras

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Abstract. In this paper, we establish the generalized Hyers–Ulam stability of Jordan homomorphisms and Jordan derivations on fuzzy Banach algebras.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [20] in 1940, concerning the stability of group homomorphisms. *Let* $(G_1, .)$ *be a group and let* $(G_2, *)$ *be a metric group with the metric* $d(., .)$. *Given* $\epsilon > 0$, *does there exist a* $\delta 0$, *such that if a mapping* $h : G_1 \longrightarrow G_2$ *satisfies the inequality* $d(h(x.y), h(x)*h(y)) < \delta$ *for all* $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \longrightarrow G_2$ *with* $d(h(x), H(x)) < \epsilon$ *for all* $x \in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [13] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Th. M. Rassias [18] proved the following theorem.

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Theorem 1.1. *Let $f : E \longrightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{2}$$

for all $x \in E$. If $p < 0$ then inequality (1) holds for all $x, y \neq 0$, and (2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1991, Z. Gajda [11] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as the generalized Hyers–Ulam stability of functional equations.

Following [1], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:

(N_1) $N(x,a) = 0$ for $a \leq 0$;

(N_2) $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;

(N_3) $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;

(N_4) $N(x+y, a+b) \geq \min\{N(x,a), N(y,b)\}$;

(N_5) $N(x,.)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x,a) = 1$;

(N_6) For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement "the norm of x is less than or equal to the real number a".

Example 1.2. *Let $(X, \|.\|)$ be a normed linear space. Then*

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0 \quad, x \in X; \\ 0, & a \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each a there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$. It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Let X be an algebra and (X,N) be complete fuzzy normed space, the pair (X,N) is said to be a fuzzy Banach algebra if for every $x, y \in X$, $a, b \in \mathbb{R}$

$$N(xy, ab) \geq \max\{N(x,a), N(y,b)\}. \tag{3}$$

Let (X,N) be a fuzzy Banach algebra and $\{x_n\}, \{y_n\}$ be convergent sequences in (X,N) such that $N - \lim_{n\to\infty} x_n = x$ and $N - \lim_{n\to\infty} y_n = y$. Then

$$\begin{aligned} N(x_n y_n - xy, 2t) &\geq \min\{N((x_n - x)y_n, t), N(x(y_n - y), t)\} \\ &\geq \min\{N(x_n - x, t), N(y_n - y, t)\} \end{aligned}$$

for all $t > 0$. Therefore $N - \lim_{n\to\infty} x_n y_n = xy$.

The generalized Hyers–Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [7, 12, 14] and [19].

Definition 1.3. *Suppose A and B are two Banach algebras. We say that a mapping $h : A \to B$ is a Jordan homomorphism if*

$$h(a+b) = h(a) + h(b)$$

and

$$h(a^2) = h(a)^2$$

for all $a, b \in A$.

Definition 1.4. *Suppose A is a Banach algebra. We say that a mapping $d : A \to A$ is a Jordan derivation if*

$$d(a+b) = d(a) + d(b)$$

and

$$d(a^2) = ad(a) + d(a)a$$

for all $a, b \in A$.

The stability of different functional equations on Banach algebras has been recently studied in [2], [3]–[6],[8]–[10] and [16]–[17].

In the present paper, we investigate the generalized Hyers–Ulam stability of Jordan homomorphisms and Jordan derivations on fuzzy Banach algebras.

2. Fuzzy stability of Jordan homomorphisms

In this section we investigate the fuzzy stability of Jordan homomorphisms.

Theorem 2.1. *Suppose (A, N) and (B, N) are two fuzzy Banach algebras and (C, N') be a fuzzy normed space. Let $\varphi : A \times A \to C$ be a function such that for some $0 < \alpha < 2$,*

$$N'(\varphi(2a, 2b), t) \geq N'(\alpha\varphi(a, b), t) \tag{4}$$

for all $a, b \in A$ and all $t > 0$. If $f : A \to B$ is a mapping such that

$$N(f(a+b) - f(a) - f(b), t) \geq N'(\varphi(a, b), t) \tag{5}$$

and

$$N(f(a^2) - f(a)^2, s) \geq N'(\varphi(a, a), s) \tag{6}$$

for all $a, b \in A$ and all $t, s > 0$. Then there exists a unique Jordan homomorphism $h : A \to B$ such that

$$N(f(a) - h(a), t) \geq N'\Big(\frac{2\varphi(a, a)}{2-\alpha}, t\Big) \tag{7}$$

where $a \in A$ and $t > 0$.

Proof. Using (5) and Theorem 3.1 of [15], the mapping $h : A \to B$ defined by $h(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ is an additive function satisfying (7). Now we only need to show that $h(a^2) = h(a)^2$ for all $a \in A$. It follows from (4) that

$$N(f(2^n a) - h(2^n a), t) \geq N'\Big(\frac{2\varphi(2^n a, 2^n a)}{2-\alpha}, t\Big) \geq N'\Big(\alpha^n \varphi(a, a), t\frac{2-\alpha}{2}\Big)$$

for all $a \in A$ and all $t > 0$. Thus

$$N(2^{-n} f(2^n a) - 2^{-n} h(2^n a), 2^{-n} t) \geq N'\Big(\varphi(a, a), t\frac{2-\alpha}{2\alpha^n}\Big)$$

for all $a \in A$ and all $t > 0$. By the additivity of h it is easy to see that

$$N(2^{-n}f(2^n a) - h(a), t) \geq N'\Big(\varphi(a, a), t\frac{2^n(2-\alpha)}{2\alpha^n}\Big) \tag{8}$$

for all $a \in A$ and all $t > 0$. Letting n to infinity in (8) and using (N_5), we see that

$$h(a) = N - \lim_{n\to\infty} 2^{-n} f(2^n a) \tag{9}$$

for all $a \in A$. Similarly, we obtain

$$h(a^2) = N - \lim_{n\to\infty} 2^{-2n} f(2^{2n} a^2) \tag{10}$$

for all $a \in A$. Using inequality (6), we get

$$N(f(2^{2n}a^2) - f(2^n a)^2, s) \geq N'(\varphi(2^n a, 2^n a), s) \geq N'(\alpha^n \varphi(a, a), s)$$

for all $a \in A$ and all $s > 0$. Thus

$$N\Big(2^{-2n}[f(2^{2n}a^2) - f(2^n a)^2], s\Big) \geq N'\Big(\varphi(a, a), \frac{2^{2n}s}{\alpha^n}\Big) \tag{11}$$

for all $a, b \in A$ and all $s > 0$. Letting n to infinity in (11) and using (N_5), we see that

$$N - \lim_{n\to\infty} 2^{-2n}[f(2^{2n}a^2) - f(2^n a)^2] = 0. \tag{12}$$

Applying (9), (10) and (12), we have

$$\begin{aligned}
h(a^2) &= N - \lim_{n\to\infty} 2^{-2n} f(2^{2n}a^2) \\
&= N - \lim_{n\to\infty} 2^{-2n}[f(2^{2n}a^2) - f(2^{2n}a^2) + f(2^n a)^2] \\
&= N - \lim_{n\to\infty} 2^{-2n} f(2^n a)^2 \\
&= \{N - \lim_{n\to\infty} 2^{-n} f(2^n a)\}^2 \\
&= h(a)^2
\end{aligned}$$

for all $a \in A$.

To prove the uniqueness of h, assume that h' is another Jordan homomorphism satisfying (7). Since both h and h' are additive, we get from (4) and (7) that

$$\begin{aligned}
N(h(a) - h'(a), t) &= N(h(2^n a) - h'(2^n a), 2^n t) \\
&= N\Big([h(2^n a) - f(2^n a)] + [f(2^n a) - h'(2^n a)], 2^n t\Big) \\
&\geq \min\Big\{N\Big(h(2^n a) - f(2^n a), \frac{2^n t}{2}\Big), N\Big(f(2^n a) - h'(2^n a), \frac{2^n t}{2}\Big)\Big\} \\
&\geq N'\Big(\frac{2\varphi(2^n a, 2^n a)}{2-\alpha}, \frac{2^n t}{2}\Big) \\
&\geq N'\Big(\varphi(a, a), \frac{2^n(2-\alpha)t}{4\alpha^n}\Big)
\end{aligned}$$

for all $a \in A$ and all $t > 0$. Letting n to infinity, we infer that

$$N(h(a) - h'(a), t) = 1$$

for all $a \in A$ and all $t > 0$. Hence (N_2) implies that $h(a) = h'(a)$ for all $a \in A$. $\square$

3. Fuzzy stability of Jordan derivations

In this section we prove the stability of Jordan derivations on fuzzy Banach algebras.

Theorem 3.1. *Let (A, N) be a fuzzy Banach algebra and (B, N') be a fuzzy normed space. Let $\varphi : A \times A \to B$ be a function such that for some $0 < \alpha < 2$,*

$$N'(\varphi(2a, 2b), t) \geq N'(\alpha\varphi(a, b), t) \tag{13}$$

for all $a, b \in A$ and all $t > 0$. Suppose that $f : A \to A$ is a function such that

$$N(f(a + b) - f(a) - f(b), t) \geq N'(\varphi(a, b), t) \tag{14}$$

and

$$N(f(a^2) - af(a) - f(a)a, s) \geq N'(\varphi(a, a), s) \tag{15}$$

for all $a, b \in A$ and all $t, s > 0$. Then there exists a unique Jordan derivation $d : A \to A$ such that

$$N(f(a) - d(a), t) \geq N'\Big(\frac{2\varphi(a, a)}{2 - \alpha}, t\Big) \tag{16}$$

where $a \in A$ and $t > 0$.

Proof. It follows from (14) and Theorem 3.1 of [15] that the mapping $d : A \to B$ defined by $d(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ is an additive function satisfying (16). Now we only need to show that d satisfies

$$d(a^2) = ad(a) + d(a)a$$

for all $a \in A$. The inequalities (13) and (16) imply that

$$N(f(2^n a) - d(2^n a), t) \geq N'\Big(\frac{2\varphi(2^n a, 2^n a)}{2 - \alpha}, t\Big) \geq N'\Big(\alpha^n \varphi(a, a), t\frac{2 - \alpha}{2}\Big)$$

for all $a \in A$ and all $t > 0$. Thus

$$N\Big(2^{-n} f(2^n a) - 2^{-n} d(2^n a), 2^{-n} t\Big) \geq N'\Big(\varphi(a, a), t\frac{2 - \alpha}{2\alpha^n}\Big)$$

for all $a \in A$ and all $t > 0$. By the additivity of d it is easy to see that

$$N(2^{-n} f(2^n a) - d(a), t) \geq N'\Big(\varphi(a, a), t\frac{2^n(2 - \alpha)}{2\alpha^n}\Big) \tag{17}$$

for all $a \in A$ and all $t > 0$. Letting n to infinity in (17) and using (N_5), we get

$$d(a) = N - \lim_{n\to\infty} 2^{-n} f(2^n a) \tag{18}$$

for all $a \in A$. Similarly, we get

$$d(a^2) = N - \lim_{n\to\infty} 2^{-2n} f(2^{2n} a^2) \tag{19}$$

for all $a \in A$. Using (13) and (15), we get

$$\begin{aligned} N\Big(f(2^{2n} a^2) - (2^n a) f(2^n a) - f(2^n a)(2^n a), s\Big) &\geq N'\Big(\varphi(2^n a, 2^n a), s\Big) \\ &\geq N'\Big(\varphi(a, a), \frac{s}{\alpha^n}\Big) \end{aligned} \tag{20}$$

for all $a \in A$ and all $s > 0$. Let $g : A \times A \to A$ be a function defined by

$$g(a, a) = f(a^2) - af(a) - f(a)a$$

for all $a \in A$. Hence, (20) implies that

$$N - \lim_{n\to\infty} 2^{-n} g(2^n a, 2^n a) = 0$$

and

$$N - \lim_{n\to\infty} 2^{-2n} g(2^n a, 2^n a) = 0 \tag{21}$$

for all $a \in A$. Since (A, N) is a fuzzy Banach algebra, applying (18), (19) and (21), we get

$$\begin{aligned} d(a^2) &= N - \lim_{n\to\infty} 2^{-2n} f(2^{2n} a^2) \\ &= N - \lim_{n\to\infty} [a2^{-n} f(2^n a) + 2^{-n} f(2^n a)a + 2^{-2n} g(2^n a, 2^n a)] \\ &= a(N - \lim_{n\to\infty} 2^{-n} f(2^n a)) + (N - \lim_{n\to\infty} 2^{-n} f(2^n a))a \\ &\quad + N - \lim_{n\to\infty} 2^{-2n} g(2^n a, 2^n a) \\ &= ad(a) + d(a)a. \end{aligned}$$

for all $a \in A$.

To prove the uniqueness property of d, assume that d' is another Jordan derivation satisfying (16). Since both d and d' are additive, we get from (13) and (16) that

$$\begin{aligned} N(d(a) - d'(a), t) &= N(d(2^n a) - d'(2^n a), 2^n t) \\ &= N\Big([d(2^n a) - f(2^n a)] + [f(2^n a) - d'(2^n a)], 2^n t\Big) \\ &\geq \min\Big\{N\Big(d(2^n a) - f(2^n a), \frac{2^n t}{2}\Big), N\Big(f(2^n a) - d'(2^n a), \frac{2^n t}{2}\Big)\Big\} \\ &\geq N'\Big(\frac{2\varphi(2^n a, 2^n a)}{2-\alpha}, \frac{2^n t}{2}\Big) \\ &\geq N'\Big(\varphi(a, a), \frac{2^n (2-\alpha) t}{4\alpha^n}\Big) \end{aligned}$$

for all $a \in A$ and all $t > 0$. Letting n to infinity in the above inequality, we get $N(d(a) - d'(a), t) = 1$ for all $a \in A$ and all $t > 0$. Hence $d(a) = d'(a)$ for all $a \in A$. □

4. Conclusion

We establish the generalized Hyers–Ulam stability of Jordan homomorphisms and Jordan derivations on fuzzy Banach algebras. We show that every approximately Jordan homomorphism (Jordan derivation) is near to an exact Jordan homomorphism (Jordan derivation).

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